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A NOTE ON THE IRRATIONALITY OF CERTAIN LUCAS INFINITE SERIES

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1. Introduction

Recently, C. Badea [1] showed that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2^n}}$$

is irrational, where L_n is the usual Lucas number. We shall extend here his result to other series, with a direct proof, and we shall also give a deeper result, namely,

$$\sum_{n=0}^{\infty} \frac{\varepsilon^n}{L_{2^n}} \notin \mathbb{Q}(\sqrt{5}), \text{ with } \varepsilon = \pm 1.$$

Consider the sequence of integers $\{w_n\}$ defined by the recurrence relation

$$(1.1) \qquad w_n = pw_{n-1} - qw_{n-2},$$

where $p \geq 1$, $q \neq 0$ are integers with $d = p^2 - 4q > 0$. Roots of the characteristic polynomial of (1.1) are

$$\alpha = \frac{p + \sqrt{d}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{d}}{2},$$

where $\alpha + \beta = p$, $\alpha\beta = q$, and $\alpha - \beta = \sqrt{d} > 0$. Note that $\alpha > |\beta|$ and $\alpha > 1$ since $\alpha^2 > \alpha|\beta| = |q| \geq 1$.

Special cases of $\{w_n\}$ which interest us here are the generalized Fibonacci $\{U_n\}$ and Lucas $\{V_n\}$ sequences defined by

$$(1.2) \qquad U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

It is easily proved that $\{U_n\}$ and $\{V_n\}$ are increasing sequences of natural numbers (for $n \geq 1$) and that

$$U_n \sim \frac{\alpha^n}{\alpha - \beta}, \; V_n \sim \alpha^n, \; U_n \leq V_n$$

for all positive integers n.

We also have

$$(1.3) \qquad U_{2n} = U_n V_n,$$

$$(1.4) \qquad \alpha U_n - U_{n+1} = -\beta^n.$$

The purpose of this paper is to establish the following result.

Theorem: We assume that the above conditions are realized and that ε is fixed ($\varepsilon = \pm 1$). We then have:

1) $\theta = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{V_{2^n}}$ is an irrational number;

2) If $\sqrt{d}$ is irrational and $|\beta| < 1$, then $1, \alpha, \theta$ are linearly independent over $\mathbb{Q}$ [or, in other words: $\theta \notin \mathbb{Q}(\sqrt{d})$].

Remark: When $q = \pm 1$, it is quite simple to prove that $|\beta| < 1$ and $\sqrt{d}$ is irrational. More generally, $|\beta| < 1$ if and only if $p + q > -1$ and $p - q > 1$ [since in that case $P(1) < 0$, $P(-1) > 0$, where P is the characteristic polynomial].

2. Preliminary Lemmas

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of integers defined by

$$S_n = \sum_{k=0}^{n} \frac{\varepsilon^k}{V_{2^k}} = \frac{p_n}{q_n}, \text{ with } q_n = \prod_{k=0}^{n} V_{2^k}.$$

By (1.3), we have

(2.1) $\quad q_n = U_{2^{n+1}}$.

We need the following lemmas.

Lemma 1: $\left|\theta - \frac{p_n}{q_n}\right| = \varepsilon^{n+1}\left(\theta - \frac{p_n}{q_n}\right)$.

Proof: The result is obvious when $\varepsilon = 1$. In the other case, since V_n is increasing, we have:

$$\frac{p_{2n}}{q_{2n}} > \theta, \quad \frac{p_{2n+1}}{q_{2n+1}} < \theta.$$

Lemma 2: $p_n q_{n-1} - p_{n-1} q_n = \varepsilon^n U_{2^n}^2$.

