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GENERAL IDENTITIES FOR LINEAR FIBONACCI AND LUCAS SUMMATIONS

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Many well known identities involving the first n terms of the Fibonacci sequence $\{F_j\}_{j=0}^{\infty}$ and the Lucas sequence $\{L_j\}_{j=0}^{\infty}$ have extensions to the sequences $\{F_{j+r}\}_{j=0}^{\infty}$, $\{L_{j+r}\}_{j=0}^{\infty}$, $\{F_{jk}\}_{j=0}^{\infty}$, and $\{L_{jk}\}_{j=0}^{\infty}$, where r and k are fixed integers. Any such result may be considered as a special case of an identity related to sequences $\{F_{jk+r}\}_{j=0}^{\infty}$ and $\{L_{jk+r}\}_{j=0}^{\infty}$, and hence it is with these latter sequences that we are principally concerned. Since the subscripts are linear functions of j, these identities are called linear Fibonacci and Lucas summations.

A variety of techniques are used in deriving many of these summations. We begin by considering several basic results which are quickly deduced from the Binet definition of the terms of the given sequences. This approach is introduced in [1] and [2], with extensions via a difference equation route given in [3]. We have

$$(0)\qquad F_{jk+r} = \frac{\alpha^{jk+r}-\beta^{jk+r}}{\alpha-\beta} \quad\text{and}\quad L_{jk+r} = \alpha^{jk+r}+\beta^{jk+r},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad\text{and}\quad \beta = \frac{1-\sqrt{5}}{2}.$$

Note that α and β are the roots of the equation $x^2 - x - 1 = 0$, and hence $\alpha+\beta = 1$ and $\alpha\beta = -1$. Using the summation formula for the first n terms of a geometric progression, the following results are obtained:

$$(1)\qquad \sum_{j=0}^{n} F_{jk+r} = F_r + F_{k+r} + F_{2k+r} + \cdots + F_{nk+r} = \left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right) + \left(\frac{\alpha^{k+r}-\beta^{k+r}}{\alpha-\beta}\right) + \left(\frac{\alpha^{2k+r}-\beta^{2k+r}}{\alpha-\beta}\right) + \cdots + \left(\frac{\alpha^{nk+r}-\beta^{nk+r}}{\alpha-\beta}\right)$$

$$= \frac{1}{\alpha-\beta}\left[\alpha^r\left(\frac{\alpha^{(n+1)k}-1}{\alpha^k-1}\right) - \beta^r\left(\frac{\beta^{(n+1)k}-1}{\beta^k-1}\right)\right] = \frac{F_{(n+1)k+r} + (-1)^{k+1}F_{nk+r} + (-1)^r F_{k-r} - F_r}{L_k - 1 + (-1)^{k+1}}.$$

Similarly, one may find

$$(2)\qquad \sum_{j=0}^{n} L_{jk+r} = \frac{L_{(n+1)k+r} + (-1)^{k+1}L_{nk+r} + (-1)^r L_{k-r} - L_r}{L_k - 1 + (-1)^{k+1}}$$

$$(3)\qquad \sum_{j=0}^{n} (-1)^j F_{jk+r} = \frac{(-1)^n F_{(n+1)k+r} + (-1)^{n+k}F_{nk+r} + (-1)^r F_{k-r} + F_r}{L_k + 1 + (-1)^k}$$

$$(4)\qquad \sum_{j=0}^{n} (-1)^j L_{jk+r} = \frac{(-1)^n L_{(n+1)k+r} + (-1)^{n+k}L_{nk+r} + (-1)^r L_{k-r} + L_r}{L_k + 1 + (-1)^k}.$$

These identities are used to simplify any summation expression that may be represented as a linear combination of Fibonacci and/or Lucas numbers. One direction to take is to observe by (0) that

$$F_{jk+r}F_{ju+v} = \frac{1}{5}\left[L_{j(k+u)+(r+v)} - (-1)^{ju+v}L_{j(k-u)+(r-v)}\right]$$

$$F_{jk+r}L_{ju+v} = F_{j(k+u)+(r+v)} + (-1)^{ju+v}F_{j(k-u)+(r-v)}$$

$$L_{jk+r}L_{ju+v} = L_{j(k+u)+(r+v)} + (-1)^{ju+v}L_{j(k-u)+(r-v)}.$$

Identities (1) to (4) yield an expression for the sum of the first $n+1$ terms $(0 \leqslant j \leqslant n)$ of each product given above. Let us explicitly consider only the second such product.

