Applied Mathematics and Computation 220 (2013) 482–486

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A new approach to generalized Fibonacci and Lucas numbers with binomial coefficients

H.H. Gulec *, N. Taskara, K. Uslu

Selcuk University, Science Faculty, Department of Mathematics, Campus, 42075 Konya, Turkey

ARTICLE INFO

Keywords:
Fibonacci number
Lucas number
Generalized Fibonacci number
Binomial coefficients

ABSTRACT

In this study, Fibonacci and Lucas numbers have been obtained by using generalized Fibonacci numbers. In addition, some new properties of generalized Fibonacci numbers with binomial coefficients have been investigated to write generalized Fibonacci sequences in a new direct way. Furthermore, it has been given a new formula for some Lucas numbers.

1. Introduction

For $a,b \in \mathbb{R}$ and $n > 2$, the well-known Fibonacci $\{F_n\}$, Lucas $\{L_n\}$ and generalized Fibonacci $\{G_n\}$ sequences are defined by $F_n = F_{n-1} + F_{n-2},\ L_n = L_{n-1} + L_{n-2}$, and $G_n = G_{n-1} + G_{n-2}$ respectively, where $F_1 = F_2 = 1,\ L_1 = 2,\ L_2 = 1$ and $G_1 = a,\ G_2 = b$. Moreover, for the first n Fibonacci numbers, it is well known that the sum of the squares is $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ and $\sum_{i=0}^{n}\binom{n-i}{i} = F_{n+1}$. Throughout this paper, we take the notations $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}_0 = \{0,1,2,\ldots\}$.

Recently, there has been a huge interest of the application for Fibonacci and Lucas numbers in almost all sciences. For rich applications of these numbers in science and nature, one can see the citations in [1–8]. For instance, the ratio of two consecutive of these numbers converges to the Golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$. Applications of Golden ratio appears in many research areas, particularly in Physics, Engineering, Architecture, Nature and Art. Benjamin et al., in [10], extended the combinatorial approach to understand relationships among generalized Fibonacci numbers. In [11], Vajda gave identities involving generalized Fibonacci numbers and binomial coefficients. All of these are special cases of the following two identities $G_{n+p} = \sum_{i=0}^{p}\binom{p}{i} G_{n-i}$ and $G_{m+(t+1)p} = \sum_{i=0}^{p}\binom{p}{i} f_t^i f_{t-1}^{p-i} G_{m+i}$. In [9], new properties of Fibonacci numbers are given and some new properties of Fibonacci numbers are investigated with binomial coefficiations. Moreover, Taskara et al., in [12], obtained new properties of Lucas numbers with binomial coefficients and gave some important consequences of these results related to the Fibonacci numbers. In [13], the authors gave a new family of k-Fibonacci numbers and established some properties of the relation to the ordinary Fibonacci numbers.

2. Main results

Fibonacci numbers arise in the solution of many combinatorial problems. In this section, we gave new formulas for Fibonacci and Lucas numbers related to generalized Fibonacci numbers and obtained some new properties of generalized Fibonacci numbers with binomial coefficients. Finally, a new formula has been given for special Lucas numbers.

The following Lemma gives new formulas for Fibonacci and Lucas numbers by using generalized Fibonacci numbers. These formulas allow us to obtain in easy form a family of Fibonacci and Lucas sequences in a new and direct way.

* Corresponding author.
E-mail addresses: hhgulec@selcuk.edu.tr (H.H. Gulec), ntaskara@selcuk.edu.tr (N. Taskara), kuslu@selcuk.edu.tr (K. Uslu).

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.05.043

Lemma 1. *For* $a^2+ab-b^2 \neq 0,\ n \in \mathbb{N}$, *we have the relations:*

(i) $F_n = \frac{aG_{n+2}-bG_{n+1}}{a^2+ab-b^2}$,

(ii) $L_n = \frac{(2a+b)G_{n+2}-(a+3b)G_{n+1}}{a^2+ab-b^2}$.

Proof.

