A generalization of Lucas polynomial sequence

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A R T I C L E   I N F O

A B S T R A C T

In this paper, we obtain a generalized Lucas polynomial sequence from the lattice paths for the Delannoy numbers by allowing weights on the steps $(1, 0)$, $(0, 1)$ and $(1, 1)$. These weighted lattice paths lead us to a combinatorial interpretation for such a Lucas polynomial sequence. The concept of Riordan arrays is extensively used throughout this paper.

1. Introduction

Horadam [4] introduced the polynomial sequence $\{W_n(x)\}$ defined recursively by

$$W_n(x) = p(x)W_{n-1}(x) + q(x)W_{n-2}(x), \quad (n \geq 2), \tag{1}$$

where

$$W_0(x) = c_0, \qquad W_1(x) = c_1 x^d, \qquad p(x) = c_2 x^d, \qquad q(x) = c_3 x^d$$

in which c_0, c_1, c_2, c_3 are constants and $d = 0$ or 1. If $W_0 = 0$ and $W_1(x) = 1$ then the Binet form of $W_n(x)$ is expressed as

$$W_n(x) = \frac{u^n(x) - v^n(x)}{u(x) - v(x)},$$

and if instead $W_0 = 2$ and $W_1(x) = p(x)$ then

$$w_n(x) := W_n(x) = u^n(x) + v^n(x),$$

where $u(x) + v(x) = p(x)$ and $u(x)v(x) = -q(x)$.

In some of the literature (e.g. see [9]), $\{W_n(x)\}$ and $\{w_n(x)\}$ are called the *Lucas polynomial sequences of the first kind* and *of the second kind*, respectively.

Special cases of Lucas polynomial sequences of both kinds are well known [4] and listed in Table 1 by their polynomial symbols and name along with the corresponding $p(x)$ and $q(x)$.

In this paper, we obtain a generalized Lucas polynomial sequence from the Riordan array which is obtained from weighted Delannoy numbers, say $D_w(n, k)$ where $w = (a, b, c)$ is a weight. This enables us to give a combinatorial interpretation for those Lucas polynomial sequences by a suitable choice of the weights a, b, c respectively. In Section 2 we develop the Riordan array $D_w(a, b, c)$ associated with the weighted Delannoy numbers. In Section 3, we obtain a generalized Lucas polynomial sequence from the row sum of the Riordan array $D_w(a, b, c)$. Finally, combinatorial interpretations for a pair of generalized Lucas polynomial sequences are given in Section 4.

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Table 1
Pairs of cognate polynomial sequences

$p(x)$	$q(x)$	$W_n(x)$	$w_n(x)$
x	1	Fibonacci, $F_n(x)$	Lucas, $L_n(x)$
$2x$	1	Pell, $P_n(x)$	Pell–Lucas, $Q_n(x)$
1	$2x$	Jacobsthal, $J_n(x)$	Jacobsthal–Lucas, $j_n(x)$
$3x$	-2	Fermat, $\mathcal{F}_n(x)$	Fermat–Lucas, $f_n(x)$
$2x$	-1	Chebyshev of 2nd kind, $U_n(x)$	Chebyshev of 1st kind, $2T_n(x)$

2. Riordan array $D_w(a, b, c)$

In this section, we develop the Riordan array introduced by Shapiro et. al. (see [6,8]) associated with weighted Delannoy numbers for our purpose.

We begin with the concept of Riordan arrays. A *Riordan array* $\{d_{n,k}\}_{n,k \in N_0}$ is defined by a pair of generating functions, $g(z) = 1 + g_1 z + g_2 z^2 + \cdots$ and $f(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots$ with $f_1 \neq 0$ such that the generic element $d_{n,k}$ is

$$d_{n,k} = [z^n] g(z)(f(z))^k,$$

where $[z^n]$ is the coefficient operator and $N_0 = \{0, 1, 2, \ldots\}$.

