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Cumulants, lattice paths, and orthogonal polynomials

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Abstract

A formula expressing free cumulants in terms of Jacobi parameters of the corresponding orthogonal polynomials is derived. It combines Flajolet's theory of continued fractions and the Lagrange inversion formula. For the converse we discuss Gessel–Viennot theory to express Hankel determinants in terms of various cumulants.

Keywords: Cumulants; Lattice path combinatorics; Motzkin paths; Łukasiewicz paths; Continued fractions; Hankel determinants; Orthogonal polynomials

1. Introduction

In his seminal paper [9], Flajolet gave a combinatorial interpretation of continued fractions and derived generating function expressions for enumeration problems of various lattice paths. This is connected to work of Karlin and McGregor and others on birth and death processes, see [10] or [14] for a survey. The basic principle has been rediscovered many times, see e.g. [11] for an application to random walks and [1] for an interpretation in noncommutative probability. Since Stieltjes' times continued fractions have also been a basic ingredient of the theory of orthogonal polynomials. A synthesis of both aspects can be found in Viennot's memoir [27] or in the survey [28].

Orthogonal polynomials and the associated Jacobi operators have been extensively studied in connection with moment problems and spectral theory [2]. Spectral theory of convolution operators was also one of Voiculescu's motivations for the development of free probability theory [32] and it would be interesting to understand the behaviour

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of orthogonal polynomials under free convolution. The special case of free projections, which is up to a translation equivalent to the case of free generators, has been considered in the literature, see e.g. [22,6,26,3].

In this note we derive a formula for free cumulants in the spirit of Flajolet's and Viennot's theory of lattice paths. After discussions with A. Lascoux, he gave a proof in terms of symmetric functions of a more general formula for the coefficients of powers of continued fractions, cf. [16,35].

The paper is organized as follows.

In Section 2 we review Flajolet's formula for the generating function of Motzkin paths.

In Section 3 we briefly discuss the relevant facts about orthogonal polynomials.

In Section 4 we review the definitions of various cumulants.

In Section 5 we prove the main result, a formula expressing free cumulants in terms of Jacobi parameters.

Finally, in Section 6 we indicate how Gessel–Viennot theory of Hankel determinants can be used to express the Jacobi parameters in terms of cumulants.

2. Enumeration of lattice paths

Definition 2.1. A *lattice path* is a sequence of points in the integer lattice $\mathbb{Z}^2$. A pair of consecutive points is called a *step* of the path. A *valuation* is a function on the set of possible steps $\mathbb{Z}^2 \times \mathbb{Z}^2$. A valuation of a path is the product of the valuations of its steps. In the rest of this paper all lattice paths will have the following properties:

(1) Starting point and end point lie on the x-axis.
(2) The y-coordinates of all points are nonnegative.
(3) In each step, the x-coordinate is incremented by one.

Thus, a path of length n will start at some point $(x_0, 0)$ and end at $(x_0+n, 0)$. The valuations will be independent of the x-coordinates of the points. Therefore the x-coordinates are redundant and we can represent a path π by the sequence of its y-coordinates $(\pi(0), \pi(1), \dots, \pi(n))$. We call a path *irreducible* if it does not touch the x-axis except at the start and at the end. Every path has a unique factorization into irreducible ones.

We will be concerned with two types of paths.

Definition 2.2.

(i) A *Motzkin path* of length n is a lattice path starting at $P_0 = (0, 0)$ and ending at $(n, 0)$, all of whose y coordinates are nonnegative and whose steps are of the three following types.
  rising step: $(1, 1)$.
  horizontal step: $(1, 0)$.
  falling step: $(1, -1)$.

Motzkin paths without horizontal steps are called *Dyck paths*. We will denote the set of Motzkin paths by $\mathscr{M}_n$ and the subset of irreducible Motzkin paths by $\mathscr{M}_n^{\mathrm{irr}}$. Motzkin paths are counted by the well-known *Motzkin numbers*.

(ii) A *Łukasiewicz path* of length n is a path starting at $(0,0)$ and ending at $(n,0)$ whose steps are of the following types:

rising step: $(1,1)$.

horizontal step: $(1,0)$.

falling steps: $(1,-k)$, $k>0$.

We denote the set of Łukasiewicz paths of length n by $\mathscr{L}_n$ and the subset of irreducible Łukasiewicz paths by $\mathscr{L}_n^{\mathrm{irr}}$. Łukasiewicz paths form a *Catalan family*, i.e., they are counted by the *Catalan numbers*.

The name Łukasiewicz path is motivated by the natural bijection to the *Łukasiewicz language*, the language understood by calculators using reverse polish notation, which was used by Raney in [23] to give a combinatorial proof of the Lagrange inversion formula.

Flajolet's formula expresses the generating function of weighted Motzkin paths as a continued fraction.

