SUMS OF PRODUCTS OF GENERALIZED FIBONACCI AND LUCAS NUMBERS

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Abstract

In this paper, we establish several formulae for sums and alternating sums of products of generalized Fibonacci and Lucas numbers. In particular, we recover and extend all results of Z. Čerin [2, 2005] and Z. Čerin and G. M. Gianella [3, 2006], more easily.

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1 Introduction and main result

Let p and q two integers such that $pq \neq 0$ and $\Delta := p^2 - 4q \neq 0$. We define sequences of generalized Fibonacci and Lucas numbers $(U_n) = (U_n^{(p,q)})$ and $(V_n) = (V_n^{(p,q)})$, for all n, by induction

$$\begin{cases} U_0 = 0,\ U_1 = 1,\ U_n = pU_{n-1} - qU_{n-2} \\ V_0 = 2,\ V_1 = p,\ V_n = pV_{n-1} - qV_{n-2} \end{cases}$$

Sequences of Fibonacci (F_n) , Lucas (L_n) , Pell (P_n) , Pell-Lucas (Q_n) , Jacobsthal (J_n) , Jacobsthal-Lucas (j_n) listed respectively A000045, A00032, A000129, A002203, A001045, A014551 in SLOANE [9] are $(F_n, L_n) = (U_n^{(1,-1)}, V_n^{(1,-1)})$, $(P_n, Q_n) = (U_n^{(2,-1)}, V_n^{(2,-1)})$, $(J_n, j_n) = (U_n^{(1,-2)}, V_n^{(1,-2)})$ for $n \geq 0$.

For r and s two integers and for all sequences $(X_m)_{m\in\mathbb{Z}}$ and $(Y_m)_{m\in\mathbb{Z}}$, let

$$S_n^{(r,s)}(X,Y) := \sum_{i=0}^{n} X_{r+2i}Y_{s+2i} \quad \text{and} \quad A_n^{(r,s)}(X,Y) := \sum_{i=0}^{n} (-1)^i X_{r+2i}Y_{s+2i},$$

for convenience, we also set $S_n^{(r,s)}(X) := S_n^{(r,s)}(X,X)$ and $A_n^{(r,s)}(X) := A_n^{(r,s)}(X,X)$.

Sums involving Fibonacci, Lucas, Pell and Pell-Lucas numbers and generalizations have been studied by several authors, for example, for trigonometric sums see Melham [6, 1999] and Belbachir & Bencherif [1, 2007], for reciprocal and powers sums see Melham [7, 1999] and [8, 2000], and for the sum of squares see Long [5, 1986], Čerin [2, 2005] and Čerin & Gianella [3, 2006].

In [2, 2005], Čerin studied $A_n^{(r,s)}(L)$ for $s = r$ and $s = r+1$ when r is odd, and in [3, 2006], Čerin and Gianella considered $S_n^{(r,s)}(Q)$ and $A_n^{(r,s)}(Q)$ for $s = r$ and $s = r+1$ when r is even.

Recently, Čerin [4, 2007] studied the sums of squares and products of Jacobsthal numbers by establishing identities for $S_n^{(r,s)}(J)$, and $A_n^{(r,s)}(J)$, for $s=r$ and $s=r+1$ when r is even. This case corresponds to $(p,q)=(1,-2)$.

Our purpose is to give simplified expressions for the sums $S_n^{(r,s)}(U)$, $S_n^{(r,s)}(V)$, $A_n^{(r,s)}(U)$ and $A_n^{(r,s)}(V)$. In all what follows, we suppose $q=\pm1$ (which gives $V_2\neq0$, $U_2\neq0$ and $U_4\neq0$).