As usual, for a pair $g(z), f(z)$ of analytic functions, we denote the array by $\mathcal{R}(d_{n,k}) = (g(z), f(z))$ where the rows and columns are indexed by $0, 1, 2, \ldots$. From this definition, $\mathcal{R}(d_{n,k})$ is an infinite, lower triangular matrix. One example of a Riordan array is the Pascal triangle $P = \mathcal{R}\left(\binom{n}{k}\right)$ for which we have $g(z) = 1/(1 - z)$ and $f(z) = z/(1 - z)$.

The concept of a Riordan array may be used in a constructive way to find the generating function of many combinatorial sums. For any sequence $\{h_k\}$ having $h(z)$ as its generating function, we have the *summation property* (SP) [8] or the fundamental theorem of Riordan arrays:

$$\sum_{k=0}^{n} d_{n,k} h_k = [z^n] g(z) h(f(z)).$$

The SP of the Riordan array will be very useful in this paper.

For $n, k \in N_0$, it is well known that the Delannoy numbers [2] denoted $D(n, k)$ count the number of unweighted lattice paths from the point $(0, 0)$ to the point (k, n) using the steps $H = (1, 0)$, $V = (0, 1)$ and $D = (1, 1)$. Now let us consider weighted lattice paths such that a horizontal step H, a vertical step V and a diagonal step D are endowed with weights a, b, and c, respectively. We call such a path a (a, b, c)-*weighted path*. The *weight* of a weighted path is the product of the weights of all its steps in the weighted path and the *length of a weighted path* is the number of steps making up the path.

It is known (also see [5]) that the total sum of the weights of all (a, b, c)-weighted paths in the lattice plane from $(0, 0)$ to (k, n) is

$$D_w(n, k) := \sum_{d \geq 0} \binom{k}{d} \binom{n + k - d}{k} a^{k-d} b^{n-d} c^d. \tag{2}$$

When $a = b = c = 1$ they reduce to the ordinary Delannoy numbers.

In [5], Razpet studied the following Lucas property of a number array $D_w(n, k)$ obtained from the (a, b, c)-weighted paths in the lattice plane from $(0, 0)$ to (n, k) with positive integer weights a, b, c:

$$D_w(\alpha p + \beta, \gamma p + \delta) \equiv D_w(\alpha, \gamma) D_w(\beta, \delta) \pmod{p},$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative integers such that $0 \leq \beta < p, 0 \leq \delta < p$ for a prime number p.

In this section, we are mainly interested in a Riordan array expression for the numbers $D_w(n, k)$ and then we obtain a generalized Lucas polynomial sequence from the row sum. Hence it leads us to a combinatorial interpretation for such a Lucas polynomial sequence.

For our purpose, let us define an infinite lower triangular array $D_w(a, b, c) = [d_{n,k}]_{n,k \in N_0}$ by

$$d_{n,k} = \begin{cases} D_w(n - k, k) & \text{if } n \geq k, \\ 0 & \text{if } n < k, \end{cases} \tag{3}$$

where $D_w(n, k)$ are the weighted Delannoy numbers given by (2). Then we obtain an infinite lower triangular array:

$$D_w(a, b, c) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ b^2 & c + 2ab & a^2 & 0 & 0 \\ b^3 & b(2c + 3ab) & a(2c + 3ab) & a^3 & 0 \\ b^4 & b^2(3c + 4ab) & c^2 + 6abc + 6b^2a^2 & a^2(3c + 4ab) & a^4 \\ & & \cdots & & \end{bmatrix}.$$

Recently, Barry [1] introduced a generalized Pascal triangle $T = [T(n, k)]_{n,k \in \mathbb{N}_0}$ associated with the sequence $\{a_n\}$ with $a_0 = 1$ which is a centrally symmetric number triangle, i.e. $T(n, k) = T(n, n - k), T(n, 0) = T(n, n) = 1$, where

$$T(n, k) = \sum_{j=0}^{n-k} \binom{k}{j} \binom{n-k}{j} a_j.$$

In particular, he explored the triangle T when $a_n = r^n (r \in Z)$. In this case $T = D_w(1, 1, r - 1)$. More generally, we have the following theorem.