$$(5)\qquad \sum_{j=0}^{n} F_{jk+r}L_{ju+v} = \sum_{j=0}^{n} F_{j(k+u)+(r+v)} + (-1)^v \sum_{j=0}^{n} (-1)^{ju}F_{j(k-u)+(r-v)}$$

$$= \frac{F_{(n+1)(k+u)+(r+v)} + (-1)^{k+u+1}F_{n(k+u)+(r+v)}}{L_{k+u} - 1 + (-1)^{k+u+1}} + \frac{(-1)^{r+v}F_{(k+u)-(r+v)} - F_{r+v}}{L_{k+u} - 1 + (-1)^{k+u+1}}$$

$$+\begin{cases} \dfrac{(-1)^v[F_{(n+1)(k-u)+(r-v)} + (-1)^{k-u+1}F_{n(k-u)+(r-v)} + (-1)^{r-v}F_{(k-u)-(r-v)} - F_{r-v}]}{L_{k-u} - 1 + (-1)^{k-u+1}} & \text{if } u \text{ is even,} \\[2ex] \dfrac{(-1)^v[(-1)^nF_{(n+1)(k-u)+(r-v)} + (-1)^{n+k-u}F_{n(k-u)+(r-v)} + (-1)^{r-v}F_{(k-u)-(r-v)} + F_{r-v}]}{L_{k-u} + 1 + (-1)^{k-u}} & \text{if } u \text{ is odd.} \end{cases}$$

Specifying k, u, r, and v as particular integers leads the reader to a countable number of interesting special cases.

The known (see [3] and [4]) generating functions for sequences $\{F_{jk+r}\}_{j=0}^{\infty}$ and $\{L_{jk+r}\}_{j=0}^{\infty}$ are now used to find several additional classes of general linear Fibonacci and Lucas summation identities. We now list these generating functions for reference—with the first be derived from (0) to show a general approach to such calculations.

$$(6)\qquad \sum_{j=0}^{\infty} F_{jk+r}x^j = \sum_{j=0}^{\infty} \frac{\alpha^{jk+r} - \beta^{jk+r}}{\alpha-\beta}x^j = \frac{1}{\alpha-\beta}\left[\alpha^r \sum_{j=0}^{\infty} \alpha^{jk}x^j - \beta^r \sum_{j=0}^{\infty} \beta^{jk}x^j\right]$$

$$= \frac{1}{\alpha-\beta}\left[\frac{\alpha^r}{1-\alpha^k x} - \frac{\beta^r}{1-\beta^k x}\right] = \frac{1}{\alpha-\beta}\left[\frac{(\alpha^r - \beta^r) + (-\alpha^r\beta^k + \alpha^k\beta^r)x}{1-(\alpha^k+\beta^k)x + \alpha^k\beta^k x^2}\right]$$

$$= \frac{F_r + (-1)^r F_{k-r}x}{1 - L_k x + (-1)^k x^2} = \frac{F_r + (F_{k+r} - F_r L_k)x}{1 - L_k x + (-1)^k x^2}\,.$$

$$(7)\qquad \sum_{j=0}^{\infty} (-1)^j F_{jk+r}x^j = \frac{F_r + (-1)^{r+1}F_{k-r}x}{1 + L_k x + (-1)^k x^2} = \frac{F_r + (F_r L_k - F_{k+r})x}{1 + L_k x + (-1)^k x^2}\,.$$

$$(8)\qquad \sum_{j=0}^{\infty} L_{jk+r}x^j = \frac{L_r + (-1)^{r+1}L_{k-r}x}{1 - L_k x + (-1)^k x^2} = \frac{L_r + (L_{k+r} - L_r L_k)x}{1 - L_k x + (-1)^k x^2}\,.$$

$$(9)\qquad \sum_{j=0}^{\infty} (-1)^j L_{jk+r}x^j = \frac{L_r + (-1)^r L_{k-r}x}{1 + L_k x + (-1)^k x^2} = \frac{L_r + (L_r L_k - L_{k+r})x}{1 + L_k x + (-1)^k x^2}\,.$$