For $n\in\mathbb{Z}$, let us define the sequences (a_n), (b_n), (c_n), (d_n) and (e_n) by the relations

$$a_n=\frac{U_{2n}}{U_2},\ b_n=\frac{d_{n+1}-1}{p^2\Delta},\ c_n=\frac{U_{4n+4}}{U_4},\ d_n=\frac{V_{4n+2}}{V_2},\ e_n=p\left(d_n-1\right).$$

These sequences, depending on p and q satisfy the recurrence relations

$$\begin{array}{lll}
a_{-1}=-1, & a_0=0, & a_n=V_2a_{n-1}-a_{n-2},\\
b_{-1}=0, & b_0=1, & b_n=V_4b_{n-1}-b_{n-2}+1,\\
c_{-1}=0, & c_0=1, & c_n=V_4c_{n-1}-c_{n-2},\\
d_{-1}=1, & d_0=1, & d_n=V_4d_{n-1}-d_{n-2},\\
e_{-1}=0, & e_0=0, & e_n=V_4e_{n-1}-e_{n-2}+p^3\left(p^2-4\right).
\end{array}$$

For $(p,q)=(1,-1)$, we have, for $n\geq0$, $(U_n,V_n)=(F_n,L_n)$ and one gets $(a_n)=(0,1,3,8,21,\ldots)$, $(b_n)=(1,8,56,385,2640,\ldots)$, $(c_n)=(1,7,48,329,2255,\ldots)$ and $(d_n)=(1,6,41,281,1926,\ldots)$ listed in SLOANE respectively as A001906, A092521, A004187, A049685.

For $(p,q)=(2,-1)$, we have, for $n\geq0$, $(U_n,V_n)=(P_n,Q_n)$ and one gets $(a_n)=(0,1,6,35,\ldots)$, $(b_n)=(1,35,1190,40426,\ldots)$, $(c_n)=(1,34,1155,39236,\ldots)$ and $(d_n)=(1,33,1121,38081,\ldots)$ listed in SLOANE respectively as A001109, A029546, A029547, A077420.

We give now, for $\varepsilon=\left(1+(-1)^n\right)/2$, the main result of the paper

Theorem 1 *For all integers r, s and $n\geq0$, we have*

$$\begin{aligned}
S_n^{(r,s)}(U) &= \sum_{i=0}^{n}U_{r+2i}U_{s+2i}=p^{-1}\Delta^{-1}\left[U_{4n+r+s+2}-U_{r+s-2}\right]-(n+1)\Delta^{-1}q^rV_{s-r},\\
S_n^{(r,s)}(V) &= \sum_{i=0}^{n}V_{r+2i}V_{s+2i}=p^{-1}\left[U_{4n+r+s+2}-U_{r+s-2}\right]+(n+1)q^rV_{s-r},\\
S_n^{(r,s)}(U,V) &= \sum_{i=0}^{n}U_{r+2i}V_{s+2i}=p^{-1}\Delta^{-1}\left[V_{4n+r+s+2}-V_{r+s-2}\right]-(n+1)\Delta^{-1}q^rU_{s-r},\\
A_n^{(r,s)}(U) &= \sum_{i=0}^{n}(-1)^iU_{r+2i}U_{s+2i}=\Delta^{-1}V_2^{-1}\left[V_{r+s-2}+(-1)^nV_{4n+r+s+2}\right]-\varepsilon\Delta^{-1}q^rV_{s-r},\\
A_n^{(r,s)}(V) &= \sum_{i=0}^{n}(-1)^iV_{r+2i}V_{s+2i}=V_2^{-1}\left[V_{r+s-2}+(-1)^nV_{4n+r+s+2}\right]+\varepsilon q^rV_{s-r},\\
A_n^{(r,s)}(U,V) &= \sum_{i=0}^{n}(-1)^iU_{r+2i}V_{s+2i}=V_2^{-1}\left[U_{r+s-2}+(-1)^nU_{4n+r+s+2}\right]-\varepsilon q^rU_{s-r}.
\end{aligned}$$