Theorem 2.1. $D_w(a, b, c)$ *has a Riordan array expression given by*

$$D_w(a, b, c) = \left(\frac{1}{1 - bz}, z\frac{a + cz}{1 - bz} \right). \tag{4}$$

Proof. Let $D_w(a, b, c) = (g(z), f(z)) = \mathcal{R}(d_{n,k})$. Clearly, $g(z) = \frac{1}{1-bz}$. Now, we show that $f(z) = z\frac{a+cz}{1-bz}$. Computing $d_{n+k,k}$ using $\frac{1}{1-bz} \left(z\frac{a+cz}{1-bz} \right)^k$, we obtain

$$\begin{aligned} d_{n+k,k} &= [z^{n+k}]\frac{1}{1 - bz} \left(z\frac{a + cz}{1 - bz} \right)^k = [z^n]\frac{1}{1 - bz} \left(\frac{a + cz}{1 - bz} \right)^k \\ &= \sum_{p=0}^{n}[z^p] \left(\frac{1}{1 - bz} \right)^{k+1} [z^{n-p}](a + cz)^k \\ &= \sum_{p=0}^{n} a^{k-(n-p)} b^p c^{n-p} \binom{k + p}{p} \binom{k}{n - p}. \end{aligned} \tag{5}$$

By setting $n - p = d$, we have

$$d_{n+k,k} = \sum_{d=0}^{n} a^{k-d} b^{n-d} c^d \binom{k}{d} \binom{n + k - d}{k} = D_w(n, k), \tag{6}$$

which proves $f(z) = z\frac{a+cz}{1-bz}$. Hence the proof is completed. ∎

From (5) and (6), since $D_w(n, k) = [z^n]\frac{(a+cz)^k}{(1-bz)^{k+1}}$, we have the following generating function for the weighted Delannoy numbers $D_w(n, k)$ (also see [5]):

$$\sum_{n,k\geq 0} D_w(n, k)z^n y^k = \sum_{k\geq 0} \frac{(a + cz)^k}{(1 - bz)^{k+1}}y^k = \frac{1}{1 - ay - bz - cyz}. \tag{7}$$

It is natural to take the row sums of any Riordan array. The row sums of $D_w(a, b, c)$ turn out to be closely related to the Lucas polynomial sequences.

Lemma 2.2. *Let* $\phi(z)$ *be the generating function for the row sums of the Riordan array* $D_w(a, b, c)$. *Then*

$$\phi(z) = \frac{1}{1 - (a + b)z - cz^2}. \tag{8}$$

Proof. By applying the SP of the Riordan array $D_w(a, b, c)$ given by (4) together with the generating function $h(z) = \frac{1}{1-z}$, we obtain immediately

$$\phi(z) = \left(\frac{1}{1 - bz} \right) \frac{1}{1 - \left(z\frac{a+cz}{1-bz} \right)} = \frac{1}{1 - (a + b)z - cz^2},$$

which completes the proof. ∎

3. Generalized Lucas polynomial sequence

Note that the numbers a, b, c appearing in the Riordan array $D_w(a, b, c)$ are weights on the steps $(1, 0), (0, 1)$ and $(1, 1)$, respectively. Now let us consider a, b, c as weight functions $a = a(x), b = b(x)$ and $c = c(x)$ independently.

Defining $\bar{p}(x)$ and $\bar{q}(x)$ by $a + b = \bar{p}(x)$ and $c = \bar{q}(x)$, we obtain a Riordan array

$$D_w(a, b, c) := D_w(\bar{p}(x), \bar{q}(x)).$$

We now define $\bar{W}_n(x)$ to be the n-th row sum of $D_w(\bar{p}(x), \bar{q}(x))$. Thus it follows from (8) that the generating function for $\bar{W}_n(x)$ may be written as

$$\phi_w(z) := \frac{1}{1 - \bar{p}(x)z - \bar{q}(x)z^2}. \tag{9}$$

In this section, we first derive the explicit formula for $\bar{W}_n(x)$ in terms of $\bar{p}(x)$ and $\bar{q}(x)$ using the Riordan array method.

