GENERALIZED LUCAS NUMBERS AND RELATIONS WITH GENERALIZED FIBONACCI NUMBERS

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Abstract. In this paper, we present a new generalization of the Lucas numbers by matrix representation using Genaralized Lucas Polynomials. We give some properties of this new generalization and some relations between the generalized order-k Lucas numbers and generalized order-k Fibonacci numbers. In addition, we obtain Binet formula and combinatorial representation for generalized order-k Lucas numbers by using properties of generalized Fibonacci numbers.

1. Introduction

There are various types of generalization of Fibonacci and Lucas numbers. For example Er[1] defined the generalized order-k Fibonacci numbers(GOkF), Kılıç[6] defined the generalized order-k Pell numbers(GOkP) and Taşçı[3] defined the generalized order-k Lucas numbers (GOkL). MacHenry[7] defined Generalized Fibonacci and Lucas Polynomials and MacHenry[8] defined matrices $A_{(k)}^{\infty}$ and $D_{(k)}^{\infty}$ depending on these polynomials. $A_{(k)}^{\infty}$ is reduced to GOkF when $t_i = 1$ and $A_{(k)}^{\infty}$ is reduced to GOkP when $t_1 = 2$ and $t_i = 1$ (for $2 \leq i \leq k$). This analogy shows the importance of the matrix $A_{(k)}^{\infty}$ and Generalized Fibonacci and Lucas polynomials in generalizations. However, Lucas generalization of Taşçı[3] is not compatible with the matrix $A_{(k)}^{\infty}$ and Generalization Fibonacci and Lucas polynomials, we studied on generalized order-k Lucas numbers $l_{k,n}$ (GOkL) and k sequences of the generalized order-k Lucas numbers $l_{k,n}^{i}$ (kSOkL) with the help of Lucas Polynomials $G_{k,n}$ and the matrix $D_{(k)}^{\infty}$. In this paper, after presenting a matrix representation of $l_{k,n}^{i}$, we derived a relations between generalized order-k Fibonacci numbers(GOkF) and GOkL, as well as relation between kSOkL and k sequences of the generalized order-k Fibonacci numbers $f_{k,n}^{\ i}$(kSOkF). Since many properties of Fibonacci numbers and it's generalizations are known, these relations are very important. Using these relations, properties of Lucas numbers and properties of it's generalizations can be obtained. In addition to obtaining these relations, we give a generalized Binet formula and combinatorial representation for kSOkL with the help of properties of generalized Fibonacci numbers.

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1.1. **Fibonacci and Lucas Numbers and Properties of Fibonacci Generalization.**

The well-known Fibonacci sequence $\{f_n\}$ is defined recursively by the equation,

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 3$$

where $f_1 = 1$, $f_2 = 1$ and Lucas sequence $\{l_n\}$ is defined recursively by the equation,

$$l_n = l_{n-1} + l_{n-2}, \text{ for } n \geq 2$$

where $l_0 = 2$, $l_1 = 1$.

Miles [10] defined generalized order-k Fibonacci numbers(GOkF) as,

$$f_{k,n} = \sum_{j=1}^{k} f_{k,n-j} \tag{1.1}$$

for $n > k \geq 2$, with boundary conditions: $f_{k,1} = f_{k,2} = f_{k,3} = \cdots = f_{k,k-2} = 0$, $f_{k,k-1} = f_{k,k} = 1$.
Er [1] defined kSOkF as; for $n > 0$, $1 \leq i \leq k$

$$f_{k,n}^{\ i} = \sum_{j=1}^{k} c_j f_{k,n-j}^{\ i} \tag{1.2}$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$f_{k,n}^{\ i} = \begin{cases} 1 & \text{if } \ i = 1 - n, \\ 0 & \quad \text{otherwise,} \end{cases}$$

where c_j $(1 \leq j \leq k)$ are constant coefficients, $f_{k,n}^{\ i}$ is the n-th term of i-th sequence of order k generalization. k-th column of this generalization involves the Miles generalization for $i = k$, i.e. $f_{k,n}^{k} = f_{k,k+n-2}$.

Er [1] showed

$$F_{n+1}^{\sim} = AF_n^{\sim}$$

where

$$A = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{k-1} & c_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

is $k \times k$ companion matrix and

$$F_n^{\sim} = \begin{bmatrix} f_{k,n}^{1} & f_{k,n}^{2} & \cdots & f_{k,n}^{k} \\ f_{k,n-1}^{1} & f_{k,n-1}^{2} & \cdots & f_{k,n-1}^{k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k,n-k+1}^{1} & f_{k,n-k+1}^{2} & \cdots & f_{k,n-k+1}^{k} \end{bmatrix} \tag{1.3}$$

is $k \times k$ matrix.

Karaduman [5] showed $F_1^{\sim} = A$ and $F_n^{\sim} = A^n$ for $c_j = 1$, $(1 \leq j \leq k)$.