The derivative of these generating functions leads to identities which are of interest in themselves, and these in turn yield additional summation results. We begin by differentiating both sides of (6) with respect to x.

$$\frac{d}{dx}\left[\sum_{j=0}^{\infty} F_{jk+r}x^j\right] = \frac{d}{dx}\left[\frac{F_r + (-1)^r F_{k-r}x}{1 - L_k x + (-1)^k x^2}\right]$$

$$\sum_{j=0}^{\infty}(j+1)F_{(j+1)k+r}x^j = \frac{d}{dx}\left[\frac{F_r}{1-L_kx+(-1)^kx^2}\right]+\frac{d}{dx}\left[\frac{(-1)^rF_{k-r}x}{1-L_kx+(-1)^kx^2}\right]$$

$$= \frac{F_r}{1-L_kx+(-1)^kx^2}\;\frac{L_k+2(-1)^{k+1}x}{1-L_kx+(-1)^kx^2}+\frac{(-1)^rF_{k-r}}{1-L_kx+(-1)^kx^2}\;\frac{1-(-1)^kx^2}{1-L_kx+(-1)^kx^2}$$

$$= \sum_{j=0}^{\infty}\frac{F_rF_{(j+1)k}}{F_k}x^j\cdot\sum_{j=0}^{\infty}L_{(j+1)k}x^j$$

$$+\sum_{j=0}^{\infty}\frac{(-1)^rF_{k-r}F_{(j+1)k}}{F_k}\cdot x^j\left[-1+\frac{2-L_kx}{1-L_kx+(-1)^kx^2}\right]$$

by special cases of (6) and (8),

$$= \sum_{j=0}^{\infty}\sum_{s=0}^{j}\frac{F_rF_{(s+1)k}}{F_k}L_{(j-s+1)k}x^j+\sum_{j=0}^{\infty}\frac{(-1)^{r+1}F_{k-r}F_{(j+1)k}}{F_k}x^j$$

$$+\sum_{j=0}^{\infty}\frac{(-1)^rF_{k-r}F_{(j+1)k}}{F_k}x^j\cdot\sum_{j=0}^{\infty}L_{jk}x^j,$$

by convolution of the series and by (8),

$$= \sum_{j=0}^{\infty}\left[\frac{(-1)^{r+1}F_{k-r}F_{(j+1)k}}{F_k}+\sum_{s=0}^{j}\frac{F_rF_{(s+1)k}L_{(j-s+1)k}}{F_k}+\sum_{s=0}^{j}\frac{(-1)^rF_{k-r}F_{(s+1)k}L_{(j-s)k}}{F_k}\right]x^j$$

By equating the corresponding coefficients of the above series, the identity

$$(10)\qquad (j+1)F_{(j+1)k+r} = \frac{1}{F_k}\left\{(-1)^{r+1}F_{k-r}F_{(j+1)k}+\sum_{s=0}^{j}F_{(s+1)k}[F_rL_{(j-s+1)k}+(-1)^rF_{k-r}L_{(j-s)k}]\right\}$$

is found, which in turn yields

$$(11)\qquad \sum_{j=0}^{n}(j+1)F_{(j+1)k+r} = \frac{1}{F_k}\sum_{j=0}^{n}\left\{(-1)^{r+1}F_{k-r}F_{(j+1)k}+\sum_{s=0}^{j}F_{(s+1)k}[F_rL_{(j-s+1)k}+(-1)^rF_{k-r}L_{(j-s)k}]\right\}.$$

Performing the same operations as above on identities (7), (8), and (9) yields results similar to (10) and (11). These results related to (8) are as follows:

$$(12)\qquad (j+1)L_{(j+1)k+r} = \frac{1}{F_k}\left\{(-1)^rL_{k-r}F_{(j+1)k}+\sum_{s=0}^{j}F_{(s+1)k}[L_rL_{(j-s+1)k}+(-1)^{r+1}L_{k-r}L_{(j-s)k}]\right\}$$

and

$$(13)\qquad \sum_{j=0}^{n}(j+1)L_{(j+1)k+r} = \frac{1}{F_k}\sum_{j=0}^{n}\left\{(-1)^rL_{k-r}F_{(j+1)k}+\sum_{s=0}^{j}F_{(s+1)k}[L_rL_{(j-s+1)k}+(-1)^{r+1}L_{k-r}L_{(j-s)k}]\right\}.$$

Taking higher order derivatives of (6), (7), (8), and (9) leads the reader to additional summation identities that are similar in form to those listed above. Further, numerous special cases of each identity given may be quickly deduced.