(i) Let us use the principle of mathematical induction on n.
For $n = 1$, it is easy to see that

$$F_1 = \frac{aG_3 - bG_2}{a^2+ab-b^2} = 1.$$

Assume that it is true for all positive integers $n = k$. That is,

$$F_k = \frac{aG_{k+2} - bG_{k+1}}{a^2+ab-b^2}. \tag{1}$$

Therefore, we have to show that it is true for $n = k+1$. Adding F_{k-1} to both sides of (1), we have

$$F_k + F_{k-1} = \frac{aG_{k+2}-bG_{k+1}}{a^2+ab-b^2} + F_{k-1} = \frac{aG_{k+2}-bG_{k+1}}{a^2+ab-b^2} + \frac{aG_{k+1}-bG_k}{a^2+ab-b^2} = \frac{1}{a^2+ab-b^2}[a(G_{k+2}+G_{k+1}) - b(G_{k+1}+G_k)]$$
$$= \frac{1}{a^2+ab-b^2}(aG_{k+3} - bG_{k+2}) = F_{k+1}$$

as required.

(ii) The proof can be also seen by using the principle of induction on n as in (i). □

For $a = b = 1$, it is obvious $F_n = G_n$. Also, for $a = 2,\ b = 1$, we can clearly see that $L_n = G_n$.

In the following theorem, for some values of $n \in \mathbb{N}$, we will formulate generalized Fibonacci numbers in terms of their different indices.

Theorem 2. *For* $n \in 2\mathbb{N},\ 3a-2b \neq 0$, *we have the relation*

$$G_{3n-4} = \frac{a^2+ab-b^2}{3a-2b}\sum_{i=0}^{\frac{n-2}{2}} 2^{2n-3-4i}\binom{n-2-i}{n-2-2i} - \frac{2a-b}{3a-2b}G_{3n-5}.$$

Proof. Let $c = \frac{a^2+ab-b^2}{3a-2b}$ be a real number and let us consider the sequence $\{S_k\}$. From [9], elements of this sequence can be written as

$$S_0 = c\left[\binom{0}{0}2^1\right] = cF_3,$$
$$S_1 = c\left[2^5\binom{2}{2} + 2^1\binom{1}{0}\right] = cF_9, \tag{2}$$
$$\vdots$$

To obtain the elements of sequence $\{S_k\}$, we can use the following method:

By forming a $(k+1)\times(k+1)$ square matrix with the rows of $\{S_k\}$ sequence and with the columns of the coefficients $c2^1,\ c2^5,\ c2^9,\ldots,c2^{2n-3}$, we see that the elements in the main diagonal and those elements above the main diagonal come from the binomial expansion as follows:

$$\begin{array}{c} \\ c2^1 \\ c2^5 \\ c2^9 \\ c2^{13} \\ c2^{17} \\ c2^{21} \\ \vdots \\ c2^{2n-3} \end{array}
\begin{array}{c} \begin{array}{cccccccc} S_0 & S_1 & S_2 & S_3 & S_4 & S_5 & \cdots & S_k \end{array} \\
\begin{bmatrix}
\binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} & \cdots & \binom{\frac{n-2}{2}}{0} \\
0 & \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \cdots & \binom{\frac{n}{2}}{2} \\
0 & 0 & \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \cdots & \binom{\frac{n}{2}+1}{4} \\
0 & 0 & 0 & \binom{6}{6} & \binom{7}{6} & \binom{8}{6} & \cdots & \binom{\frac{n}{2}+2}{6} \\
0 & 0 & 0 & 0 & \binom{8}{8} & \binom{9}{8} & \cdots & \binom{\frac{n}{2}+3}{8} \\
0 & 0 & 0 & 0 & 0 & \binom{10}{10} & \cdots & \binom{\frac{n}{2}+4}{10} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-2}
\end{bmatrix} \end{array}$$

