SOME PROPERTIES OF PARTIAL DERIVATIVES OF GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

Gospava B. Djordjević
Department of Mathematics, University of Niš, 16000 Leskovac, Yugoslavia
(Submitted February 1999-Final Revision October 1999)

1. INTRODUCTION

In [5], Hongquan Yu and Chuanguang Liang considered the partial derivative sequences of the bivariate Fibonacci polynomials $U_n(x, y)$ and the bivariate Lucas polynomials $V_n(x, y)$. Some properties involving second-order derivative sequences of the Fibonacci polynomials $U_n(x)$ and Lucas polynomials $V_n(x)$ are established in [1] and [2]. These results may be extended to the k^{th} derivative case (see [4]).

In this paper we shall consider the partial derivative sequences of the generalized bivariate Fibonacci polynomials $U_{n,m}(x, y)$ and the generalized bivariate Lucas polynomials $V_{n,m}(x, y)$. We shall use the notation $U_{n,m}$ and $V_{n,m}$ instead of $U_{n,m}(x, y)$ and $V_{n,m}(x, y)$, respectively. These polynomials are defined by

$$U_{n,m} = xU_{n-1,m} + yU_{n-m,m}, \quad n \geq m, \tag{1.1}$$

with $U_{0,m} = 0$, $U_{n,m} = x^{n-1}$, $n = 1, 2, \ldots, m-1$, and

$$V_{n,m} = xV_{n-1,m} + yV_{n-m,m}, \quad n \geq m, \tag{1.2}$$

with $V_{0,m} = 2$, $V_{n,m} = x^n$, $n = 1, 2, \ldots, m-1$.

For $p = 0$ and $q = -y$, the polynomials $U_{n,m}$ are the known polynomials $\phi_n(0, -y; x)$ [3].

From (1.1) and (1.2), we find some first members of the sequences $U_{n,m}$ and $V_{n,m}$, respectively. These polynomials are given in the following table.

TABLE 1

n	$U_{n,m}$	$V_{n,m}$
0	0	2
1	1	x
2	x	x^2
3	x^2	x^3
$\vdots$	$\vdots$	$\vdots$
$m-1$	x^{m-2}	x^{m-1}
m	x^{m-1}	$x^m + 2y$
$m+1$	$x^m + y$	$x^{m+1} + 3xy$
$\vdots$	$\vdots$	$\vdots$
$2m-1$	$x^{2m-2} + (m-1)x^{m-2}y$	$x^{2m-1} + (m+1)x^{m-1}y$
$2m$	$x^{2m-1} + mx^{m-1}y$	$x^{2m} + (m+2)x^m y + 2y^2$
$\vdots$	$\vdots$	$\vdots$

The partial derivatives of $U_{n,m}$ and $V_{n,m}$ are defined by

$$U_{n,m}^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} U_{n,m} \quad \text{and} \quad V_{n,m}^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} V_{n,m}, \quad k \geq 0, \ j \geq 0.$$

Also, we find that $U_{n,m}$ and $V_{n,m}$ possess the following generating functions:

$$F = (1 - xt - yt^m)^{-1} = \sum_{n=1}^{\infty} U_{n,m} t^{n-1} \tag{1.3}$$

and

$$G = (2 - xt^{m-1})(1 - xt - yt^m)^{-1} = \sum_{n=0}^{\infty} V_{n,m} t^n. \tag{1.4}$$

From (1.3) and (1.4), we get the following representations of $U_{n,m}$ and $V_{n,m}$, respectively:

$$U_{n,m} = \sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k} x^{n-1-mk} y^k \tag{1.5}$$

and

$$V_{n,m} = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} x^{n-mk} y^k. \tag{1.6}$$

If $m = 2$, then polynomials $U_{n,m}$ and $V_{n,m}$ are the known polynomials U_n and V_n ([5]), respectively.