The relationships between binomial coefficients and terms of sequences $\{F_{jk+r}\}_{j=0}^{\infty}$ and $\{L_{jk+r}\}_{j=0}^{\infty}$ take the form of rather simple but elegant summation identities. To begin we return to definition (0).

$$(14)\qquad F_{jk+r} = \frac{\alpha^{jk}\alpha^r-\beta^{jk}\beta^r}{\alpha-\beta} = \frac{(\alpha^2-1)^{jk}\alpha^r-(\beta^2-1)^{jk}\beta^r}{\alpha-\beta} =$$

$$= \frac{\sum_{t=0}^{jk} \binom{jk}{t}(\alpha)^{2t+r}(-1)^{jk-t} - \sum_{t=0}^{jk} \binom{jk}{t}(\beta)^{2t+r}(-1)^{jt}}{\alpha-\beta} = \sum_{t=0}^{jk} \binom{jk}{t}(-1)^{jk-t}\left[\frac{\alpha^{2t+r}-\beta^{2t+r}}{\alpha-\beta}\right]$$

$$= \sum_{t=0}^{jk} \binom{jk}{t}(-1)^{jk-t}F_{2t+r} = \sum_{t=0}^{jk} \binom{jk}{t}(-1)^{jk-t}F_{t+r/2}L_{t+r/2}.$$

If $j = 2j'$, then an even more elementary summation results.

(15)
$$F_{2j'k+r} = \frac{(\alpha^2)^{j'k}\alpha^r - (\beta^2)^{j'k}\beta^r}{\alpha-\beta} = \frac{(\alpha+1)^{j'k}\alpha^r - (\beta+1)^{j'k}\beta^r}{\alpha-\beta} = \sum_{t=0}^{j'k} \binom{j'k}{t}F_{t+r}.$$

For the Lucas numbers, the corresponding results are

(16)
$$L_{jk+r} = \sum_{t=0}^{jk} \binom{jk}{t}(-1)^{jk-t}L_{2t+r} = \sum_{t=0}^{jk} \binom{jk}{t}(-1)^{jk-t}\sum_{s=0}^{t}\binom{t}{s}L_{s+r}$$

and

(17)
$$L_{2j'k+r} = \sum_{t=0}^{j'k}\binom{j'k}{t}L_{t+r}.$$

Taking the view that a summation identity is "improved" by reducing the number of addends (even if the addends become more complicated), we now consider several methods of approach in an attempt to find additional "improved" results linking binomial coefficients and Fibonacci and Lucas numbers.

The column generators of the columns in the left-justified Pascal Triangle shown below are most useful in this endeavor, as was first shown by V. E. Hoggatt, Jr., in [5]

$$\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots \\
\dfrac{1}{1-x} & \dfrac{x}{(1-x)^2} & \dfrac{x^2}{(1-x)^3} & \dfrac{x^3}{(1-x)^4} & \cdots
\end{array}$$
Column Generators

That is, defining the binomial coefficient $\binom{n}{j} = 0$ for $n < j$ we observe

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\binom{n}{0}x^n, \quad \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty}\binom{n}{1}x^n, \quad \frac{x^j}{(1-x)^{j+1}} = \sum_{n=0}^{\infty}\binom{n}{j}x^n, \quad j \geq 0.$$

Hence,
$$\sum_{j=0}^{\infty} F_{jk+r}\frac{x^j}{(1-x)^{j+1}} = \sum_{j=0}^{\infty} F_{jk+r}\sum_{n=0}^{\infty}\binom{n}{j}x^n = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}F_{jk+r}x^n.$$

By identity (6),

$$\sum_{j=0}^{\infty} F_{jk+r}x^j = \frac{F_r + (-1)^r F_{k-r}x}{1 - L_k x + (-1)^k x^2}\quad,$$

and thus, we also have

$$\sum_{j=0}^{\infty} F_{jk+r}\frac{x^j}{(1-x)^{j+1}} = \frac{1}{1-x}\,\frac{F_r + (-1)^r F_{k-r}(x/1-x)}{1 - L_k(x/1-x) + (-1)^k (x/1-x)^2} = \frac{F_r + [-F_r + (-1)^r F_{k-r}]x}{1 - [2+L_k]x + [1 + L_k + (-1)^k]x^2}\,.$$