Kalman [2] derived the Binet formula by using Vandermonde matrix, for λ_i $(1 \leq i \leq k)$ are roots of the polynomial

$$P(x; t_1, t_2, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k \tag{1.4}$$

($t_{1,\dots,}t_k$ are constants)

$$f_{k,n}^{k}=\sum_{i=1}^{k}\frac{(\lambda_i)^n}{P'(\lambda_i)} \tag{1.5}$$

where $f_{k,n}^{k}$ is (for $c_j=1$, $1\le j\le k$ and $i=k$) k-th sequences of kSOkF and $P'(x)$ is derivative of the polynomial (1.4).

Kılıç [5] studied $F_n^{\sim}$ and $f_{k,n}^{k}$ and gave some formulas and properties concerning kSOkF. One of these is Binet formula for kSOkF. For roots of (1.4) named as λ_i $(1\le i\le k)$,

$$V=\begin{bmatrix}\lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1}\\ \lambda_1^{k-2} & \lambda_2^{k-2} & \dots & \lambda_k^{k-2}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_1 & \lambda_2 & \dots & \lambda_k\\ 1 & 1 & \dots & 1\end{bmatrix} \text{ and } d_k^i=\begin{bmatrix}\lambda_1^{k-i+n}\\ \lambda_2^{k-i+n}\\ \vdots\\ \lambda_k^{k-i+n}\end{bmatrix} \tag{1.6}$$

where V is a $k\times k$ Vandermonde matrix and $V_j^{(i)}$ is a $k\times k$ matrix obtained from V by replacing j-th column of V by d_k^i, Binet formula of $f_{k,n}^{k}$ is;

$$f_{k,n}^{k}=t_{1k}=\frac{\det(V_k^{(1)})}{\det(V)}. \tag{1.7}$$

1.2. **Generalized Fibonacci and Lucas Polynomials.** MacHenry [7] defined generalized Fibonacci polynomials ($F_{k,n}(t)$), Lucas polynomials ($G_{k,n}(t)$) and obtained important relations between generalized Fibonacci and Lucas polynomials, where t_i $(1\le i\le k)$ are constant coefficients of the core polynomial (1.4). $F_{k,n}(t)$ defined inductively by

$$\begin{aligned}F_{k,n}(t) &= 0,\ n<0\\ F_{k,0}(t) &= 1\\ F_{k,1}(t) &= t_1\\ F_{k,n+1}(t) &= t_1F_{k,n}(t)+\dots+t_kF_{k,n-k+1}(t)\end{aligned}$$

where $t=(t_1,t_2,\dots,t_k)$, $k\in\mathbb{N}$, n is an integer and $G_{k,n}(t_1,t_2,\dots,t_k)$ defined by

$$\begin{aligned}G_{k,n}(t) &= 0,\ n<0\\ G_{k,0}(t) &= k\\ G_{k,1}(t) &= t_1\\ G_{k,n+1}(t) &= t_1G_{k,n}(t)+\dots+t_kG_{k,n-k+1}(t).\end{aligned} \tag{1.8}$$

In addition, in [9] authors obtained $F_{k,n}(t)$ and $G_{k,n}(t)$ $(n,k\in\mathbb{N},\ n\ge 1)$ as

$$F_{k,n}(t)=\sum_{a\vdash n}\binom{|a|}{a_{1,\dots,}a_k}t_1^{a_1}\dots t_k^{a_k} \tag{1.9}$$

and

$$G_{k,n}(t)=\sum_{a\vdash n}\frac{n}{|a|}\binom{|a|}{a_{1,\ldots,a_k}}t_1^{a_1}\ldots t_k^{a_k} \tag{1.10}$$

where a_i are nonnegative integers for all i $(1\leq i\leq k)$, with initial conditions given by

$$F_{k,0}(t)=1,\ F_{k,-1}(t)=0,\ \ \ldots,\ F_{k,-k+1}(t)=0$$

and

$$G_{k,0}(t)=k,\ G_{k,-1}(t)=0,\ \ \cdots,\ G_{k,-k+1}(t)=0.$$

In this paper, the notations $a\vdash n$ and $|a|$ are used instead of $\sum_{j=1}^{k}ja_j=n$ and $\sum_{j=1}^{k}a_j$, respectively. A combinatorial representation for Fibonacci polynomials is given in [9] as

$$F_{2,n}(t)=\sum_{j=0}^{\lceil\frac{n}{2}\rceil}(-1)^j\binom{n-j}{j}F_1^{n-2j}(-t_2)^j \tag{1.11}$$

for $n\in\mathbb{Z}$, where $\lceil\frac{n}{2}\rceil=k$, either $n=2k$ or $n=2k-1$.

In [8], matrices $A_{(k)}^{\infty}$ and $D_{(k)}^{\infty}$ are defined by using the following matrix,

$$A_{(k)}=\begin{bmatrix} 0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 1\\ t_k & t_{k-1} & t_{k-2} & \ldots & t_1 \end{bmatrix}.$$