It is clear that the multipliers $c2^1,\ c2^5,\ c2^9,\ldots,c2^{2n-3}$ of $S_0,\ S_1,\ S_2,\ldots,S_k$ have the binomial coefficients as:

$$S_k = c\left[2^{2n-3}\binom{n-2}{n-2} + 2^{2n-7}\binom{n-3}{n-4} + 2^{2n-11}\binom{n-4}{n-6} + \cdots + 2^1\binom{\frac{n-2}{2}}{0}\right]. \tag{3}$$

If the Eq. (3) can be written as a sum, then we have

$$S_k = c\sum_{i=0}^{\frac{n-2}{2}} 2^{2n-3-4i}\binom{n-2-i}{n-2-2i}. \tag{4}$$

We can obtain the elements of sequence $\{S_k\}$ from (2) as

$$S_k = \{2c, 34c, \ldots, cF_{3n-3}\}.$$

Hence, by using (i) of Lemma 1, it is obvious that the equality

$$S_k = G_{3n-4} + \frac{2a-b}{3a-2b}G_{3n-5} \tag{5}$$

holds. As a result of (4) and (5), the proof is completed. □

From the above theorem, the following consequences can be clearly seen.

Corollary 3. *For $n \in 2\mathbb{N}$, it is obvious that the following results hold:*

(i) *For $a = b = 1$, the Fibonacci sequence with binomial coefficients in [9] is obtained as*

$$F_{3n+3} = \sum_{i=0}^{\frac{n}{2}} 2^{2n+1-4i}\binom{n-i}{n-2i}.$$

(ii) For $a = 2$ and $b = 1$, we have obtained the Lucas sequences with binomial coefficients given in [12] as

$$L_{3n+2} = \frac{5}{4}\sum_{i=0}^{\frac{n}{2}} 2^{2n+1-4i}\binom{n-i}{n-2i} - \frac{3}{4}L_{3n+1},$$

$$L_{3n+3} = \sum_{i=0}^{\frac{n}{2}} 2^{2n+1-4i}\binom{n-i}{n-2i} + F_{3n+1},$$

$$L_{3n+4} = \left[5\left(\sum_{i=0}^{\frac{n}{2}} 2^{2n+1-4i}\binom{n-i}{n-2i}\right)^2 - 4\right]^{1/2}.$$

Lemma 4 [9]. *For $\frac{n+1}{2},\ \frac{k}{3} \in \mathbb{Z},\ \frac{k}{3} - 1 = n$ and $n \geqslant 1$,*

$$F_{n+1} = \sum_{i=0}^{\frac{n-1}{2}} \binom{n-i}{n-2i}.$$

In the following theorem, when n is odd, we formulate generalized Fibonacci numbers in terms of their different indices.

Theorem 5. *If $a \neq 0$ and $n \in \mathbb{N}$ is odd, then we have*

$$G_{n+3} = \frac{a^2 + ab - b^2}{a}\sum_{i=0}^{\frac{n-1}{2}} \binom{n-i}{n-2i} + \frac{b}{a}G_{n+2}.$$

Proof. From Lemmas 1 and 4, the proof of this theorem is obvious. □

By using above theorem, the following result can be easily obtained.

Corollary 6. *For $a = 2$ and $b = 1$, it is clearly seen that*

$$L_{2n+3} = 5\left(\sum_{i=0}^{\frac{n-1}{2}} \binom{n-i}{n-2i}\right)^2 + 2$$

holds for Lucas sequences with binomial coefficients.

In addition to Theorem 2, we may also obtain different generalized Fibonacci numbers as in the following.

Theorem 7. *For $n \in \mathbb{N}_0$ and $a \neq 0$ we write the equation*

$$G_{2n+4} = \frac{a^2+ab-b^2}{a}\sum_{i=0}^{n}\binom{n+1+i}{1+2i} + \frac{b}{a}G_{2n+3}.$$

Proof. Let us $c = \frac{a^2+ab-b^2}{a}$. For $n \in \mathbb{N}_0$, we have the following iteration:

$$\begin{aligned} G_4 &= c\binom{1}{1} + \frac{b}{a}(a+b), \\ G_6 &= c\left[\binom{2}{1} + \binom{3}{3}\right] + \frac{b}{a}(2a+3b), \\ G_8 &= c\left[\binom{3}{1} + \binom{4}{3} + \binom{5}{5}\right] + \frac{b}{a}(5a+8b), \\ &\vdots \end{aligned} \tag{6}$$

