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Expansion of Fibonacci and Lucas Polynomials: An Answer to Prodinger's Question

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Abstract

We give an answer to a recent question of Prodinger, which consists of finding q-analogues of identities related to Fibonacci and Lucas polynomials.

1 Introduction

The bivariate polynomials of Fibonacci and Lucas, denoted respectively by (U_n) and (V_n), are defined by

$$\begin{cases} U_0 = 0,\ U_1 = 1, \\ U_n = tU_{n-1} + zU_{n-2} \quad (n \geq 2), \end{cases} \qquad \text{and} \qquad \begin{cases} V_0 = 2,\ V_1 = t, \\ V_n = tV_{n-1} + zV_{n-2} \ (n \geq 2). \end{cases}$$

It is well-known (see, for example, [3]) that

$$U_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} t^{n-2k} z^k, \qquad V_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} t^{n-2k} z^k \ \ (n \geq 1).$$

In [1], Belbachir and Bencherif proved the following five formulae:

$$2U_{2n+1} = \sum_{k=0}^{n} a_{n,k} t^k V_{2n-k}, \ \text{ with } \ a_{n,k} = 2 \sum_{j=0}^{n} (-1)^{j+k} \binom{j}{k} - (-1)^{n+k} \binom{n}{k}, \tag{1}$$

$$V_{2n} = \sum_{k=1}^{n} b_{n,k} t^k V_{2n-k}, \text{ with } \ b_{n,k} = (-1)^{k+1} \binom{n}{k}, \tag{2}$$

$$V_{2n-1} = \sum_{k=1}^{n} c_{n,k} t^k U_{2n-k}, \ \text{ with } \ c_{n,k} = 2 (-1)^{k+1} \binom{n}{k} - [k=1], \tag{3}$$

$$2V_{2n-1} = \sum_{k=1}^{n} d_{n,k} t^k V_{2n-1-k}, \ \text{ with } \ d_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k}, \tag{4}$$

$$2U_{2n} = \sum_{k=1}^{n} e_{n,k} t^k V_{2n-k}, \ \text{ with} \tag{5}$$

$$e_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k} + \sum_{j=0}^{n-1} (-1)^{j+k-1} \binom{j}{k-1} - \frac{1}{2} (-1)^{n+k} \binom{n-1}{k-1}$$

and they also gave the following simple sixth one:

$$V_{2n} = 2U_{2n+1} - xU_{2n}. \tag{6}$$

The results presented in [1] are companion generalizations for Chebyshev polynomials [2].

Without loss of generality, we suppose that $t = 1$. We refer here to the modified polynomials given by Prodinger in the introduction of [5].

In order to give q-analogues of these instances, Prodinger considered Cigler's [4] suggestion to replace U_n and V_n, respectively, with

$$\mathbf{F}_{n+1}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q z^k, \qquad \mathbf{L}_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{[n]_q}{[n-k]_q} z^k,$$

with the following q-notations

$$[n]_q = 1 + q + \cdots + q^{n-1}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Prodinger [5] found the following two q-analogues for identities (4) and (6) respectively

$$\mathbf{F}_{2n}(z) = \sum_{k=1}^{n} \beta_{n,k} \mathbf{F}_{2n-k}(z), \quad \text{with} \quad \beta_{n,k} = (-1)^{k+1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{7}$$

$$\mathbf{L}_{2n-1} = \sum_{k=1}^{n} \delta_{n,k} \mathbf{L}_{2n-1-k}(z), \quad \text{with} \quad \delta_{n,k} = (-1)^{k+1} \frac{q^{\binom{k}{2}}}{1+q^{n-1}} \left(\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \right), \tag{8}$$

and left the three others as a challenge.

We give an answer to the remaining three instances and we find a q-analogue for identity (6).

2 An answer to Prodinger's question

Here, we find a q-analogue for the identities (6), (3), (1) and (5), which can be considered as an answer to Prodinger's question [5].

Theorem 1. *The following q-identities hold*

$$\mathbf{L}_{2n}(z) = \mathbf{F}_{2n+1}\left(\frac{z}{q}\right) + \mathbf{F}_{2n+1}(z) - \mathbf{F}_{2n}(z), \tag{9}$$

$$\mathbf{L}_{2n-1}(z) = \sum_{k=1}^{n} \beta_{n,k} \left(\mathbf{F}_{2n-k}(z) + \mathbf{F}_{2n-k}\left(\frac{z}{q}\right) \right) - \mathbf{F}_{2n-1}(z), \tag{10}$$

$$\mathbf{F}_{2n+1}(z) = \frac{1}{1-q^{2n}} \left(\mathbf{L}_{2n}(qz) - q^{2n} \mathbf{L}_{2n}(z) \right), \tag{11}$$

$$\mathbf{F}_{2n}(z) = \frac{1}{1-q^{2n-1}} \left(\mathbf{L}_{2n-1}(qz) - q^{2n-1} \mathbf{L}_{2n-1}(z) \right). \tag{12}$$

Proof. We have

$$\begin{aligned}
\mathbf{F}_{n+1}\left(\frac{z}{q}\right) + \mathbf{F}_{n+1}(z) - \mathbf{F}_{n}(z) &= \sum\nolimits_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + q^k - q^k \frac{[n-2k]_q}{[n-k]_q} \right) z^k \\
&= \sum\nolimits_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + q^k - q^k \frac{1-q^{n-2k}}{1-q^{n-k}} \right) z^k \\
&= \sum\nolimits_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{1-q^n}{1-q^{n-k}} z^k \\
&= \sum\nolimits_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{[n]_q}{[n-k]_q} z^k = \mathbf{L}_n(z).
\end{aligned}$$

For the second identity $\sum_{k=1}^{n} \beta_{n,k} \left(\mathbf{F}_{2n-k}(z) + \mathbf{F}_{2n-k}\left(\frac{z}{q}\right) \right) - \mathbf{F}_{2n-1}(z) = \mathbf{F}_{2n}(z) + \mathbf{F}_{2n}\left(\frac{z}{q}\right) - \mathbf{F}_{2n-1}(z) = \mathbf{L}_{2n-1}(z).$

To establish (11) and (12), we have

$$\begin{aligned}\mathbf{L}_n(qz) - q^n \mathbf{L}_n(z) &= \sum\nolimits_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{[n]_q}{[n-k]_q} \left(q^k - q^n\right) z^k \\ &= \sum\nolimits_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (1-q^n) z^k \\ &= (1-q^n) \mathbf{F}_{n+1}(z),\end{aligned}$$

and then take n odd and n even respectively. □

Remark 2. We have to notice that the expressions of relations (11) and (12) do not give, for $q = 1$, the initial formulae as given in relations (1) and (5). We obtain, when $q \to 1$, the well known identity $U_{n+1}(z) = V_n(z) - zU_{n-1}(z)$, for n odd and n even respectively. However, they express, as in Belbachir's and Bencherif's work [1], an odd Fibonacci polynomial in terms of Lucas polynomials starting with a smallest even index, and an even Fibonacci polynomial in terms of Lucas polynomials starting with a smallest odd index.

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