Proof: $\frac{\varepsilon^n}{V_{2^n}} = S_n - S_{n-1} = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n q_{n-1}}$. Hence, by (2.1) and (1.3),

$$p_n q_{n-1} - p_{n-1} q_n = \frac{\varepsilon^n}{V_{2^n}} q_n q_{n-1} = \frac{\varepsilon^n}{V_{2^n}} U_{2^{n+1}} U_{2^n} = \varepsilon^n U_{2^n}^2.$$

Lemma 3: For all positive integers n and k, we have

$$\frac{U_{2^{n+1}}}{V_{2^{n+k+1}}} \le \left(\frac{1}{V_{2^{n+1}}}\right)^k.$$

Proof: Using (1.3), we can show that

$$U_{2^{n+1}} \prod_{i=1}^{k} V_{2^{n+i}} = U_{2^{n+k+1}} \le V_{2^{n+k+1}}$$

and so

$$\frac{U_{2^{n+1}}}{V_{2^{n+k+1}}} \le \frac{1}{\prod_{i=1}^{k} V_{2^{n+i}}} \le \left(\frac{1}{V_{2^{n+1}}}\right)^k,$$

since V_n is increasing.

Lemma 4: $\lim_{n\to\infty} |q_n\theta - p_n| = \frac{1}{\alpha - \beta}$, where $\{p_n\}$ and $\{q_n\}$ are defined as above.

Proof: $\left|\theta - \frac{p_n}{q_n}\right| = \varepsilon^{n+1}\left(\theta - \frac{p_n}{q_n}\right) = \varepsilon^{n+1}(\theta - S_n)$

$$= \varepsilon^{n+1} \sum_{k=0}^{\infty} \frac{\varepsilon^{n+k+1}}{V_{2^{n+k+1}}} = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{V_{2^{n+k+1}}}.$$

Hence,
$$|q_n\theta - p_n| = \sum_{k=0}^{\infty} \frac{\varepsilon^k q_n}{V_{2^{n+k+1}}} = \sum_{k=0}^{\infty} \frac{\varepsilon^k U_{2^{n+1}}}{V_{2^{n+k+1}}} = \frac{U_{2^{n+1}}}{V_{2^{n+1}}} + R_n,$$
with $R_n = \sum_{k=1}^{\infty} \frac{\varepsilon^k U_{2^{n+1}}}{V_{2^{n+k+1}}}$.

However, by Lemma 3, we have
$$|R_n| \le \sum_{k=1}^{\infty} \frac{U_{2^{n+1}}}{V_{2^{n+k+1}}} \le \sum_{k=1}^{\infty} \left(\frac{1}{V_{2^{n+1}}}\right)^k = \frac{1}{V_{2^{n+1}} - 1},$$
so that $\lim_{n\to\infty} R_n = 0$ and
$$\lim_{n\to\infty}|q_n\theta - p_n| = \lim_{n\to\infty} \frac{U_{2^{n+1}}}{V_{2^{n+1}}} = \frac{1}{\alpha - \beta}.$$

3. Proof of the First Part of the Theorem

Recall that a convergent sequence of integers is stationary, and suppose that $\theta = a/b$ (a and b integers, $b > 0$). By Lemma 4, the sequence of positive integers $|q_n a - p_n b|$ tends to the limit $c = b/(\alpha - \beta)$. When $(\alpha - \beta)$ is irrational, this is clearly impossible. In the other case we have, for all large n, since the sequence is stationary,
$$\left|q_n \frac{a}{b} - p_n\right| = \varepsilon^{n+1}\left(q_n \frac{a}{b} - p_n\right) = \frac{1}{\alpha - \beta},$$
and so, for all large n,
$$q_n \frac{a}{b} - p_n = \frac{\varepsilon^{n+1}}{\alpha - \beta}. \tag{3.1}$$
Using (3.1) for n and $n - 1$, we have
$$p_n q_{n-1} - p_{n-1}q_n = \frac{\varepsilon^n}{\alpha - \beta}(q_n - \varepsilon q_{n-1}).$$
By (2.1), (1.3), and Lemma 2, we obtain
$$U_{2^n}^2 = \frac{1}{\alpha - \beta}(U_{2^{n+1}} - \varepsilon U_{2^n}) = \frac{U_{2^n}}{\alpha - \beta}(V_{2^n} - \varepsilon),$$
and so
$$U_{2^n} = \frac{1}{\alpha - \beta}(V_{2^n} - \varepsilon).$$
It follows from this and (1.2) that
$$\alpha^{2^n} - \beta^{2^n} = \alpha^{2^n} + \beta^{2^n} - \varepsilon \quad \text{or} \quad \beta^{2^n} = \varepsilon/2,$$
for all large n. This is clearly impossible, since
$$\lim_{n\to+\infty} |\beta|^{2^n} \in \{0, 1, +\infty\}.$$
This concludes the proof.

