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Some Trigonometric Identities Involving Fibonacci and Lucas Numbers

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Abstract

In this paper, using the number of spanning trees in some classes of graphs, we prove the identities:

$$F_n = \frac{2^{n-1}}{n}\sqrt{\prod_{k=1}^{n-1}(1-\cos\frac{k\pi}{n}\cos\frac{3k\pi}{n})},\quad n \geq 2,$$

$$\prod_{k=0}^{n-1}(1+4\sin^2\frac{k\pi}{n}) = L_{2n}-2 = F_{2n+2}-F_{2n-2}-2,\quad n \geq 1,$$

where F_n and L_n denote the Fibonacci and Lucas numbers, respectively. Also, we give a new proof for the identity:

$$F_n = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(1+4\sin^2\frac{k\pi}{n}) = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(1+4\cos^2\frac{k\pi}{n}),\quad n \geq 4.$$

1 Introduction

Let F_n and L_n denote the Fibonacci and Lucas numbers respectively. That is, $F_{n+2} = F_{n+1} + F_n$, for $n \geq 1$ with $F_1 = F_2 = 1$, and $L_{n+2} = L_{n+1} + L_n$, for $n \geq 1$ with $L_1 = 1$ and $L_2 = 3$.

In this paper, we derive the identities:

$$F_n = \frac{2^{n-1}}{n}\sqrt{\prod_{k=1}^{n-1}(1-\cos\frac{k\pi}{n}\cos\frac{3k\pi}{n})},\quad n\geq 2, \tag{1}$$

$$\prod_{k=0}^{n-1}(1+4\sin^2\frac{k\pi}{n}) = L_{2n}-2 = F_{2n+2}-F_{2n-2}-2,\quad n\geq 1. \tag{2}$$

To prove identity (1), we apply the number of spanning trees in a special class of graphs known as circulant graphs. Identity (2) is derived from the number of spanning trees in a wheel.

Applying the same technique to a graph known as fan gives us a new proof for the following identity:

$$F_n = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(1+4\sin^2\frac{k\pi}{n}) = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(1+4\cos^2\frac{k\pi}{n}),\quad n\geq 4, \tag{3}$$

appeared in [6] and its corresponding references.

Also, applying this technique to the path P_n and the cycle C_n gives us a new proof for the well-known identities:

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}},\qquad n\geq 2, \tag{4}$$

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}},\qquad n\geq 2. \tag{5}$$

2 Techniques and Proofs

For a graph G, a *spanning tree* in G is a tree which has the same vertex set as G. The number of spanning trees in a graph (network) G, denoted by $t(G)$, is an important invariant of the graph (network). It is also an important measure of reliability of a network. In the sequel, we assume our graphs are loopless but multiple edges are allowed.

A famous and classic result on the study of $t(G)$ is the following theorem, known as the *Matrix-tree Theorem*. The *Laplacian matrix* of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ and $A(G)$ are the degree matrix and the adjacency matrix of G, respectively. Since this theorem is first proved by Kirchhoff [7], $L(G)$ is also known as the *Kirchhoff matrix* of the graph G.

Theorem 1. *For every connected graph G, $t(G)$ is equal to any cofactor of $L(G)$.*

The number of spanning trees of a connected graph G can be expressed in terms of the eigenvalues of $L(G)$. Since by definition, $L(G)$ is a real symmetric matrix, it therefore has n non-negative real eigenvalues, where n is the number of vertices of G. Anderson and Morley

[1, Theorem 1] proved that the multiplicity of 0 as an eigenvalue of $L(G)$ equals the number of components of G. Therefore, the Laplacian matrix of a connected graph G has 0 as an eigenvalue with multiplicity one.

Theorem 2. *([5]) Suppose G is a connected graph with n vertices. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $L(G)$, with $\lambda_n = 0$. Then $t(G) = \frac{1}{n}\lambda_1 \cdots \lambda_{n-1}$.*

As the first example, we prove identity (4).

Proof of identity (4). Consider the path P_n. It is known that the eigenvalues of the Laplacian matrix of P_n are $2 - 2\cos\frac{k\pi}{n}$ $(0 \leq k \leq n-1)$ (see, e.g., [4]). On the other hand, we know that $t(P_n) = 1$, therefore by using Theorem (2) we obtain (4). □

Now, we state some more definitions and theorems.

Definition 3. An $n \times n$ matrix $C = (c_{ij})$ is called a *circulant matrix* if its entries satisfy $c_{ij} = c_{1,\, j-i+1}$, where subscripts are reduced modulo n and lie in the set $\{1, 2, \ldots, n\}$.

Definition 4. Let $1 \leq s_1 < s_2 < \cdots < s_k < \frac{n}{2}$, where n and s_i $(1 \leq i \leq k)$ are positive integers. An *undirected circulant graph* $C_n(s_1, s_2, \ldots, s_k)$ is a $2k$-regular graph with vertex set $V = \{0, 1, \ldots, n-1\}$ and edge set $E = \{\{i, i + s_j \pmod n\} \mid i = 0, 1, \ldots, n-1,\ j = 1, 2, \ldots, k\}$.