From Table 1, using induction on n, we can prove that

$$V_{n,m} = U_{n+1,m} + yU_{n+1-m,m}, \quad n \geq m-1. \tag{1.7}$$

2. SOME PROPERTIES OF $U_{n,m}^{(k,j)}$ AND $V_{n,m}^{(k,j)}$

We shall consider the partial derivatives $U_{n,m}^{(k,j)}$ and $V_{n,m}^{(k,j)}$. Namely, we shall prove the following theorem.

Theorem 2.1: The polynomials $U_{n,m}^{(k,j)}$ and $V_{n,m}^{(k,j)}$ $(n \geq 0, k \geq 0, j \geq 0)$ satisfy the following identities:

$$V_{n,m}^{(k,j)} = U_{n+1,m}^{(k,j)} + jU_{n+1-m,m}^{(k,j-1)} + yU_{n+1-m,m}^{(k,j)}; \tag{2.1}$$

$$U_{n,m}^{(k,j)} = kU_{n-1,m}^{(k-1,j)} + xU_{n-1,m}^{(k,j)} + jU_{n-m,m}^{(k,j-1)} + yU_{n-m,m}^{(k,j)}; \tag{2.2}$$

$$V_{n,m}^{(k,j)} = kV_{n-1,m}^{(k-1,j)} + xV_{n-1,m}^{(k,j)} + jV_{n-m,m}^{(k,j-1)} + yV_{n-m,m}^{(k,j)}; \tag{2.3}$$

$$V_{n,m}^{(k,j)} = \sum_{i=j}^{[(n-k)/m]} (n-(m-2)i) \frac{(n-(m-1)i)!}{(i-j)!(n-k-mi)!} x^{n-k-mi} y^{i-j}. \tag{2.4}$$

Proof: Differentiating (1.7), (1.1), and (1.2), first k-times with respect to x, then j-times with respect to y, we get (2.1), (2.2), and (2.3), respectively.

Also, if we differentiate (1.6) with respect to x, then with respect to y, we get (2.4).

Remark 2.1: If $m = 2$, then identities (2.1)-(2.4) become identities (i)-(iv) in [5].

Theorem 2.2: Let $k \geq 0,\ j \geq 0$. Then, we have:

$$\sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m} = \frac{1}{k+j+1} U_{n,m}^{(k+1,j)}; \tag{2.5}$$

$$\sum_{i=0}^{n} V_{i,m}^{(k,0)} V_{n-i,m}^{(0,j)} = \left((k+j+1) \binom{k+j}{j} \right)^{-1} (2 - xt^{m-1})^2 (2t^{-1} - t^{m-3} + yt^{2m-3}) U_{n+1,m}^{(k+j,j)}; \tag{2.6}$$

$$\sum_{i=0}^{n} U_{i+1,m}^{(0,j-1)} V_{n-i,m}^{(0,k)} = \left((j+k) \binom{j+k-1}{j-1} t^m \right)^{-1} V_{n,m}^{(0,j+k)}; \tag{2.7}$$

$$\sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m}^{(l,p)} = \left((k+j+p+l+1) \binom{k+j+p+l}{k+j} \right)^{-1} U_{n,m}^{(k+l+1,j+p)}. \tag{2.8}$$

Proof: Differentiating (1.3) k-times with respect to x, then j-times with respect to y, we get

$$F^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} F = \frac{(k+j)!\, t^{k+mj}}{(1 - xt - yt^m)^{k+j+1}} = \sum_{n=1}^{\infty} U_{n,m}^{(k,j)} t^{n-1}. \tag{i}$$

From (i), we have

$$F^{(0,0)} F^{(k,j)} = \frac{(k+j)!\, t^{k+jm}}{(1 - xt - yt^m)^{k+j+2}} = \sum_{n=1}^{\infty} \sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m} t^{n-2}.$$

Hence, we conclude that

$$\sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m} = \frac{(k+j)!\, t^{k+1+jm}}{(1 - xt - yt^m)^{k+j+2}}$$

$$= \frac{(k+j+1)!\, t^{k+1+jm}}{(k+j+1)(1 - xt - yt^m)^{k+j+2}} = \frac{1}{k+j+1} U_{n,m}^{(k+1,j)}.$$

By the last equalities, we get (2.5)

In a similar way, we can obtain (2.6), (2.7), and (2.8).