Theorem 3.1. *For any two functions $\bar{p}(x)$ and $\bar{q}(x)$, the polynomials $\bar{W}_n(x)$ may be expressed as powers of $\bar{p}(x)$ and $\bar{q}(x)$:*

$$\bar{W}_n(x) = \sum_{k=0}^{\lceil(n-1)/2\rceil} \binom{n-k}{k} (\bar{p}(x))^{n-2k}(\bar{q}(x))^k. \tag{10}$$

Proof. First, note that

$$\frac{1}{1 - \bar{p}(x)z - \bar{q}(x)z^2} = \left(\frac{1}{1 - \bar{q}(x)z^2}\right)\left(\frac{1}{1-w}\right), \quad \text{with } w = \frac{\bar{p}(x)z}{1 - \bar{q}(x)z^2}.$$

Hence by the SP, $\bar{W}_n(x)$ may be expressed by the n-th row sum of the following Riordan array:

$$\left(\frac{1}{1 - \bar{q}(x)z^2}, \frac{\bar{p}(x)z}{1 - \bar{q}(x)z^2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{p} & 0 & 0 & 0 & 0 \\ \bar{q} & 0 & \bar{p}^2 & 0 & 0 & 0 \\ 0 & 2\bar{q}\bar{p} & 0 & \bar{p}^3 & 0 & 0 \\ \bar{q}^2 & 0 & 3\bar{q}\bar{p}^2 & 0 & \bar{p}^4 & 0 \\ 0 & 3\bar{q}^2\bar{p} & 0 & 4\bar{q}\bar{p}^3 & 0 & \bar{p}^5 \\ & & \cdots & & & \end{bmatrix},$$

where $\bar{p} = \bar{p}(x)$ and $\bar{q} = \bar{q}(x)$. By inspection (10) follows. ∎

Theorem 3.2. *Let $\bar{W}_n(x)$ be the polynomials as defined by (10). Then for any two functions $\bar{p}(x)$ and $\bar{q}(x)$, the polynomials $\bar{W}_n(x)$ satisfy the following recurrence relation for $n \geq 2$:*

$$\bar{W}_n(x) = \bar{p}(x)\bar{W}_{n-1}(x) + \bar{q}(x)\bar{W}_{n-2}(x), \tag{11}$$

where $\bar{W}_0(x) = 1$ and $\bar{W}_1(x) = \bar{p}(x)$.

Proof. Let n be an odd number. Then $\lceil\frac{n-2}{2}\rceil = \lceil\frac{n-1}{2}\rceil$. Applying the Vandermonde convolution yields the following:

$$\begin{aligned}
\bar{p}(x)\bar{W}_{n-1}(x) + \bar{q}(x)\bar{W}_{n-2}(x) &= \sum_{k=0}^{\lceil\frac{n-2}{2}\rceil} \binom{n-k-1}{k}(\bar{p}(x))^{n-2k}(\bar{q}(x))^k + \sum_{k=0}^{\lceil\frac{n-3}{2}\rceil} \binom{n-k-2}{k}(\bar{p}(x))^{n-2k-2}(\bar{q}(x))^{k+1} \\
&= (\bar{p}(x))^n + \sum_{k=1}^{\lceil\frac{n-1}{2}\rceil}\left\{\binom{n-k-1}{k} + \binom{n-k-1}{k-1}\right\}(\bar{p}(x))^{n-2k}(\bar{q}(x))^k \\
&= \sum_{k=0}^{\lceil\frac{n-1}{2}\rceil}\binom{n-k}{k}(\bar{p}(x))^{n-2k}(\bar{q}(x))^k = \bar{W}_n(x).
\end{aligned}$$