They also record the orbit of the k-th row vector of $A_{(k)}$ under the action of $A_{(k)}$, below $A_{(k)}$, and the orbit of the first row of $A_{(k)}$ under the action of $A_{(k)}^{-1}$ on the first row of $A_{(k)}$ is recorded above $A_{(k)}$, and consider the $\infty\times k$ matrix whose row vectors are the elements of the doubly infinite orbit of $A_{(k)}$ acting on any one of them. For $k=3$, $A_{(k)}^{\infty}$ looks like this

$$A_{(3)}^{\infty}=\begin{bmatrix} \cdots & \cdots & \cdots\\ S_{(-n,1^2)} & -S_{(-n,1)} & S_{(-n)}\\ \cdots & \cdots & \cdots\\ S_{(-3,1^2)} & -S_{(-3,1)} & S_{(-3)}\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ t_3 & t_2 & t_1\\ \cdots & \cdots & \cdots\\ S_{(n-1,1^2)} & -S_{(n-1,1)} & S_{(n-1)}\\ S_{(n,1^2)} & -S_{(n,1)} & S_{(n)}\\ \cdots & \cdots & \cdots \end{bmatrix}$$

and

$$A^n_{(k)} = \begin{bmatrix} (-1)^{k-1}S_{(n-k+1,1^{k-1})} & \cdots & (-1)^{k-j}S_{(n-k+1,1^{k-j})} & \cdots & S_{(n-k+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{k-1}S_{(n,1^{k-1})} & \cdots & (-1)^{k-j}S_{(n,1^{k-j})} & \cdots & S_{(n)} \end{bmatrix}$$

where

$$S_{(n-r,1^r)} = (-1)^r \sum_{j=r+1}^{n} t_j S_{(n-j)},\ 0 \le r \le n. \tag{1.12}$$

Derivative of the core polynomial $P(x;t_1,t_2,\dots,t_k) = x^k - t_1x^{k-1} - \cdots - t_k$ is $P'(x) = kx^{k-1} - t_1(k-1)x^{k-2} - \cdots - t_{k-1}$, which is represented by the vector $(-t_{k-1},\dots,-t_1(k-1),k)$ and the orbit of this vector under the action of $A_{(k)}$ gives the standard matrix representation $D^{\infty}_{(k)}$.

Right hand column of $A^{\infty}_{(k)}$ contains sequence of the generalized Fibonacci polynomials $F_{k,n}(t)$ and $tr(A^n_{(k)}) = G_{k,n}(t)$ for $n \in \mathbb{Z}$, where $G_{k,n}(t)$ is the sequence of the generalized Lucas polynomials, which is also a t-linear recursion. In addition, the right hand column of $D^{\infty}_{(k)}$ contains sequence of the generalized Lucas polynomials $G_{k,n}(t)$.

It is clear that, for $t_i = 1$ and $c_i = 1$ $(1 \le i \le k)$ $S_{(n)} = f^1_{k,n}$ where $f^1_{k,n}$, is the n-th term of the first sequence of kSOkF. Moreover, the matrix $A^{\infty}_{(k)}$ involves the generalization (1.2).

Example 1.1. *We give matrix $A^{\infty}_{(k)}$ for k=3 and the matrix $D^{\infty}_{(k)}$ for $k = 4$, while* $t_1 = t_2 = \cdots = t_k = 1$

$$A^{\infty}_{(3)} = \begin{bmatrix} \cdots & \cdots & \cdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ \cdots & \cdots & \cdots \end{bmatrix} \text{ and } D^{\infty}_{(4)} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 7 & 1 & 0 & -1 \\ -1 & 6 & 0 & -1 \\ -1 & -2 & 5 & -1 \\ -1 & -2 & -3 & 4 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

2. Generalizations of Lucas Numbers

For $t_s = 1$, $1 \le s \le k$, the Lucas polynomials $G_{k,n}(t)$ and $D^{\infty}_{(k)}$ together are reduced to

$$l_{k,n} = \sum_{j=1}^{k} l_{k,n-j} \tag{2.1}$$

with boundary conditions

$$l_{k,1-k} = l_{k,2-k} = \ldots = l_{k,-1} = -1 \text{ and } l_{k,0} = k,$$

which is called generalized order-k Lucas numbers(GOkL). When k=2, it is reduced to ordinary Lucas numbers.

In this paper, we study on positive direction of $D^{\infty}_{(k)}$ for $t_s = 1$, $1 \le s \le k$, which can be written explicitly as

$$l_{k,n}^{i}=\sum_{j=1}^{k} l_{k,n-j}^{i} \tag{2.2}$$

for $n>0$ and $1\leq i\leq k$, with boundary conditions

$$l_{k,n}^{i}=\begin{cases} -i & \text{if } i-n<k, \\ -2n+i & \text{if } i-n=k, \\ k-i-1 & \text{if } i-n>k \end{cases}$$

for $1-k\leq n\leq 0$, where $l_{k,n}^{i}$ is the n-th term of i-th sequence. This generalization is called k sequences of the generalized order-k Lucas numbers(kSOkL).

Although names are the same, the initial conditions of this generalization are different from the generalizations in [3]. These initial conditions arise from Lucas Polynomials and $D_{(k)}^{\infty}$.

When $i=k=2$, we obtain ordinary Lucas numbers and $l_{k,n}^{k}=l_{k,n}$.