The Laplacian matrix of $C_n(s_1, s_2, \ldots, s_k)$ is clearly a circulant matrix. By a direct using of Theorem 4.8 of [12], we obtain the following lemma:

Lemma 5. *The nonzero eigenvalues of $L(C_n(s_1, s_2, \ldots, s_k))$ are*

$$2k - \omega^{s_1 j} - \cdots - \omega^{s_k j} - \omega^{-s_1 j} - \cdots - \omega^{-s_k j},\ \ 1 \leq j \leq n-1,$$

where $\omega = e^{\frac{2\pi i}{n}}$.

With combining Theorem 2 and the lemma above, we obtain the following corollary:

Corollary 6. *The number of spanning trees in $G = C_n(s_1, s_2, \ldots, s_k)$ is equal to:*

$$t(G) = \frac{1}{n}\prod_{j=1}^{n-1}\Big(\sum_{i=1}^{k}(2 - 2\cos\frac{2js_i\pi}{n})\Big).$$

Proof of identity (1). Consider the *square cycle* $C_n(1,2)$. We can use Corollary 6 to obtain the number of spanning trees of $C_n(1,2)$. On the other hand, Kleitman and Golden [8] proved that $t(C_n(1,2)) = nF_n^2$. Now, with a little additional algebraic manipulation, identity (1) follows. □

Proof of identity (5). Look at the cycle $C_n(1) = C_n$. We know that $t(C_n) = n$, therefore by applying Corollary 6 to it, (5) follows. □

Definition 7. The join $W_n = C_n \bigvee K_1$ of a cycle C_n and a single vertex is referred to as a *wheel* with n *spokes*. Similarly, the join $\mathcal{F}_n = P_n \bigvee K_1$ of a path P_n and a single vertex is called a *fan*.

Sedlacek [11] and later Myers [10] showed that $t(W_n) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2,\ n \geq 1$. Also, Bibak and Shirdareh Haghighi [2, 3] proved that $t(\mathcal{F}_n) = F_{2n},\ n \geq 1$.

Now, we find the number of spanning trees in W_n and $\mathcal{F}_n$ by applying Theorem 2. We first need to determine the eigenvalues of $L(W_n)$ and $L(\mathcal{F}_n)$.

Theorem 8. *([9]) Let G_1 and G_2 be simple graphs on disjoint sets of r and s vertices, respectively. If $S(G_1) = (\mu_1, \ldots, \mu_r)$ and $S(G_2) = (\nu_1, \ldots, \nu_s)$ are the eigenvalues of $L(G_1)$ and $L(G_2)$ arranged in nonincreasing order, then the eigenvalues of $L(G_1 \bigvee G_2)$ are $n = r+s$; $\mu_1 + s, \ldots, \mu_{r-1} + s$; $\nu_1 + r, \ldots, \nu_{s-1} + r$; and 0.*

Since the eigenvalues of $L(C_n)$ are $2 - 2\cos\frac{2k\pi}{n}$ $(0 \leq k \leq n-1)$ (by Lemma 5), and the eigenvalues of $L(P_n)$ are $2 - 2\cos\frac{k\pi}{n}$ $(0 \leq k \leq n-1)$, therefore, by Theorem 8 we can determine the eigenvalues of $L(W_n)$ and $L(\mathcal{F}_n)$.

Theorem 9. *The eigenvalues of $L(W_n)$ are $n+1$, 0 and $1 + 4\sin^2\frac{k\pi}{n}$ $(1 \leq k \leq n-1)$, and the eigenvalues of $L(\mathcal{F}_n)$ are $n+1$, 0 and $1 + 4\sin^2\frac{k\pi}{2n}$ $(1 \leq k \leq n-1)$ (or $n+1$, 0 and $1 + 4\cos^2\frac{k\pi}{2n}$ $(1 \leq k \leq n-1)$).*

Proofs of the identities (2) and (3). By Theorems 2 and 9, the number of spanning trees of W_n and $\mathcal{F}_n$ are, respectively,

$$t(W_n) = \prod_{k=0}^{n-1}(1 + 4\sin^2\frac{k\pi}{n}), \quad n \geq 1,$$

$$t(\mathcal{F}_n) = \prod_{k=1}^{n-1}(1 + 4\sin^2\frac{k\pi}{2n}) = \prod_{k=1}^{n-1}(1 + 4\cos^2\frac{k\pi}{2n}), \quad n \geq 2.$$

On the other hand, as we already referred, $t(W_n) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2,\ n \geq 1$ and $t(\mathcal{F}_n) = F_{2n},\ n \geq 1$. Therefore, we obtain (2) and (3). □

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