By a similar argument, we can establish (11) for an even number n, which completes the proof. ∎

Corollary 3.3. *In determinant form we have*

$$\bar{W}_n(x) = \det \begin{bmatrix} \bar{p}(x) & -1 & 0 & \cdots & 0 \\ \bar{q}(x) & \bar{p}(x) & -1 & \ddots & \vdots \\ 0 & \bar{q}(x) & \bar{p}(x) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & \bar{q}(x) & \bar{p}(x) \end{bmatrix}_{n\times n}. \tag{12}$$

Proof. The Laplace expansion of the first row and the three-term recurrence relation for $\bar{W}_n(x)$ given by (11) along with induction yields (12). ∎

We note that $\bar{W}_n(x)$ has the same recurrence relation (1) as $W_n(x)$, differing only in the initial conditions. By a similar argument as that used in [4], from (11) we can derive

$$\bar{W}_n(x) = \frac{\bar{u}^{n+1}(x) - \bar{v}^{n+1}(x)}{\bar{u}(x) - \bar{v}(x)},$$

where

$$\bar{u}(x) = \frac{\bar{p}(x) + \sqrt{\bar{p}^2(x) + 4\bar{q}(x)}}{2}, \quad \text{and} \quad \bar{v}(x) = \frac{\bar{p}(x) - \sqrt{\bar{p}^2(x) + 4\bar{q}(x)}}{2}$$

giving $\bar{u}(x) + \bar{v}(x) = \bar{p}(x)$ and $\bar{u}(x)\bar{v}(x) = -\bar{q}(x)$.

Let us define the polynomial sequences $\{\bar{w}_n(x)\}$ by

$$\bar{w}_n(x) = \bar{u}^{n+1}(x) + \bar{v}^{n+1}(x).$$

We note that the two functions $p(x)$ and $q(x)$ used in (1) are monomial of degree ≤ 1 whereas $\bar{p}(x)$ and $\bar{q}(x)$ used in (11) are any functions. It is easy to show that when $\bar{p}(x) = p(x)$ and $\bar{q}(x) = q(x)$ we have $\bar{W}_n(x) = W_{n+1}(x)$ and $\bar{w}_n(x) = w_{n+1}(x)$ for $n \in \mathbb{N}_0$. In this sense, we shall call $\{\bar{W}_n(x)\}$ and $\{\bar{w}_n(x)\}$ *generalized Lucas polynomial sequences of the first kind* and *of the second kind*, respectively.

The following theorem for an expression for the polynomial sequence $\{\bar{w}_n(x)\}$ follows from (10) and

$$\bar{w}_n(x) = \bar{q}(x)\bar{W}_{n-1}(x) + \bar{W}_{n+1}(x), \quad (n \geq 1), \tag{13}$$

where $\bar{w}_0(x) = \bar{p}(x)$ and $\bar{W}_0(x) = 1$.

Theorem 3.4. *Let $\{\bar{w}_n(x)\}$ be the generalized Lucas polynomial sequence of the second kind. Then $\bar{w}_n(x)$ may be expressed in terms of powers of $\bar{p}(x)$ and $\bar{q}(x)$:*

$$\bar{w}_n(x) = \sum_{k=0}^{\lceil n/2 \rceil} \left\{ \binom{n-k}{k-1} + \binom{n-k+1}{k} \right\} (\bar{p}(x))^{n-2k+1}(\bar{q}(x))^k. \tag{14}$$

With the Riordan array notation, (13) can be written as the following matrix form:

$$(1 + \bar{q}(x)z^2, z)[\bar{W}_0(x), \bar{W}_1(x), \ldots]^{\mathrm{T}} = [1, \bar{w}_0(x), \bar{w}_1(x), \ldots]^{\mathrm{T}}.$$

By applying the SP to the Riordan array $(1 + \bar{q}(x)z^2, z)$ together with the generating function $\phi_w(z)$ for $\bar{W}_n(x)$, we obtain the generating function for $\{\bar{w}_n(x)\}$:

$$\frac{1 + \bar{q}(x)z^2}{1 - \bar{p}(x)z - \bar{q}(x)z^2} = \sum_{n \geq 0} \bar{w}_{n-1}(x)z^n, \quad (\bar{w}_{-1} := 1). \tag{15}$$

Now, we are interested in pairs of generalized Lucas polynomial sequences similar to the pairs of cognate polynomial sequences given by Table 1.