Example 2.1. *Substituting $k=3$ and $i=2$ we obtain the generalized order-3 Lucas sequence as;*

$$l_{3,-2}^{2}=0,\ l_{3,-1}^{2}=4,\ l_{3,0}^{2}=-2,\ l_{3,1}^{2}=2,\ l_{3,2}^{2}=4,\ l_{3,3}^{2}=4,\ \ldots$$

Lemma 2.2. *Matrix multiplication and (2.2) can be used to obtain*

$$L_{n+1}^{\sim}=A_1L_{n}^{\sim}$$

where

$$A_1=\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{k\times k}=\begin{bmatrix} 1 & 1 & \ldots & 1 \\ & & & 0 \\ & I & & \vdots \\ & & & 0 \end{bmatrix}_{k\times k} \tag{2.3}$$

where I is $(k-1)\times(k-1)$ identity matrix and we define a $k\times k$ matrix $L_{n}^{\sim}$ as;

$$L_{n}^{\sim}=\begin{bmatrix} l_{k,n}^{1} & l_{k,n}^{2} & \cdots & l_{k,n}^{k} \\ l_{k,n-1}^{1} & l_{k,n-1}^{2} & \cdots & l_{k,n-1}^{k} \\ \vdots & \vdots & \ddots & \vdots \\ l_{k,n-k+1}^{1} & l_{k,n-k+1}^{2} & \cdots & l_{k,n-k+1}^{k} \end{bmatrix} \tag{2.4}$$

which is contained by $k\times k$ block of $D_{(k)}^{\infty}$ for $t_i=1$, $1\leq i\leq k$.

Lemma 2.3. *Let A_1 and $L_{n}^{\sim}$ be as in (2.3) and (2.4), respectively. Then $L_{n+1}^{\sim}=A_1^{n+1}L_{0}^{\sim}$, where*

$$L_{0}^{\sim}=\begin{bmatrix} -1 & -2 & -3 & \ldots & -(k-2) & -(k-1) & k \\ -1 & -2 & -3 & \ldots & -(k-2) & k+1 & -1 \\ \vdots & \vdots & \vdots & \ldots & k+2 & 0 & -1 \\ -1 & -2 & 2k-3 & \ldots & 1 & 0 & -1 \\ -1 & 2k-2 & k-4 & \ldots & \vdots & \vdots & \vdots \\ 2k-1 & k-3 & k-4 & \ldots & 1 & 0 & -1 \end{bmatrix}_{k\times k}.$$

Proof. It is clear that $L_1^{\sim} = A_1 L_0^{\sim}$ and $L_{n+1}^{\sim} = A_1 L_n^{\sim}$. So by induction and properties of matrix multiplication, we have $L_{n+1}^{\sim} = A^{n+1} L_0^{\sim}$. □

Lemma 2.4. *Let $F_n^{\sim}$ and $L_n^{\sim}$ be as in (1.3) and (2.4), respectively. Then*

$$L_n^{\sim} = F_n^{\sim} L_0^{\sim}.$$

Proof. Proof is trivial from $F_n^{\sim} = A_1^n$(see [4]) and Lemma 2.3. □

Example 2.5. *From Lemma 2.4 for $k = 2$, we have*

$$\begin{bmatrix} l_{2,n}^1 & l_{2,n}^2 \\ l_{2,n-1}^1 & l_{2,n-1}^2 \end{bmatrix} = \begin{bmatrix} f_{2,n}^1 & f_{2,n}^2 \\ f_{2,n-1}^1 & f_{2,n-1}^2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}.$$

Therefore, $l_{2,n}^2 = 2f_{2,n}^1 - f_{2,n}^2$. Since $f_{2,n}^1 = f_{2,n+1}^2$ for all $n \in \mathbb{Z}$, then we have

$$l_{2,n}^2 = 2f_{2,n+1}^2 - f_{2,n}^2$$

where $l_{2,n}^2$ and $f_{2,n}^2$ are ordinary Lucas and Fibonacci numbers, respectively. For $k = 3$, we have

$$\begin{bmatrix} l_{3,n}^1 & l_{3,n}^2 & l_{3,n}^3 \\ l_{3,n-1}^1 & l_{3,n-1}^2 & l_{3,n-1}^3 \\ l_{3,n-2}^1 & l_{3,n-2}^2 & l_{3,n-2}^3 \end{bmatrix} = \begin{bmatrix} f_{3,n}^1 & f_{3,n}^2 & f_{3,n}^3 \\ f_{3,n-1}^1 & f_{3,n-1}^2 & f_{3,n-1}^3 \\ f_{3,n-2}^1 & f_{3,n-2}^2 & f_{3,n-2}^3 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ -1 & 4 & -1 \\ 5 & 0 & -1 \end{bmatrix}.$$

Therefore, $l_{3,n}^3 = 3f_{3,n}^1 - f_{3,n}^2 - f_{3,n}^3$. Since for k=3, $f_{3,n}^1 = f_{3,n+1}^3$ and $f_{3,n}^2 = f_{3,n-1}^1 + f_{3,n-1}^3 = f_{3,n}^3 + f_{3,n-1}^3$ for all $n \in \mathbb{Z}$, we have

$$l_{3,n}^3 = 3f_{3,n+1}^3 - 2f_{3,n}^3 - f_{3,n-1}^3.$$

Theorem 2.6. *For $i = k$, $n \geq 0$ and $c_1 = \cdots = c_k = 1$,*

$$l_{k,n}^k = kf_{k,n+1}^k - (k-1)f_{k,n}^k - \cdots - f_{k,n-k+2}^k = kf_{k,n+1}^k - \sum_{j=2}^{k}(k-j+1)f_{k,n+2-j}^k \tag{2.5}$$

where $l_{k,n}^i$ and $f_{k,n}^i$ kSOkL and kSOkF, respectively.