Examples:

a) $\sum_{n=0}^{\infty} \frac{\varepsilon^n}{L_{2^n}}$ is irrational (the case $\varepsilon = 1$ is Badea's).

b) $\sum_{n=0}^{\infty} \frac{\varepsilon^n}{2^{2^n} + 1}$ is irrational (the case $\varepsilon = 1$ was discovered by Golomb [2]).

4. Proof of the Second Part of the Theorem

Suppose that we can find a relation

(4.1) $k_0 + k_1\alpha + k_2\theta = 0,\ k_i \in \mathbb{Q}$.

We can limit ourselves to the case of $k_i \in Z$. Replacing n by 2^{n+1} in (1.4) and putting $x_n = U_{2^{n+1}+1}$, we have

(4.2) $\lim_{n\to\infty}(\alpha q_n - x_n) = 0,$

since $|\beta| < 1$.

By (4.1), it follows that

$$k_0 q_n + k_1(q_n\alpha - x_n) + k_2(q_n\theta - p_n) + k_1 x_n + k_2 p_n = 0$$

or, for all positive integers n,

$$k_1(q_n\alpha - x_n) + k_2(q_n\theta - p_n) \in Z.$$

Hence, by Lemma 1,

$$k_1\varepsilon^{n+1}(q_n\alpha - x_n) + k_2|q_n\theta - p_n| \in Z.$$

Using Lemma 4 and (4.2), it follows that

$$\lim_{n\to\infty}(k_1\varepsilon^{n+1}(q_n\alpha - x_n) + k_2|q_n\theta - p_n|) = \frac{k_2}{\alpha - \beta} \in Z.$$

Thus, we have $k_2 = 0$ (since $\alpha - \beta$ is irrational) and, by (4.1),

$$k_1 = k_0 = 0,$$

since $\alpha = (p + \sqrt{d})/2$ is irrational. This concludes the proof.

Example: $\sum_{n=0}^{\infty} \frac{\varepsilon^n}{L_{2^n}} \notin \mathbb{Q}(\sqrt{5})$.

Corollary: Let r be a positive integer. With the hypotheses of the theorem, we have:

1) $\theta_r = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{V_{r \cdot 2^n}}$ is an irrational number;

2) If $\sqrt{d}$ is irrational and $|\beta| < 1$, then $1, \alpha, \theta_r$ are linearly independent over $\mathbb{Q}$.

Define the sequence $\{V_n'\}$ by

$$V_n' = V_{rn} = (\alpha^r)^n + (\beta^r)^n.$$

$\{V_n'\}$ is the Lucas generalized sequence, with real roots α^r and β^r, which is associated with the recurrence

$$W_n' = (\alpha^r + \beta^r)W_{n-1}' - \alpha^r\beta^r W_{n-2}' = V_r W_{n-1}' - q^r W_{n-2}'.$$

We can apply the result of the Theorem to the sequence $\{V_{2^n}'\}$. In fact, we have

$$V_r \ge V_1 = p \ge 1, \quad |\beta|^r < 1 \quad (\text{since } |\beta| < 1)$$

and the discriminant d' of the recurrence is

$$d' = V_r^2 - 4q^r = (\alpha^r - \beta^r)^2 = (\alpha - \beta)^2 U_r^2.$$

From this, we have

$$\sqrt{d'} = (\alpha - \beta)U_r = \sqrt{d}U_r.$$

Thus, $\sqrt{d'}$ is an irrational number because $\sqrt{d}$ is.

References

1. C. Badea. "The Irrationality of Certain Infinite Series." *Glasgow Math. J.* *29* (1987):221-28.
2. S. W. Golomb. "On the Sum of the Reciprocals of the Fermat Numbers and Related irrationalities." *Can. J. Math.* *15* (1963):475-78.

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Announcement

FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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The FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of St. Andrews, St. Andrews, Scotland from July 20 to July 24, 1992. This Conference is sponsored jointly by the Fibonacci Association and The University of St. Andrews.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1992. Manuscripts are due by May 30, 1992. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

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