Theorem 3.5. *Let $\bar{p}(x) = ax + b$ and $\bar{q}(x) = cx$. Then*

(i) $\bar{W}_n(x) = \sum_{k=0}^{n} \left(\sum_{d=0}^{k} \binom{k}{d}\binom{n-d}{k} a^{k-d}b^{n-k-d}c^d \right) x^k, \quad n \geq 0$;

(ii) $\bar{w}_{n-1}(x) = \sum_{k=0}^{n} \left(\sum_{d=0}^{k} \frac{n}{n-d} \binom{n-k}{d}\binom{n-d}{k-d} a^{k-d}b^{n-k-d}c^d \right) x^k, \quad n \geq 1$.

Proof. (i) By applying the SP of the Riordan array $D_w(a, b, c)$ together with $h(z) = \frac{1}{1-xz}$, we obtain

$$\sum_{k=0}^{n} D_w(n-k, k)x^k = [z^n]\left(\frac{1}{1-bz}\right)\frac{1}{1 - x\left(\frac{z(a+cz)}{1-bz}\right)}$$

$$= [z^n]\frac{1}{1 - (ax+b)z - cxz^2} = [z^n]\frac{1}{1 - \bar{p}(x)z - \bar{q}(x)z^2}$$

$$= \bar{W}_n(x).$$

The last equality follows from (9) and this finishes the proof of (i).

(ii) By substituting $\bar{p}(x) = ax + b$ and $\bar{q}(x) = cx$ into (14) and then applying the binomial theorem, we can establish (ii) by elementary algebraic calculations. ∎

Example 3.6. Let $\bar{p}(x) = x + 1$ and $\bar{q}(x) = x$. Then we obtain the generalized Lucas polynomial sequence of the first kind with the same coefficients as the Delannoy numbers. Denote it as the *Delannoy polynomial sequence* $\{D_n(x)\}$:

$$D_n(x) = \sum_{k=0}^{n} D(n-k, k)x^k = \sum_{k=0}^{n}\sum_{d=0}^{k} \binom{k}{d}\binom{n-d}{k} x^k.$$

From (9) we have the generating function for the Delannoy polynomials:

$$\frac{1}{1 - (x + 1)z - xz^2} = \sum_{n=0}^{\infty} D_n(x)z^n.$$

The first few polynomials are:

$D_0(x) = 1,$
$D_1(x) = x + 1,$
$D_2(x) = x^2 + 3x + 1,$
$D_4(x) = x^3 + 5x^2 + 5x + 1.$

Now we consider a cognate polynomial sequence to the Delannoy polynomials. From (15) we obtain the generalized Lucas polynomial sequence of the second kind $\{C_n(x)\}$ with the generating function:

$$\frac{1 + xz^2}{1 - (x + 1)z - xz^2} = \sum_{n \geq 0} C_n(x)z^n.$$

The first few polynomials are:

$C_0(x) = 1,$
$C_1(x) = x + 1,$
$C_2(x) = x^2 + 4x + 1,$
$C_3(x) = x^3 + 6x^2 + 6x + 1.$