Corollary 2 *For all integers r, s and $n \geq 0$, we have*

$$\begin{aligned}
\Delta S_n^{(r,s)}(U) &= a_{n+1}V_{2n+r+s} - (n+1)\,q^r V_{s-r}, && (1)\\
S_n^{(r,s)}(V) &= a_{n+1}V_{2n+r+s} + (n+1)\,q^r V_{s-r}, && (2)\\
S_n^{(r,s)}(U,V) &= a_{n+1}U_{2n+r+s} - (n+1)\,q^r U_{s-r}, && (3)\\
\Delta A_n^{(r,s)}(U) &= \begin{cases} d_m V_{4m+r+s} - q^r V_{s-r} & \text{if } n = 2m \\ -p\Delta c_m U_{4m+r+s+2} & \text{if } n = 2m+1 \end{cases}, && (4)\\
A_n^{(r,s)}(V) &= \begin{cases} d_m V_{4m+r+s} + q^r V_{s-r} & \text{if } n = 2m \\ -p\Delta c_m U_{4m+r+s+2} & \text{if } n = 2m+1 \end{cases}, && (5)\\
A_n^{(r,s)}(U,V) &= \begin{cases} d_m U_{4m+r+s} - q^r U_{s-r} & \text{if } n = 2m \\ -pc_m V_{4m+r+s+2} & \text{if } n = 2m+1 \end{cases}. && (6)
\end{aligned}$$

Corollary 3 *For all integers r, s, t and $n \geq 0$, we have*

$$S_n^{(s,s)}(U) - q^{s-r}S_n^{(r,r)}(U) = \Delta^{-1}\left(S_n^{(s,s)}(V) - q^{s-r}S_n^{(r,r)}(V)\right) = a_{n+1}U_{s-r}U_{2n+r+s+t}, \tag{7}$$

$$S_n^{(s,s+t)}(V) + \Delta q^{s-r}S_n^{(r,r+t)}(U) = a_{n+1}V_{s-r}V_{2n+r+s+t}. \tag{8}$$

2 Proof of the main result

We shall use the following Lemmas

Lemma 4 *For all integers n, m and h, we have*

1. $U_{-n} = -q^{-n}U_n$,
2. $V_{-n} = q^{-n}V_n$,
3. $\Delta U_n U_m = V_{n+m} - q^m V_{n-m}$,
4. $V_n V_m = V_{n+m} + q^m V_{n-m}$,
5. $U_n V_m = U_{n+m} + q^m U_{n-m}$,
6. $V_n U_m = U_{n+m} - q^m U_{n-m}$,
7. $U_n U_{m+h} - U_{n+h}U_m = q^m U_h U_{n-m}$,
8. $V_n V_{m+h} - V_{n+h}V_m = -q^m \Delta U_h U_{n-m}$,
9. $V_n V_{m+h} - \Delta U_{n+h}U_m = q^m V_h V_{n-m}$,
10. $U_n V_{m+h} - U_{n+h}V_m = -q^m U_h V_{n-m}$.

Lemma 5 *For all integers r and $n \geq 0$, we have*

1. $\Delta U_2 \sum_{i=0}^n U_{r+4i} = V_{4n+r+2} - V_{r-2} = \Delta U_{2n+r}U_{2n+2}$,
2. $U_2 \sum_{i=0}^n V_{r+4i} = U_{4n+r+2} - U_{r-2} = V_{2n+r}U_{2n+2}$,
3. $V_2 \sum_{i=0}^n (-1)^i U_{r+4i} = (-1)^n U_{4n+r+2} + U_{r-2} = \begin{cases} U_{2n+r}V_{2n+2} & \text{if } n \text{ is even} \\ -V_{2n+r}U_{2n+2} & \text{if } n \text{ is odd} \end{cases}$,
4. $V_2 \sum_{i=0}^n (-1)^i V_{r+4i} = (-1)^n V_{4n+r+2} + V_{r-2} = \begin{cases} V_{2n+r}V_{2n+2} & \text{if } n \text{ is even} \\ -\Delta U_{2n+r}U_{2n+2} & \text{if } n \text{ is odd} \end{cases}$.

