The Binet Forms for the Fibonacci and Lucas Numbers

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Abstract

The Binet forms for the well known Fibonacci and Lucas Sequences $\{F_n\}$ and $\{L_n\}$are discussed, in detail, in this paper.

Keywords: Fibonacci F_n, Lucas L_n, Binet forms

I. Introduction

Among numerical Sequences, real valued or complex valued, the Fibonacci sequence F_n defined by the recurrence relation,

$$F_{n+2} = f_{n+1} + f_n \ , \ n \geq 0 \ , \ f_0 = 0 \ , f_1 = 1 \qquad (1.1)$$

has found not only an historic importance, but also an applicable value from mathematics point of view.

It is one of the two shining stars, the second being Lucas Sequence defined by the recurrence relation,

$$L_{n+2} = L_{n+1} + L_n \ , \quad n \geq 0 \ , \ L_0 = 2 \ , \ L_1 = 1 \qquad (1.2)$$

Fibonacci sequence is more famous than the Lucas Sequence. In 1202, the Italian mathematician Leonardo Fibonacci, also known as Leonardo of pisono, introduced the Fibonacci sequence $\{F_n\}$,with the help of the rabbit problem [4].

Changing the initial values give in the definition of Fibonacci sequence, different sequences can be obtained. These are known as Fibonacci like sequences. One such a sequence is the Lucas Sequence $\{L_n\}$,

$$L_n = L_{n-2} + L_{n-1} \ , L_1 = 1, L_2 = 3, n \geq 3,$$

named after E Lucas (1842 - 1891).

In the next section we give the detail proof of the Binet formula for $\{F_n\}$ and its extension for $\{L_n\}$.

II. The Binet forms for the Fibonacci and Lucas Numbers

The Sequence $\{F_n\}$ is recursively defined by

$F_n = F_{n-2} + F_{n-1}, \ F_1 = 1, \ F_2 = 1, \ n \geq 3$ is well known as the Fibonacci Sequence with the same recurrence relation, changing initial values, we get the Lucas Sequence defined by

$L_n = L_{n-2} + L_{n-1}, \ L_1 = 1, \ L_2 = 3, \ n \geq 3,$ The two sequences are related by the formula $L_n = F_{n-1} + F_{n+1}, \ n \geq 1$.

More relations between the sequences can be found in Hoggatt [2].

Let $\propto$ and β be the roots of the quadratic equation.

$$x^2 - x - 1 = 0 \ ,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ & $\beta = \frac{1-\sqrt{5}}{2}$.

The two roots play an important role in studying the Fibonacci and Lucas Sequences. The number alpha is called the golden ratio.

$\propto+\beta=1, \propto-\beta=\sqrt{5}, \propto\beta=-1, \propto^2=\propto+1$&

$\beta^2=\beta+1$ (1.2)

The equation (1.1) is called the recurrence relation for Fibonacci sequence. The Binet form for the Fibonacci numbers, is given by

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}, \text{where } n=1,2,3,\ldots\ldots\ldots\ldots$$

We give below the proof of this formula.

Proof : We will prove (1.3) by using mathematics induction on n.

When n = 1, it is clear that

$$F_1=1=\frac{\alpha-\beta}{\alpha-\beta}$$

Hence, $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ is true for n = 1 Let k be an arbitrary positive integer. Suppose , $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ for any positive integer form 1 to k. Then we must show that equation also holds when n = k + 1.

Thus, by the inductive hypothesis, we have

$$F_k=\frac{\alpha^k-\beta^k}{\alpha-\beta}$$ and

$$F_{k-1}=\frac{\alpha^{k-1}-\beta^{k-1}}{\alpha-\beta}$$

Hence it follows that

$$F_{k+1}=F_k+F_{k-1}$$

$$=\frac{\alpha^k-\beta^k}{\alpha-\beta}+\frac{\alpha^{k-1}-\beta^{k-1}}{\alpha-\beta}$$

$$=\frac{\alpha^{k-1}(\alpha+1)-\beta^{k-1}(\beta+1)}{\alpha-\beta}$$

$$=\frac{\alpha^{k-1}\alpha^2-\beta^{k-1}\beta^2}{(1.3)-\beta} \quad \text{by } (1.2)$$

$$=\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}$$

Thus, by the principle of mathematical induction, $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ is true for all positive integer n and the proof is complete.

However, in 2004, another proof are also given by, sury [3] based by (1.3).

Therefore the Binet form for the Lucas numbers is given by

$$L_n=\alpha^n+\beta^n, \text{where } n=1,2,3,\ldots\ldots\ldots\ldots$$

Proof : we have prove (1.4) by using mathematical induction on n.

When n = 1, then

$L_1=1=\alpha+\beta$ by (1.2). Hence

$L_n=\alpha^n+\beta^n$ is true for n = 1.

We now suppose that $L_n=\alpha^n+\beta^n$ is true for any integer n form 1 to k, where k is an arbitrary positive integer. Then, we will show the above equation also holds when n = k + 1.

Thus, we have

$$
\begin{aligned}
L_{k+1} &= L_k + L_{k-1} \\
&= \alpha^k + \beta^k + \alpha^{k-1} + \beta^{k-1} \\
&= \alpha^{k-1}(\alpha+1) + \beta^{k-1}(\beta+1) \\
&= \alpha^{k-1}\alpha^2 + \beta^{k-1}\beta^2 \qquad \text{by} \quad (1.2) \\
&= \alpha^{k+1} + \beta^{k+1}.
\end{aligned}
$$

Hence, by the principle of mathematical induction,

$L_n = \alpha^n + \beta^n$ is true for all positive integers n.

III. Conclusion

In the present article, we have listed only various types of Binet form available in the literature. Further a few basic result on Binet Form for The Fibonacci and the Lucas numbers have been derived. There are short proof are covered here. The article is partly based on the contents of paper [1], [2], [3] are cited in the bibliography.

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