It follows from (13) that

$$C_{n+1}(x) = xD_{n-1}(x) + D_{n+1}(x), \quad n \geq 1. \tag{16}$$

By consulting [7] we find [(see A102413)] that the coefficients of x^k in the polynomial $C_n(x)$ are the same as the *corona numbers*, $C(n, k)$, which count the number of k-matchings of the corona $C_n \circ K_1$ of the cycle C_n and the complete graph K_1. That is, $C_n \circ K_1$ is the graph with $2n$ vertices obtained from C_n adding one edge to each vertex of C_n. For more information on corona graphs, see [3]. We shall call $\{C_n(x)\}$ the *corona polynomial sequence*. Note that the corona numbers have an explicit form from (ii) of Theorem 3.5:

$$C(n, k) = \sum_{i=0}^{k} \frac{n}{n - i} \binom{n - i}{k - i} \binom{n - k}{i}.$$

Furthermore, corona polynomials $C_n(x)$ can be interpreted as the total sum of weights of all k-matchings in a weighted corona $C_n \circ K_1$ where each matching edge has a weight x.

Corollary 3.7. *The corona numbers $C(n, k)$ may be expressed in terms of the Delannoy numbers $D(n, k)$:*

$$C(n, k) = D(n - k - 1, k - 1) + D(n - k, k), \quad n \geq 2, \ k \geq 1. \tag{17}$$

Proof. From (16), we have

$$\sum_{k=0}^{n} C(n, k)x^k = C_n(x) = xD_{n-2}(x) + D_n(x)$$

$$= x \sum_{k=0}^{n-2} D(n - k - 2, k)x^k + \sum_{k=0}^{n} D(n - k, k)x^k$$

$$= \sum_{k=1}^{n-1} D(n - k - 1, k - 1)x^k + \sum_{k=0}^{n} D(n - k, k)x^k$$

$$= D(n, 0) + \sum_{k=1}^{n-1} \{D(n - k - 1, k - 1) + D(n - k, k)\}x^k + D(0, n)x^n.$$

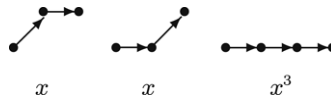Comparing the coefficients of x^k for $k \geq 1$ yields (17). ∎

Fig. 1. $(x, 0, 1)$-weighted paths and their weights.

4. Combinatorial interpretations and examples

In this section, we obtain combinatorial interpretations for a pair of generalized Lucas polynomial sequences.

Theorem 4.1. *Let* $\{\bar{W}_n(x)\}$ *and* $\{\bar{w}_n(x)\}$ *be a pair of Lucas polynomial sequences. Then*

(i) $\bar{W}_n(x) = \sum_{k=0}^{n} \omega_{k,n-k}(x)$ $(n \geq 0)$;
(ii) $\bar{w}_n(x) = \sum_{k=0}^{n} \omega_{k,n-k}(x) + \bar{q}(x) \sum_{k=0}^{n-2} \omega_{k,n-k-2}(x)$, $(n \geq 2)$,

where $\omega_{k,n-k}(x) := D_w(n-k, k)$ *is the sum of weights of* $(a(x), b(x), c(x))$-*weighted paths from* $(0, 0)$ *to* $(k, n-k)$ *using the steps* $(1, 0), (0, 1)$ *and* $(1, 1)$ *for which* $a(x) + b(x) = \bar{p}(x)$ *and* $c(x) = \bar{q}(x)$.

Proof. Since $\bar{W}_n(x)$ is the n-th row sum of the Riordan array $D_w(\bar{p}(x), \bar{q}(x))$, (i) and (ii) immediately follow from (2) and (13), respectively.   ■

Combinatorial interpretations for pairs of Lucas polynomial sequences listed by Table 1 may be obtained from Theorem 4.1 by setting $\bar{p}(x) = p(x)$ and $\bar{q}(x) = q(x)$ so that $a(x) + b(x) = p(x)$ and $c(x) = q(x)$. Also explicit forms for those polynomial sequences may be simply determined by (10) or (14). For instance, the Chebyshev polynomial of the second kind $U_n(x)$ is obtained in the form:

$$U_{n+1}(x) = \sum_{k=0}^{\lceil (n-1)/2 \rceil} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