Proof. We use mathematical induction to prove the following equality

$$l_{k,n}^k = kf_{k,n+1}^k - \sum_{j=2}^{k}(k-j+1)f_{k,n+2-j}^k.$$

It is easy to obtain $l_{k,0}^k = k$, $f_{k,0}^k = 0$ and $f_{k,1}^k = 1$ for all $k \in \mathbb{Z}^+$ with $k \geq 2$, from the definition of kSOkL and kSOkF. So, the equation (2.5) is true for $n = 0$, i.e.,

$$l_{k,0}^k = kf_{k,1}^k - (k-1)f_{k,0}^k - \cdots - f_{k,-k+2}^k = k.1 + 0 = k.$$

Suppose that the equation holds for all positive integers less than or equal to n i.e., for integer n

$$l_{k,n}^k = kf_{k,n+1}^k - \sum_{j=2}^{k}(k-j+1)f_{k,n+2-j}^k$$

then from (1.2) and (2.2), for $c_1 = \cdots = c_k = 1$, we get;

$$\begin{aligned}
l^k_{k,n+1} &= l^k_{k,n} + l^k_{k,n-1} + l^k_{k,n-2} + \cdots + l^k_{k,n-k+1} \\
&= (kf^k_{k,n+1} - (k-1)f^k_{k,n} - \cdots - f^k_{k,n-k+2}) + \\
&\quad (kf^k_{k,n} - (k-1)f^k_{k,n-1} - \cdots - f^k_{k,n-k+1}) + \\
&\quad \ldots + (kf^k_{k,n-k+2} - (k-1)f^k_{k,n-k+1} - \cdots - f^k_{k,n-2k+3}) \\
&= kf^k_{k,n+2} - (k-1)f^k_{k,n+1} - \cdots - f^k_{k,n-k+3} \\
&= kf^k_{k,n+2} - \sum_{j=2}^{k}(k-j+1)f^k_{k,n+3-j}.
\end{aligned}$$

So, the equation holds for $(n+1)$ and proof is complete. □

Since $f^k_{k,n} = f_{k,n+k-2}$ and $l^k_{k,n} = l_{k,n}$ the following relation is obvious

$$l_{k,n} = kf_{k,n+k-1} - \sum_{j=2}^{k}(k-j+1)f_{k,n+k-j}$$

where $f_{k,n}$ is the n-th GOkF as in (1.1), $l_{k,n}$ is GOkL as in (2.1) and $l^k_{k,n}$ is the n-th term of k-th sequences of the kSOkL as in (2.2).

The following theorem shows that equation (2.5) is valid for Generalized Fibonacci and Lucas Polynomials as well.

Theorem 2.7. *For $k \geq 2$ and $n \geq 0$,*

$$G_{k,n}(t) = kF_{k,n}(t) - \sum_{j=2}^{k}(k-j+1)t_{j-1}F_{k,n+1-j}(t)$$

where $F_{k,n}(t)$ and $G_{k,n}(t)$ are the Generalized Fibonacci and Lucas Polynomials, respectively.

Proof. Proof is by induction as Theorem 2.6. □

Theorem 2.8. *For $i = k$ and $n \geq 0$,*

$$l^k_{k,n} = \sum_{j=1}^{k} j f^k_{k,n+1-j} \tag{2.6}$$

where $l^i_{k,n}$ and $f^{\,i}_{k,n}$ are the kSOkL and kSOkF respectively.

Proof. Proof is by induction as Theorem 2.6. □

Lemma 2.9. *For $k \geq 2$, i-th sequences of kSOkL in terms of k-th sequences of kSOkL is*

$$l^i_{k,n} = \begin{cases} l^k_{k,n-1} & \text{if } i = 1 \\ \sum\limits_{m=1}^{i} l^k_{k,n-m} & \text{if } 1 < i < k \\ l^k_{k,n} & \text{if } i = k \end{cases}. \tag{2.7}$$

Theorem 2.10. *i-th sequences of kSOkL can be written in terms of k-th sequences of kSOkF (which is GOkF with index iteration) in different ways;*

i)For $k \geq 3$ and $1 \leq i \leq k$

$$l^i_{k,n} = \sum_{j=1}^{k+i-1} d_j f^k_{k,n-j}$$

where $1 \leq i \leq k$, $n \geq 0$ and constant coefficient

$$d_j = \begin{cases} \frac{j(j+1)}{2} & \text{if } 1 \leq j \leq i \\ \frac{j(j+1)}{2} - \frac{(j-i)(j-i+1)}{2} & \text{if } i+1 \leq j \leq k-1 \\ \frac{k(k+1)}{2} - \frac{(j-i)(j-i+1)}{2} & \text{if } k \leq j \leq k+i-1 \end{cases}$$