Proofs. For Lemma 4, we use Binet's forms of U_n and $V_n : U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n = \alpha^n + \beta^n$ where α and β are the roots of $x^2 - px - q = 0$. Lemma 5 follows from relations 3. 4. 5. 6. of Lemma 4. We obtain Theorem 1 and Corollary 2 from relations 3. 4. 5. 6. of Lemma 4 and Lemma 5, and Corollary 3 from relations (1), (2) and 3. 4. of Lemma 4. □

3 Applications: extension of Čerin & Gianella results

The following Theorem is a generalization of Cerin's Theorems 1.1, 1.2 and 1.3 cited in [2]

Theorem 6 *For all integers $m \geq 0$ and k, we have*

$$-p^2q + V_{2k}^2 = \Delta U_{2k-1}U_{2k+1} \quad \text{and} \quad -qV_2^2 + V_{2k-1}^2 = \Delta U_{2k-3}U_{2k+1}, \tag{9}$$

$$\delta_n + A_n^{(2k,2k)}(V) = \begin{cases} \Delta d_m U_{2k+2m+1}U_{2k+2m-1} & \text{if } n = 2m, \\ -p\Delta c_m U_{2k+2m+3}V_{2k+2m-1} & \text{if } n = 2m+1, \end{cases} \tag{10}$$

$$\theta_n + A_n^{(2k-1,2k-1)}(V) = \begin{cases} \Delta d_m U_{2k+2m-1}^2 & \text{if } n = 2m, \\ -p\Delta c_m U_{2k+2m+1}V_{2k+2m-1} & \text{if } n = 2m+1, \end{cases} \tag{11}$$

$$\xi_n + A_n^{(2k-1,2k)}(V) = \begin{cases} \Delta d_m U_{2k+2m-1}U_{2k+2m} & \text{if } n = 2m, \\ -p\Delta c_m U_{2k+2m}V_{2k+2m+1} & \text{if } n = 2m+1. \end{cases} \tag{12}$$

Where (δ_n), (θ_n) and (ξ_n) are defined as follows: $(\delta_{2m}, \delta_{2m+1}) = \left(-qV_{2m+1}^2, -pq\Delta U_{4m+4}\right)$; $(\theta_{2m}, \theta_{2m+1}) = \left(-2q(1+d_m), -p^2q\Delta c_m\right)$; $\left(\xi_{2m}, \xi_{2m+1}\right) = \left(-pq(1+d_m), -p\Delta c_m\right)$.

The relations $\delta_m = \delta_{m-2} - p^2q\Delta V_{2m}$ and $\theta_m = \theta_{m-2} - p^2q\Delta d_{(m-1)/2}$ for m odd, and $\theta_m = -\theta_{m-2} - 2qV_{m/2}^2$ for m even, are easily established. Then, one verifies that we obtain Theorems of [2] when $(p,q) = (1,-1)$.

Proof. For (9), we use relation 9. of Lemma 4 with $(n,m,h) = (2k, 2k-1, 1)$ and $(n,m,h) = (2k-1, 2k-3, 2)$ respectively. For relations (10), (11) and (12), we use relation (5) for $(r,s) = (2k,2k)$ resp. $(r,s) = (2k-1, 2k-1)$ and $(r,s) = (2k-1, 2k)$ and noticing that, using relations 3. 4. 5. 6. of Lemma 4, we have $\begin{cases} U_{4k+4m+2} = U_{2k+2m+3}V_{2k+2m-1} - qU_4, \\ V_{4k+4m} = \Delta U_{2k+2m+1}V_{2k+2m-1} + qV_2 \text{ and } V_{4m+2} + 2q = V_{2m+1}^2, \end{cases}$ resp.

$$\begin{cases} U_{4k+4m} = U_{2k+2m+1}V_{2k+2m-1} - qU_2, \\ V_{4k+4m-2} = \Delta U_{2k+2m-1}^2 + 2q, \end{cases}, \quad \begin{cases} U_{4k+4m+1} = U_{2k+2m}V_{2k+2m+1} + 1, \\ V_{4k+4m-1} = \Delta U_{2k+2m-1}U_{2k+2m} + pq. \end{cases} \qquad \square$$