There are two cases of the above identity to consider:

(i) k even. Then

$$\sum_{j=0}^{\infty} F_{jk+r}\frac{x^j}{(1-x)^{j+1}} = \frac{F_r + [-F_r + (-1)^r F_{k-r}]x}{1 - (2+L_k)(1-x)x} = \{F_r + [-F_r + (-1)^r F_{k-r}]x\}\sum_{j=0}^{\infty}(2+L_k)^j(1-x)^j x^j$$

$$= \sum_{j=0}^{\infty}\left\{F_r \sum_{s=0}^{j}(2+L_k)^j \binom{j}{s}(-1)^s x^{s+j} + [-F_r + (-1)^r F_{k-r}]\sum_{s=1}^{j+1}(2+L_k)^j\binom{j}{s-1}(-1)^{s-1}x^{s+j}\right\}.$$

Now let $m = s + j$. Then we have

$$= \sum_{m=0}^{\infty}\left\{F_r \sum_{s=0}^{[m/2]}(2+L_k)^{m-s}\binom{m-s}{s}(-1)^s + [-F_r + (-1)^r F_{k-r}]\sum_{s=1}^{[(m+1)/2]}(2+L_k)^{m-s}\binom{m-s}{s-1}(-1)^{s-1}\right\}x^m.$$

Hence, equating like coefficients of x in the above two series yields, for k *even*,

(18) $$\sum_{j=0}^{n}\binom{n}{j}F_{jk+r} = F_r\sum_{s=0}^{[n/2]}(2+L_k)^{n-s}\binom{n-s}{s}(-1)^s + [-F_r + (-1)^r F_{k-r}]\sum_{s=1}^{[(n+1)/2]}(2+L_k)^{n-s}\binom{n-s}{s-1}(-1)^{s-1}.$$

(ii) k odd. Then

$$\sum_{j=0}^{\infty} F_{jk+r}\frac{x^j}{(1-x)^{j+1}} = \frac{F_r + [-F_r + (-1)^r F_{k-r}]x}{1 - [(2+L_k) - L_k x]x} = \{F_r + [-F_r + (-1)^r F_{k-r}]x\}\sum_{j=0}^{\infty}[(2+L_k) - L_k x]^j x^j$$

$$= \{F_r + [-F_r + (-1)^r F_{k-r}]x\}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{j}{s}(2+L_k)^{j-s}(L_k x)^s(-1)^s x^j = \sum_{m=0}^{\infty}\left\{F_r\sum_{s=0}^{[m/2]}(-1)^s\binom{m-s}{s}(2+L_k)^{m-2s}L_k^s\right.$$

$$\left. + [-F_r + (-1)^r F_{k-r}]\sum_{s=1}^{[(m+1)/2]}(-1)^{s-1}\binom{m-s}{s-1}(2+L_k)^{m-2s+1}L_k^{s-1}\right\}x^m$$

and thus, for k odd,

(19) $$\sum_{j=0}^{n}\binom{n}{j}F_{jk+r} = F_r\sum_{s=0}^{[n/2]}(-1)^s\binom{n-s}{s}(2+L_k)^{n-2s}L_k^s$$

$$+ [-F_r + (-1)^r F_{k-r}]\sum_{s=1}^{[(n+1)/2]}(-1)^{s-1}\binom{n-s}{s-1}(2+L_k)^{n-2s+1}L_k^{s-1}\quad.$$

Using the column generators in the left-justified Pascal Triangle with generating functions (7), (8), and (9) leads to three pairs of summation identities which are similar in form to (18) and (19).

Several special cases of (18) and (19) are given which show the inherent simplicity of these identities.