Corollary 2.1: If $k = l$, $j = p$, from (2.8) we get

$$\sum_{i=0}^{n} U_{i,m}^{(k,j)} U_{n-i,m}^{(k,j)} = \left((2k+2j+1) \binom{2k+2j}{k+j} \right)^{-1} U_{n,m}^{(2k+1,2j)}.$$

Furthermore, we are going to prove the following general result.

Theorem 2.3: Let $k \geq 0, j \geq 0, s \geq 0$. Then

$$\sum_{i_1+i_2+\cdots+i_s=n} U_{i_1,m}^{(k,j)} U_{i_2,m}^{(k,j)} \cdots U_{i_s,m}^{(k,j)} = \frac{((k+j)!)^s}{(sk+sj+s-1)!} U_{n,m}^{(sk+s-1,sj)}. \tag{2.10}$$

Proof: From (i), i.e.,

$$F^{(k,j)} = \frac{(k+j)!\, t^{k+mj}}{(1 - xt - yt^m)^{k+j+1}} = \sum_{n=1}^{\infty} U_{n,m}^{(k,j)} t^{n-1},$$

we find:

$$F^{(k,j)}F^{(k,j)}\dots F^{(k,j)} = \frac{((k+j)!)^s t^{sk+sjm}}{(1-xt-yt^m)^{sk+sj+s}}$$
$$= \sum_{n=1}^{\infty} \sum_{i_1+i_2+\cdots+i_s=n} U_{i_1,m}^{(k,j)} U_{i_2,m}^{(k,j)} \dots U_{i_s,m}^{(k,j)} t^{n-s}.$$

Hence, we get

$$\sum_{n=1}^{\infty} \sum_{i_1+i_2+\cdots+i_s=n} U_{i_1,m}^{(k,j)} U_{i_2,m}^{(k,j)} \dots U_{i_s,m}^{(k,j)} t^{n-1} = \frac{((k+j)!)^s t^{sk+s-1+sjm}}{(1-xt-yt^m)^{sk+sj+s}}$$
$$= \frac{((k+j)!)^s}{(sk+sj+s-1)!} U_{n,m}^{(sk+s-1,\, sj)}.$$

The equality (2.10) follows from the last equalities.

Remark 2.2: We can prove that

$$\frac{((k+j)!)^s}{(sk+sj+s-1)!} = \prod_{i=2}^{s} \left((i\alpha-1) \binom{i\alpha-2}{\alpha-1} \right)^{-1},$$

where $\alpha = k+j+1$. So (2.10) takes the following form,

$$\sum_{i_1+i_2+\cdots+i_s=n} U_{i_1,m}^{(k,j)} U_{i_2,m}^{(k,j)} \dots U_{i_s,m}^{(k,j)} = \prod_{i=2}^{s} \left((i\alpha-1) \binom{i\alpha-2}{\alpha-1} \right)^{-1} U_{n,m}^{(sk+s-1,\, sj)},$$

where $\alpha = k+j+1$.

REFERENCES

1. G. B. Djordjević. "On a Generalization of a Class of Polynomials." *The Fibonacci Quarterly* **36.2** (1998):110-17.
2. P. Filipponi & A. F. Horadam. "Second Derivative Sequences of Fibonacci and Lucas Polynomials." *The Fibonacci Quarterly* **31.3** (1993):194-204.
3. P. Filipponi & A. F. Horadam. "Addendum to 'Second Derivative Sequences of Fibonacci and Lucas Polynomials'." *The Fibonacci Quarterly* **32.2** (1994):110.
4. Jun Wang. "On the k^{th} Derivative Sequences of Fibonacci and Lucas Polynomials." *The Fibonacci Quarterly* **33.2** (1995):174-78.
5. Hongquan Yu & Chuanguang Liang. "Identities Involving Partial Derivatives of Bivariate Fibonacci and Lucas Polynomials." *The Fibonacci Quarterly* **35.1** (1997):19-23.

AMS Classification Numbers: 11B39, 11B83

❖❖❖