ii)For $k \geq 2$ and $1 \leq i \leq k$

$$l^i_{k,n} = \begin{cases} kf^k_{k,n} - \sum\limits_{j=2}^{k}(k-j+1)f^k_{k,n+1-j} & \text{if } i=1 \\ \sum\limits_{m=1}^{i} kf^k_{k,n-m+1} - \sum\limits_{m=1}^{i}\sum\limits_{j=2}^{k}(k-j+1)f^k_{k,n-m-j+2} & \text{if } 1<i<k \\ kf^k_{k,n+1} - \sum\limits_{j=2}^{k}(k-j+1)f^k_{k,n+2-j} & \text{if } i=k \end{cases}$$

iii)For $k \geq 2$ and $1 \leq i \leq k$

$$l^i_{k,n} = \begin{cases} \sum\limits_{j=1}^{k} jf^k_{k,n-j} & \text{if } i=1 \\ \sum\limits_{m=1}^{i}\sum\limits_{j=1}^{k} jf^k_{k,n-m-j+1} & \text{if } 1<i<k \\ \sum\limits_{j=1}^{k} jf^k_{k,n+1-j} & \text{if } i=k \end{cases}.$$

Proof. i)Proof is from Theorem 2.8 and Lemma 2.9.
ii)Proof is from Theorem 2.6 and Lemma 2.9.
iii)Proof is from Theorem 2.8 and Lemma 2.9. □

Example 2.11. *Let us obtain $l^i_{k,n}$ for $k=4$, $n=4$ and $i=3$ by using Theorem (2.10 iii).*

$$\begin{aligned} l^3_{4,4} &= \sum_{m=1}^{3}\sum_{j=1}^{4} j.f^4_{4,4-m-j+1} = \sum_{m=1}^{3}(f^4_{4,4-m} + 2f^4_{4,3-m} + 3f^4_{4,2-m} + 4f^4_{4,1-m}) \\ &= f^4_{4,3} + 2f^4_{4,2} + 3f^4_{4,1} + 4f^4_{4,0} + f^4_{4,2} + 2f^4_{4,1} + f^4_{4,1} = 11 \end{aligned}$$

since $f^4_{4,0}=0$, $f^4_{4,1}=f^4_{4,2}=1$ and $f^4_{4,3}=2$.

Theorem 2.12. *Let $l^i_{k,n}$ and $f^i_{k,n}$ be the kSOkL and kSOkF, respectively. Then, for $m,n \in \mathbb{Z}$ and $1 \leq i \leq k-1$,*

$$l^i_{n+m} = \sum_{j=1}^{i}(l^k_{m-j}\sum_{s=1}^{j} f^s_n) + \sum_{j=i+1}^{k}(l^k_{m-j}\sum_{s=j-i+1}^{j} f^s_n) + \sum_{j=k+1}^{k+i-1}(l^k_{m-j}\sum_{s=j-i+1}^{k} f^s_n)$$

where we assume that, the sum is equal to zero, if the subscript is greater than the superscript in the sum.

Proof. We know that $L_n^{\sim} = F_n^{\sim} L_0^{\sim}$ (Lemma 2.4), so we can write that

$$L_{n+m}^{\sim} = F_{n+m}^{\sim} L_0^{\sim} = A_1^{n+m} L_0^{\sim} = A_1^n A_1^m L_0^{\sim} = A_1^n L_m^{\sim} = F_n^{\sim} L_m^{\sim}.$$

From this matrix product and Lemma 2.9 we obtain

$$\begin{aligned} l_{k,n+m}^i &= f_{k,n}^1 l_{k,m}^i + \cdots + f_{k,n}^k l_{k,m-k+1}^i \\ &= f_{k,n}^1 (l_{k,m-1}^k + \cdots + l_{k,m-i}^k) + \cdots + f_{k,n}^k (l_{k,m-k}^k + \cdots + l_{k,m-k-i-1}^k) \\ &= l_{k,m-1}^k f_n^1 + l_{k,m-2}^k (f_{k,n}^1 + f_{k,n}^2) + \cdots l_{k,m-i}^k (f_{k,n}^1 + f_{k,n}^2 + \cdots + f_{k,n}^i) + \\ &\quad l_{k,m-i-1}^k (f_{k,n}^2 + f_{k,n}^3 + \cdots + f_{k,n}^{i+1}) + \cdots + l_{k,m-k}^k (f_{k,n}^{k-i+1} + \cdots + f_{k,n}^k) + \\ &\quad l_{k,m-k-1}^k (f_{k,n}^{k-i+2} + \cdots + f_{k,n}^k) + \cdots + l_{k,m-k-i-1}^k f_{k,n}^k \\ &= \sum_{j=1}^{i} (l_{k,m-j}^k \sum_{t=1}^{j} f_{k,n}^t) + \sum_{j=i+1}^{k} (l_{k,m-j}^k \sum_{t=j-i+1}^{j} f_{k,n}^t) + \sum_{j=k+1}^{k+i-1} (l_{k,m-j}^k \sum_{t=j-i+1}^{k} f_{k,n}^t). \end{aligned}$$