Letting $r = 0$ and $k = 2$ in (18) gives

$$\sum_{j=0}^{n}\binom{n}{j}F_{j2} = \sum_{s=0}^{[(n+1)/2]}(-1)^{s-1}5^{n-s}\binom{n-s}{s-1}.$$

If $k = r = 1$ in (19), then

$$\sum_{j=0}^{n} \binom{n}{j} F_{j-1} = \sum_{s=0}^{[n/2]} (-1)^s \binom{n-s}{s} 3^{n-2s} + \sum_{s=1}^{[(n+1)/2]} (-1)^s \binom{n-s}{s-1} 3^{n-2s+1}.$$

Taking $r = 0$ and $k = 3$ in (19) yields

$$\sum_{j=0}^{n} \binom{n}{j} F_{j3} = 2 \sum_{s=1}^{[(n+1)/2]} (-1)^{s-1} \binom{n-s}{s-1} 6^{n-2s+1} 4^{s-1} .$$

More generally, we deduce from (18) and (19) that

$$\sum_{j=0}^{n} \binom{n}{j} F_{jk} = \begin{cases} F_k \displaystyle\sum_{s=1}^{[(n+1)/2]} (-1)^{s-1} (2+L_k)^{n-s} \binom{n-s}{s-1}, & \text{for } k \text{ even}, \\ F_k \displaystyle\sum_{s=1}^{[(n+1)/2]} (-1)^{s-1} (1+L_k)^{n-2s+1} \binom{n-s}{s-1} L_k^{s-1}, & \text{for } k \text{ odd} \end{cases}$$

One of the nicest results linking binomial coefficients and Fibonacci numbers is given in [6]. Here, using the fact that

$$Q^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad \text{for any integer } n, \text{ and}$$

$$Q^{nk+r} = \sum_{j=0}^{k} \binom{k}{j} Q^{j+r} F_n^j F_{n-1}^{k-j} ,$$

the identity

$$F_{nk+r} = \sum_{j=0}^{k} \binom{k}{j} F_{j+r} F_n^j F_{n-1}^{k-j} \tag{20}$$

is deduced by equating upper right elements in the previous matrix equation. This identity is actually a special case of the next result.

Since for any integer t,

$$Q^n = Q^{n-t} Q^t = (F_{n-t} Q + F_{n-t-1} I) Q^t,$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it follows that

$$Q^{nk+r} = Q^r [F_{n-t} Q + F_{n-t-1} I) Q^t]^k = Q^{tk+r} \sum_{j=0}^{k} \binom{k}{j} Q^j F_{n-t}^j F_{n-t-1}^{k-j}, \quad \text{for } t \neq n,$$

$$= \sum_{j=0}^{k} \binom{k}{j} Q^{tk+r+j} F_{n-t}^j F_{n-t-1}^{k-j} .$$

By equating the upper right elements in this matrix equation we obtain, for any integer $t \neq n$,

$$F_{nk+r} = \sum_{j=0}^{k} \binom{k}{j} F_{tk+r+j} F_{n-t}^j F_{n-t-1}^{k-j} . \tag{21}$$

The companion results for Lucas numbers are deduced by either using the identity $L_m = F_{m+1} + F_{m-1}$ or the matrix result

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m-1} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix}, \quad \text{for any integer } m.$$

They are

$$L_{nk+r} = \sum_{j=0}^{k} \binom{k}{j} L_{j+r} F_n^j F_{n-1}^{k-j} , \tag{22}$$

and

$$L_{nk+r} = \sum_{j=0}^{k} \binom{k}{j} L_{tk+r+j} F_{n-t}^{j} F_{n-t-1}^{k-j}, \quad \text{for } t \neq n.$$

The final approach we take to find additional linear Fibonacci and Lucas identities is via exponential generating functions. This productive technique stems from the Maclaurin series expansion for e^x:

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and hence $e^{\alpha x} = 1 + \frac{(\alpha x)}{1!} + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \cdots$ and $e^{\beta x} = 1 + \frac{(\beta x)}{1!} + \frac{(\beta x)^2}{2!} + \frac{(\beta x)^3}{3!} + \cdots$.