Example 4.2. Let us consider the Fibonacci polynomial sequence $\{F_n(x)\}$. First note that $F_n(x)$ has the following generating function:

$$\frac{z}{1 - xz - z^2} = \sum_{n \geq 0} F_n(x) z^n.$$

By using (12), the Fibonacci polynomial has also a beautiful determinantal expression:

$$F_{n+1}(x) = \det \begin{bmatrix} x & -1 & 0 & \cdots & 0 \\ 1 & x & -1 & \ddots & \vdots \\ 0 & 1 & x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & 1 & x \end{bmatrix}_{n \times n}, \quad (n \geq 1).$$

Of course, by substituting $a = x, b = 0, c = 1$ we may immediately obtain the Fibonacci polynomials from the row sums of the Riordan array $D_w(x, 0, 1)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 1 & x^2 & 0 & 0 \\ 0 & 0 & 2x & x^3 & 0 \\ 0 & 0 & 1 & 3x^2 & x^4 \\ & & \cdots & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ 1 + x^2 \\ 2x + x^3 \\ 1 + 3x^2 + x^4 \\ \vdots \end{bmatrix} = \begin{bmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ F_5(x) \\ \vdots \end{bmatrix}. \tag{18}$$

Now let us consider a combinatorial interpretation for $F_4(x) = 2x + x^3$. Since $F_4(x) = W_4(x) = \bar{W}_3(x)$ when $p(x) = a(x) + b(x) = x$ and $q(x) = c(x) = 1$, we may take $a(x) = x, b(x) = 0$ and $c(x) = 1$. By (i) of Theorem 4.1, we have $F_4(x) = \omega_{0,3}(x) + \omega_{1,2}(x) + \omega_{2,1}(x) + \omega_{3,0}(x)$. Note that $\omega_{0,3}(x) = 0$, $\omega_{1,2}(x) = 0$, while $\omega_{2,1}(x) = 2x$ and $\omega_{3,0}(x) = x^3$, see Fig. 1.

Example 4.3. Let us consider the Pell–Lucas polynomial sequence $\{Q_n(x)\}$. First note that $Q_n(x)$ has the following generating function for $n \geq 1$:

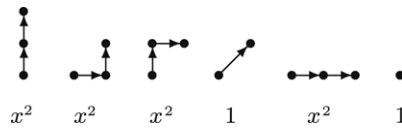$$\frac{1 + z^2}{1 - 2xz - z^2} = \sum_{n \geq 0} Q_n(x) z^n.$$

Fig. 2. $(x, x, 1)$-weighted paths and their weights.

Moreover, by (14) $Q_{n+1}(x)$ has the following explicit formula:

$$Q_{n+1}(x) = \sum_{k=0}^{\lceil n/2 \rceil} \left\{ \binom{n-k}{k-1} + \binom{n-k+1}{k} \right\} (2x)^{n-2k+1}.$$

Now let us consider a combinatorial interpretation for $Q_2(x) = 2 + 4x^2$. Since $p(x) = a(x) + b(x) = 2x$ and $q(x) = c(x) = 1$, we may take $a(x) = x$, $b(x) = x$ and $c(x) = 1$. By (ii) of Theorem 4.1, we have $Q_2(x) = \omega_{0,2}(x) + \omega_{1,1}(x) + \omega_{2,0}(x) + 1 \cdot \omega_{0,0}(x)$. Note that $\omega_{0,2}(x) = x^2$, $\omega_{1,1}(x) = 2x^2 + 1$, $\omega_{2,0}(x) = x^2$ and $\omega_{0,0}(x) = 1$, see Fig. 2.

5. Concluding remarks

We have used the Delannoy polynomials in a Riordan group context and shown how many results follow in a natural manner. The key point is that the Riordan group works over any integral domain and not just the integers. Often the rows provide polynomial sequences that mesh with the column generating functions of the Riordan group.

Acknowledgments

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