The coefficients of a and b are Fibonacci numbers in (6). Hence, by iterating this procedure, we have

$$G_{2n+4} = c\left[\binom{n+1}{1} + \binom{n+2}{3} + \cdots + \binom{2n+1}{2n+1}\right] + \frac{b}{a}(aF_{2n+1} + bF_{2n+2}). \tag{7}$$

Furthermore, considering the equality $G_{2n+3} = aF_{2n+1} + bF_{2n+2}$, we rewrite (7)

$$G_{2n+4} = c\left[\binom{n+1}{1} + \binom{n+2}{3} + \cdots + \binom{2n+1}{2n+1}\right] + \frac{b}{a}G_{2n+3}$$

or using the summation symbol, we have

$$G_{2n+4} = c\sum_{i=0}^{n}\binom{n+1+i}{1+2i} + \frac{b}{a}G_{2n+3}. \qquad \square$$

From Theorem 7, the following result can be seen.

Corollary 8. *For some $a, b \in \mathbb{R}$, it is obvious that the following results hold:*

(i) *For $a = b = 1$, the Fibonacci sequence with binomial coefficients in [9] is obtained as*

$$F_{2n+2} = \sum_{i=0}^{n}\binom{n+1+i}{1+2i}.$$

(ii) *For $a = 2$ and $b = 1$, we have*

$$L_{2n+3} = \left[5\left(\sum_{i=0}^{n}\binom{n+1+i}{1+2i}\right)^2 + 4\right]^{1/2}$$

for the Lucas sequences with binomial coefficients.

In the following theorem, for $n \geqslant 2$, we give special Lucas numbers in terms of their different indices.

Theorem 9. *For $n \geqslant 2$ and $n \in \mathbb{N}$, we have*

$$L_{3n-4} = \frac{5}{76}2^{2n+1} + \frac{1}{76}\sum_{i=1}^{n-1}2^{2n-2(i+1)}(L_{3i} + L_{3i+2}) - \frac{47}{76}L_{3n-5}.$$

Proof. Let us use the induction on n.

For $n = 2$, it is easy to see that

$$L_2 = \frac{5}{76}2^5 + \frac{1}{76}\sum_{i=1}^{1}2^{4-2(i+1)}(L_{3i} + L_{3i+2}) - \frac{47}{76}L_1 = 1.$$

Assume it is true for all positive integers $n = m - 1$. That is,

$$L_{3m-7} = \frac{5}{76}2^{2m-1} + \frac{1}{76}\sum_{i=1}^{m-2}2^{2m-2-2(i+1)}(L_{3i} + L_{3i+2}) - \frac{47}{76}L_{3m-8}. \tag{8}$$

Therefore, we have to show that it is true for $n = m$. In other words,

$$L_{3m-4} = \frac{5}{76}2^{2m+1} + \frac{1}{76}\sum_{i=1}^{m-1}2^{2m-2(i+1)}(L_{3i} + L_{3i+2}) - \frac{47}{76}L_{3m-5}.$$

If we multiply both sides of (8) with 4, then we have

$$4L_{3m-7} + \frac{188}{76}L_{3m-8} = \frac{20}{76}2^{2m-1} + \frac{4}{76}\sum_{i=1}^{m-2}2^{2m-2-2(i+1)}(L_{3i} + L_{3i+2}). \tag{9}$$

By considering the equality $L_n = L_{n+2} - L_{n+1}$, we can rewrite on the left hand side of above equality as

$$\frac{1}{76}(116L_{3m-7} + 188L_{3m-6}) = \frac{20}{76}2^{2m-1} + \frac{4}{76}\sum_{i=1}^{m-2}2^{2m-2-2(i+1)}(L_{3i} + L_{3i+2}).$$