Theorem 7 *For all integers $n \geq 0$ and r, s, t and k, the following equalities hold*

$$S_n^{(s,s+t)}(V) = \lambda_n + a_{n+1}V_{s-r}V_{2n+r+s+t}, \quad \text{with } \lambda_n = -q^{r-s}\Delta S_n^{(r,r+t)}(U), \tag{13}$$

$$A_n^{(2k,2k)}(V) = \begin{cases} d_m V_{2k+2m}^2 - 2p^2\Delta b_{m-1} & \text{if } n = 2m \\ p^2\Delta c_m\left(1 - a_{k+m+1}V_{2k+2m}\right) & \text{if } n = 2m+1 \end{cases}, \tag{14}$$

$$A_n^{(2k+1,2k+1)}(V) = \begin{cases} -\Delta U_{2m+1}^2 + d_m V_{2k+2m}V_{2k+2m+2} & \text{if } n = 2m \\ -p^2\Delta c_m a_{k+m+1}V_{2k+2m+2} & \text{if } n = 2m+1 \end{cases}, \tag{15}$$

$$A_n^{(2k,2k+1)}(V) = \begin{cases} d_m V_{2k+2m}V_{2k+2m+1} - e_m & \text{if } n = 2m \\ -p\Delta c_m\left(U_{2k+2m+3}V_{2k+2m} - p^2 + q\right) & \text{if } n = 2m+1 \end{cases}. \tag{16}$$

Proof. For (13) use (8). For (14), (15) and (16), we use (5) when $r = s = 2k$ resp. $r = s = 2k+1$ and $(r,s) = (2k, 2k+1)$ using, for $t = 0$ resp. $t = 2$ and $t = 1$, relations $V_{4k+4m+t} = V_{2k+2m+t}V_{2k+2m} - V_t$ and $U_{4k+4m+t+2} = U_{2k+2m+2-r(r-2)}V_{2k+2m+r(r-1)} - U_{(2-r)(2r+1)}$, derived from relations 4. and 5. of Lemma 4. For (15), we also use $V_2d_m - 2q = V_{4m+2} - 2q = \Delta U_{2m+1}^2$ derived from 3. of Lemma 4. $\square$

Notice that from the first relation of Theorem 1, $\lambda_n = -p^{-1}q^{s-r}\left(U_{4n+2r+t+2} - U_{2r+t-2}\right) + (n+1)q^sV_t$, we have also $e_m = pV_2^{-1}\left(V_{4m+2} - V_{-2}\right) = p^3\Delta\sum_{j=0}^{m} c_{j-1}$ using first relation of Lemma 5.

For $(p,q) = (2,-1)$, we obtain Theorems 1, 2, 3, 4, 5, 6 and 7 of Čerin and Gianella cited in [3]: relation (13), with $(s,t) = (2k,0)$ and $r \in \{0,2,1,-1\}$ give respectively Theorem 1 and relations (2.3), (2.4) and (2.5), with $(s,t) = (2k+1,0)$ and $r \in \{2,3\}$ give Theorems 2 and 3, and with $(s,t) = (2k,1)$ and $r = 0$ give Theorem 4. Relations (14), (15) and (16) give Theorems 5, 6 and 7.

Relations 8. 9. of Lemma 4 allow us to obtain immediately the following Theorem

Theorem 8 *For all integers $n,\ m,\ r,\ s$, we have*

$$V_nV_m = V_{n+r}V_{m-r} + q^n\Delta U_rU_{m-n-r} = \Delta U_{n+s}U_{m-s} + q^{m-s}V_sV_{n-m+s}.$$

For $(p,q)=(2,-1)\,,\ n=2k,\ m=2k+1,\ r=3$ and $s=2$, and by setting $P_n^\star = 2P_n$ for all n, one gets $Q_{2k}Q_{2k+1} = Q_{2k+3}Q_{2k-2} - 80 = 8P_{2k+2}P_{2k-1} - 12 = 2\left(P_{2k+2}^\star P_{2k-1}^\star - 6\right)$ which is Theorem 8 of [3], where Čerin and Gianella called $\left(P_n^\star\right)_n$ the Pell sequence instead of $\left(P_n\right)_n$.

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