Example 2.13. *Let us obtain $l_{k,n+m}^i$ for $k = 5$, $i = 3$, $n = 3$ and $m = 4$ by using Theorem 2.12;*

$$\begin{aligned} l_{5,3+4}^3 &= l_7^3 = \sum_{j=1}^{3} (l_{5,4-j}^5 \sum_{t=1}^{j} f_{5,3}^t) + \sum_{j=4}^{5} (l_{5,4-j}^5 \sum_{t=j-2}^{j} f_{5,3}^t) + \sum_{j=6}^{7} (l_{5,4-j}^5 \sum_{t=j-2}^{5} f_{5,3}^t) \\ &= l_{5,3}^5 f_{5,3}^1 + l_{5,2}^5 (f_{5,3}^1 + f_{5,3}^2) + l_{5,1}^5 (f_{5,3}^1 + f_{5,3}^2 + f_{5,3}^3) + l_{5,0}^5 (f_{5,3}^2 + f_{5,3}^3 + f_{5,3}^4) \\ &\quad + l_{5,-1}^5 (f_{5,3}^3 + f_{5,3}^4 + f_{5,3}^5) + l_{5,-2}^5 (f_{5,3}^4 + f_{5,3}^5) + l_{5,-3}^5 f_{5,3}^5 \\ &= 28 + 24 + 12 + 55 - 9 - 5 - 2 = 103. \end{aligned}$$

□

2.0.1. *Binet Formula.* We have the following corollary by (1.5) and (Theorem 2.10 *iii*).

Corollary 2.14. *For $1 \le i \le k$ and $m, n \in \mathbb{Z}^+$,*

$$l_{k,n}^i = \begin{cases} \sum_{j=1}^{k} j \sum_{i=1}^{k} \frac{(\lambda_i)^{n-j}}{P'(\lambda_i)} & \text{for } i = 1 \\ \sum_{m=1}^{i} \sum_{j=1}^{k} j \sum_{i=1}^{k} \frac{(\lambda_i)^{n-m-j+1}}{P'(\lambda_i)} & \text{for } 1 < i < k \\ \sum_{j=1}^{k} j \sum_{i=1}^{k} \frac{(\lambda_i)^{n-j+1}}{P'(\lambda_i)} & \text{for } i = k \end{cases}.$$

where $l_{k,n}^i$ is the kSOkL.

We have the following corollary by (1.7) and (Theorem 2.10 *iii*).

Corollary 2.15. *Let $l_{k,n}^i$ be the kSOkL. Then, for $1 \le i \le k$ and $m, n \in \mathbb{Z}^+$,*

$$l_{k,n}^i = \begin{cases} \sum_{j=1}^{k} j \frac{det(V_{k,n-j}^{(1)})}{det(V)} & \text{for } i = 1 \\ \sum_{m=1}^{i} \sum_{j=1}^{k} j \frac{det(V_{k,n-m-j+1}^{(1)})}{det(V)} & \text{for } 1 < i < k \\ \sum_{j=1}^{k} j \frac{det(V_{k,n-j+1}^{(1)})}{det(V)} & \text{for } i = k \end{cases}$$

where $V^{(1)}_{k,n-s}$ *is a new notation for (1.6) which depends on n, i.e.,* $V^{(1)}_{k,n-s}$ *is a* $k\times k$ *matrix obtained from V by replacing k-th column of V by*

$$d^{(1)}_{k,n-s} = \begin{bmatrix} \lambda_1^{k-1+n-s} \\ \lambda_2^{k-1+n-s} \\ \vdots \\ \lambda_k^{k-1+n-s} \end{bmatrix}.$$

2.1. **Combinatorial Representation of the Generalized Order-k Fibonacci and Lucas Numbers.** In this subsection, we obtain some combinatorial representations of i-th sequences of kSOkF and kSOkL with the help of combinatorial representations of Generalized Fibonacci and Lucas Polynomials.
i-th sequences of kSOkF can be stated in terms of k-th sequences of kSOkF as follows. For $c_i = 1$ $(1 < i < k)$,

$$f^{\,i}_{k,n} = \sum_{m=1}^{k-i+1} f^{k}_{k,n-m+1}.$$

For $t_i = 1$ $(1 < i < k)$, $F_{k,n-1}(t)$ is reduced to sequence $f^k_{k,n}$. So for $t_i = 1$ $(1 < i < k)$, $f^{\,i}_{k,n} = \sum\limits_{m=1}^{k-i+1} F_{k,n-m}(t)$ and using (1.9) we have

$$f^{\,i}_{k,n} = \sum_{m=1}^{k-i+1} \sum_{a\vdash(n-m)} \binom{|a|}{a_1,\ldots,a_k}.$$

It is obvious that, for $t_i = 1$ $(1 < i < k)$, $F_{k,n}(t) = f^1_{k,n}$ and $F_{k,n}(t) = f^k_{k,n+1}$, respectively. Then, for all $m, n \in \mathbb{Z}^+$,

$$f^{\,i}_{k,n} = \begin{cases} \sum\limits_{a\vdash n} \binom{|a|}{a_1,\ldots,a_k} & \text{if } i=1 \\ \sum\limits_{m=1}^{k-i+1} \sum\limits_{a\vdash(n-m)} \binom{|a|}{a_1,\ldots,a_k} & \text{if } 1<i<k \\ \sum\limits_{a\vdash(n-1)} \binom{|a|}{a_1,\ldots,a_k} & \text{if } i=k \end{cases}. \tag{2.8}$$