It follows that the basic Fibonacci and Lucas generating functions are

$$\sum_{n=0}^{\infty} F_n \frac{x^n}{n!} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} \quad \text{and} \quad \sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x}.$$

The exponential generating functions of the sequences of interest in this paper are found to be

$$\sum_{n=0}^{\infty} F_{nk+r} \frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta} \tag{24}$$

$$\sum_{n=0}^{\infty} L_{nk+r} \frac{x^n}{n!} = \alpha^r e^{\alpha^k x} + \beta^r e^{\beta^k x} \tag{25}$$

$$\sum_{n=0}^{\infty} (-1)^n F_{nk+r} \frac{x^n}{n!} = \frac{\alpha^r e^{-\alpha^k x} - \beta^r e^{-\beta^k x}}{\alpha - \beta} \tag{26}$$

$$\sum_{n=0}^{\infty} (-1)^n L_{nk+r} \frac{x^n}{n!} = \alpha^r e^{-\alpha^k x} + \beta^r e^{-\beta^k x}. \tag{27}$$

Convoluting series (24) and (25) and equating like coefficients yields an interesting identity. We proceed as follows:

$$\sum_{n=0}^{\infty} F_{nk+r} \frac{x^n}{n!} \sum_{n=0}^{\infty} L_{nk+r} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} F_{jk+r} L_{(n-j)k+r} \frac{x^n}{n!}$$

and

$$\left(\frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta}\right)(\alpha^r e^{\alpha^k x} + \beta^r e^{\beta^k x}) = \frac{\alpha^{2r} e^{2\alpha^k x} - \beta^{2r} e^{2\beta^k x}}{\alpha - \beta} = \frac{\alpha^{2r} \sum_{n=0}^{\infty} (2\alpha^k)^n \frac{x^n}{n!} - \beta^{2r} \sum_{n=0}^{\infty} (2\beta^k)^n \frac{x^n}{n!}}{\alpha - \beta}$$

$$= \sum_{n=0}^{\infty} 2^n \left(\frac{\alpha^{nk+2r} - \beta^{nk+2r}}{\alpha - \beta}\right) \frac{x^n}{n!} = \sum_{n=0}^{\infty} 2^n F_{nk+2r} \frac{x^n}{n!}.$$

Hence,

$$\sum_{j=0}^{n} \binom{n}{j} F_{jk+r} L_{(n-j)k+r} = 2^n F_{nk+2r}. \tag{28}$$

Many additional identities may be deduced using the generating functions (24), (25), (26), and (27). By convoluting each of (24) and (25) with itself, the following results are deduced:

$$\sum_{j=0}^{n} \binom{n}{j} F_{jk+r} F_{(n-j)k+r} = \frac{1}{5} [2^n L_{nk+2r} + 2(-1)^{r+1} L_k^n] \tag{29}$$

$$\sum_{j=0}^{n} \binom{n}{j} L_{jk+r} L_{(n-j)k+r} = 2^n L_{nk+2r} + 2(-1)^r L_k^n. \tag{30}$$

We invite the reader to explore the special cases of the results given and also to use the procedures introduced to discover additional identities.

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AN INEQUALITY FOR A CLASS OF POLYNOMIALS

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1. INTRODUCTION

Recently, Klamkin and Newman [1], using double induction, proved that

$$\sum_{k=1}^{n} A_k^3 \leqslant \left(\sum_{k=1}^{n} A_k\right)^2 \qquad (n = 1, 2, \cdots), \tag{1.1}$$

where A_k is a non-decreasing sequence with $A_0 = 0$ and $A_k - A_{k-1} \leqslant 1$. For $A_k = k$, (1.1) gives the well known elementary identity

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \qquad (n = 1, 2, \cdots). \tag{1.2}$$

Our inequality (2.1) for polynomials in a single variable x gives (1.1) for $x = 1$.

2. A POLYNOMIAL INEQUALITY

Our first general result is given by

Theorem 1. Let C_k be a non-decreasing sequence with $C_0 = 0$ and $C_k - BC_{k-1} \leqslant 1$, $k = 1, 2, \cdots$, where B is a constant, $0 \leqslant B \leqslant 1$. Then, for $x \geqslant 1$, we have the inequality

$$\sum_{k=1}^{n} C_k^3 x^k \leqslant \left(\sum_{k=1}^{n} C_k x^k\right)^2 \qquad (n = 1, 2, \cdots). \tag{2.1}$$

Proof. We will use double induction. For $n = 1$, (2.1) requires that $C_1^3 x \leqslant C_1^2 x^2$, or $C_1^2 x(C_1 - x) \leqslant 0$, which is true, since $C_1 \leqslant 1$ and $x \geqslant 1$. Assuming (2.1) is true for $k = 1, 2, \cdots, n$, we must now show that