Hence, by recurring this procedure, we have

$$L_{3m-4} - \frac{1}{76}(L_{3m-1} + L_{3m-3} - 47L_{3m-5}) = \frac{20}{76}2^{2m-1} + \frac{4}{76}\sum_{i=1}^{m-2}2^{2m-2-2(i+1)}(L_{3i} + L_{3i+2}),$$

$$L_{3m-4} - \frac{1}{76}(L_{3m-3} + L_{3m-1}) + \frac{47}{76}L_{3m-5} = \frac{5}{76}2^{2m+1} + \frac{1}{76}\sum_{i=1}^{m-2}2^{2m-2(i+1)}(L_{3i} + L_{3i+2}).$$

Consequently, if the last equation is rearranged, then we obtain

$$L_{3m-4} + \frac{47}{76}L_{3m-5} = \frac{5}{76}2^{2m+1} + \frac{1}{76}\left[\sum_{i=1}^{m-2}2^{2m-2(i+1)}(L_{3i} + L_{3i+2}) + L_{3m-3} + L_{3m-1}\right],$$

$$L_{3m-4} = \frac{5}{76}2^{2m+1} + \frac{1}{76}\sum_{i=1}^{m-1}2^{2m-2(i+1)}(L_{3i} + L_{3i+2}) - \frac{47}{76}L_{3m-5}$$

which ends up the induction. Therefore we have the required formulate on L_{3n-4}. □

3. Conclusion

The aim of this article has been to express a new formulas for Fibonacci and Lucas numbers to obtain in a easy form a family of those numbers in a new and direct way. Recently, some novel properties of Fibonacci and Lucas numbers were given with binomial coefficients in [9,12,13]. In the present article, in addition those novel properties, we improved those results and gave a new formula for special Lucas numbers. We hope those results will be considerably useful for future work of the application of Fibonacci and Lucas numbers.

References

[1] T. Koshy, Fibonacci and Lucas Numbers with Applications, John Wiley and Sons Inc., NY, 2001.
[2] A. Stakhov, Fibonacci matrices, a generalization of the ‘Cassini formula’, and a new coding theory, Chaos Soliton Fract. 30 (2006) 56–66.
[3] S. Falcoń, A. Plaza, The k-Fibonacci sequence and the Pascal 2-triangle, Chaos Soliton Fract. 33 (1) (2007) 38–49.
[4] D.G.M. Farrokhi, Some remarks on the equation $F_n = kF_m$ in Fibonacci numbers, J. Integer Seq. 10 (2007).
[5] M.Z. Spivey, Combinatorial sums and finite differences, Discrete Math. 307 (2007) 3130–3146.
[6] H. Civciv, A note on the determinants of pentadiagonal matrices with the generalized Fibonacci and Lucas numbers, Appl. Math. Comput., in press, doi: 10.1016/ j.amc.2008.06.007.
[7] M. Akbulak, D. Bozkurt, On the order-m generalized Fibonacci k-numbers, Chaos Soliton Fract. 42 (3) (2009) 1347–1355.
[8] A. Demir, N. Omur, Y.T. Ulutas, Parametrized Fibonacci search method with k-Lucas numbers, Appl. Math. Comput. 198 (1) (2008) 355–360.
[9] H.H. Gulec, N. Taskara, On the properties of Fibonacci numbers with binomial coefficients, Int. J. Contemp. Math. Sci. 4 (25) (2009) 1251–1256.
[10] Arthur T. Benjamin, Jennifer J. Quinn, Francis Edward Su, Phased tilings and generalized Fibonacci identities, Fibonacci Quart. 38 (3) (2000) 282–288.
[11] S. Vajda, Fibonacci and Lucas Numbers, and the Golden Section, Theory and applications, John Wiley and Sons, New York, 1989.
[12] N. Taskara, K. Uslu, H.H. Gulec, On the properties of Lucas numbers with binomial coefficients, Appl. Math. Lett. 23 (2010) 68–72.
[13] M. El-Mikkawy, T. Sogabe, A new family of k-Fibonacci numbers, Appl. Math. Comput. 215 (12) (2010) 4456–4461.