Lemma 2.16. [5] *Let* $f^k_{k,n}$ *be the k-th sequences of kSOkF, then,*

$$f^k_{k,n} = \sum_{m\vdash(n-1+k)} \frac{m_k}{|m|} \times \binom{|m|}{m_1,\ldots,m_k}$$

where $m = (m_1, m_2, \ldots, m_k)$ *nonnegative integers satisfying* $m_1 + 2m_2 + \ldots + km_k = n-1+k$. *In addition for* $0 \le i \le n-1$

$$f^k_{k,n-i} = \sum_{m\vdash(n-i+k-1)} \frac{m_k}{|m|} \times \binom{|m|}{m_1,\ldots,m_k}$$

where the summation is over nonnegative integers satisfying $m_1 + 2.m_2 + \ldots + k.m_k = n-1-i+k$.

Then we have the following corollary using (Theorem 2.10. *iii*).

Corollary 2.17. *Let $l_{k,n}^i$ be the kSOkL, then, for $m,n \in \mathbb{Z}^+$,*

$$l_{k,n}^i = \begin{cases} \sum\limits_{i=1}^{k} j \sum\limits_{m \vdash (n-j+k-1)} \frac{m_{jk}}{|m|} \times \binom{|m|}{m_{j1},\ldots,m_{jk}} & \text{if } i=1 \\ \sum\limits_{m=1}^{i} \sum\limits_{j=1}^{k} j \sum\limits_{t \vdash (n-m-j+k)} \frac{t_{mjk}}{|t|} \times \binom{|t|}{t_{mj1},\ldots,t_{mjk}} & \text{if } 1<i<k \\ \sum\limits_{i=1}^{k} j \sum\limits_{m \vdash (n-j+k)} \frac{m_{jk}}{|m|} \times \binom{|m|}{m_{j1},\ldots,m_{jk}} & \text{if } i=k \end{cases}$$

where $t = (t_{mj1}, \ldots, t_{mjk})$ and $m = (m_{j1}, m_{j2}, \ldots, m_{jk})$.

Corollary 2.18. *Let $l_{k,n}^i$ be the kSOkL, then, for all $m,n \in \mathbb{Z}^+$*

$$l_{k,n}^i = \begin{cases} \sum\limits_{a \vdash (n-1)} \frac{n-1}{|a|} \binom{|a|}{a_1,\ldots,a_k} & \text{if } i=1 \\ \sum\limits_{m=1}^{i} \sum\limits_{a \vdash (n-m)} \frac{n-m}{|a|} \binom{|a|}{a_1,\ldots,a_k} & \text{if } 1<i<k \\ \sum\limits_{a \vdash n} \frac{n}{|a|} \binom{|a|}{a_1,\ldots,a_k} & \text{if } i=k \end{cases}.$$

Proof. For $t_i = 1 (1 \le i \le k)$, $G_{k,n}$ is reduced to $l_{k,n}^k$. Since $l_{k,n}^k = \sum\limits_{a \vdash n} \frac{n}{|a|} \binom{|a|}{a_1,\ldots,a_k}$ from (1.10) and by using (2.7) the proof is completed. □

Corollary 2.19. *Let $l_{k,n}^i$ be the kSOkL, then, for $1 \le i \le k$ and $m,n \in \mathbb{Z}^+$*

$$l_{k,n}^i = \begin{cases} \sum\limits_{j=1}^{k} j \sum\limits_{a \vdash (n-1-j)} \binom{|a|}{a_1,\ldots,a_k} & \text{if } i=1 \\ \sum\limits_{m=1}^{i} \sum\limits_{j=1}^{k} j \sum\limits_{a \vdash (n-m-j)} \binom{|a|}{a_1,\ldots,a_k} & \text{if } 1<i<k \\ \sum\limits_{j=1}^{k} j \sum\limits_{a \vdash (n-j)} \binom{|a|}{a_1,\ldots,a_k} & \text{if } i=k \end{cases}.$$

Proof. Proof is trivial from (1.9), (2.7). □

Corollary 2.20. *Let $l_{2,n}^2$ be the second sequence of the 2SO2L, then,*

$$l_{2,n}^2 = \sum_{j=1}^{2} j \sum_{s=0}^{\lceil \frac{n-j}{2} \rceil} \binom{n-j-s}{s}$$

where $\binom{n}{s}$ is combinations s of n objects, such that $\binom{n}{s} = 0$ if $n < s$.

Proof. In (1.11), $F_{2,n}(t) = \sum\limits_{j=0}^{\lceil \frac{n}{2} \rceil} (-1)^j \binom{n-j}{j} F_1^{n-2j}(t)(-t_2)^j$ and for $t_i = 1$ and $c_i = 1$ $(1 \le i \le k)$, $F_{2,n-1}(t)$ is reduced to sequence $f_{k,n}^2$. Proof is completed by using $f_{k,n}^2(c_i = 1 \text{ for } 1 \le i \le k)$ and (2.10. *iii*). □

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