Theorem 2.3 (Flajolet [9]). *Let*

$$\mu_n = \sum_{\pi \in \mathscr{M}_n} v(\pi), \tag{2.1}$$

where the sum is over the set of Motzkin paths $\pi = (\pi(0)\cdots\pi(n))$ *of length* n. *Here* $\pi(j)$ *is the level after the* j*th step*, *and the valuation of a path is the product of the valuations of its steps* $v(\pi)=\prod_1^n v_i$, *the latter being*

$$v_i = v(\pi(i-1),\pi(i)) = \begin{cases} 1 & \text{if the } i\text{th step rises}, \\ a_{\pi(i-1)} & \text{if the } i\text{th step is horizontal}, \\ \lambda_{\pi(i-1)} & \text{if the } i\text{th step falls}. \end{cases} \tag{2.2}$$

Then, *the generating function*

$$M(z) = \sum_{n=0}^{\infty} \mu_n z^n$$

has the continued fraction expansion

$$M(z) = \cfrac{1}{1-\alpha_0 z - \cfrac{\lambda_1 z^2}{1-\alpha_1 z - \cfrac{\lambda_2 z^2}{\ddots}}}. \tag{2.3}$$

3. Orthogonal polynomials

A sequence of (formal) orthogonal polynomials is a sequence of monic polynomials $P_n(x)$ of degree $\deg P_n = n$ together with a linear functional μ on the space of polynomials with moments $\mu(x^n) = \mu_n$ such that $\mu(P_m(x)P_n(x)) = \delta_{mn}s_n$ for some coefficients s_n. We will always assume that $s_0 = 1$, that is $\mu(x^0) = 1$. It is then easy to see that such a sequence satisfies a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + a_n P_n(x) + \lambda_n P_{n-1}(x)$$

with $\lambda_0 = 0$. The numbers a_n, λ_n are the so-called *Jacobi parameters*. One verifies easily that the polynomials $P_n(x) = D_n(x)/\Delta_{n-1}$ satisfy the orthogonality condition, where

$$D_n(x) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & & \mu_{n+1} \\ \vdots & & & \vdots \\ \mu_{n-1} & & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix} \tag{3.1}$$

and

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & & \mu_{n+1} \\ \vdots & & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \tag{3.2}$$

is the nth *Hankel determinant*.

Coming from an operator theoretic background, the most natural way to make the connection to Flajolet's formula is perhaps via matrices. The Jacobi matrix model for the moment functional is

$$J = \begin{bmatrix} \alpha_0 & 1 & & & \\ \lambda_1 & \alpha_1 & 1 & & \\ & \lambda_2 & \alpha_2 & 1 & \\ & \ddots & \ddots & \ddots & \end{bmatrix},$$

that is, if we denote the basis of the vector space by $e_0, e_1, \ldots$, with inner product $\langle e_m, e_n \rangle = \delta_{mn}s_n$, then $Je_n = e_{n+1} + \alpha_n e_n + \lambda_n e_{n-1}$. Then it is easy to see that $P_n(J)\, e_0 = e_n$

and therefore J satisfies $\langle J^n e_0, e_0\rangle = \mu_n$. Expanding the matrix power yields

$$\mu_n = \sum_{i_1,\dots,i_{n-1}\geqslant 0} J_{0,i_1} J_{i_1,i_2} \dots J_{i_{n-1},0},$$

and because of tridiagonality, the sum is restricted to indices $|i_{j+1} - i_j| \leqslant 1$. The summands $J_{0,i_1} J_{i_1,i_2} \cdots J_{i_{n-1},0}$ can be interpreted as the valuations of the Motzkin paths $(0, i_1, i_2, \dots, i_{n-1}, 0)$ with weights $v(y, y') = J_{y+1,y'+1}$, where $J_{i,i-1} = \lambda_i$, $J_{i,i} = \alpha_i$, $J_{i,i+1} = 1$ and this is exactly Flajolet's formula (2.1).

4. Cumulants

Cumulants linearize convolution of probability measures coming from various notions of independence.

Definition 4.1 (Voiculescu et al. [32]). A *noncommutative probability space* is pair $(\mathscr{A}, \varphi)$ of a (complex) unital algebra $\mathscr{A}$ and a unital linear functional φ. The elements of $\mathscr{A}$ are called (*noncommutative*) random variables. The collection of moments $\mu_n(a) = \varphi(a^n)$ of such a random variable $a \in \mathscr{A}$ will be called its *distribution* and denoted $\mu_a = (\mu_n(a))_n$.

Thus noncommutative probability follows the general "quantum" philosophy of replacing function algebras by noncommutative algebras. We will review several notions of independence below. Convolution is defined as follows. Let a and b be "independent" random variables. Then the convolution of the distributions of a and b is defined to be the distribution of the sum $a+b$. In all the examples below, the distribution of the sum of "independent" random variables only depends on the individual distributions of the summands and therefore convolution is well defined and the nth moment $\mu_n(a+b)$ is a polynomial function of the moments of a and b of order less or equal to n. For our purposes it is sufficient to axiomatize cumulants as follows.

Definition 4.2. Given a notion of independence on a noncommutative probability space $(\mathscr{A}, \varphi)$, a sequence of maps $a \mapsto k_n(a)$, $n = 1, 2, \dots$, is called a *cumulant sequence* if it satisfies

(1) additivity: if a and b are "independent" random variables, then $k_n(a+b) = k_n(a) + k_n(b)$.
(2) homogenity: $k_n(\lambda a) = \lambda^n k_n(a)$.
(3) $k_n(a)$ is a polynomial in the first n moments of a with leading term $\mu_n(a)$. This ensures that conversely the moments can be recovered from the cumulants.

We will review here *free*, *classical* and *Boolean* cumulants via their matrix models. The reader interested in "q-analogues" is referred to the q-Toeplitz matrix models of Nica [19,20] which yield similar valuations on Łukasiewicz paths.

4.1. Free cumulants

Free probability was introduced by Voiculescu in [29] and has seen rapid development since, see [32] and the more recent survey [31].

Definition 4.3. Given a noncommutative probability space $(\mathscr{A}, \varphi)$, the subalgebras $\mathscr{A}_i \subseteq \mathscr{A}$ are called *free independent* (or *free* for short) if

$$\varphi(a_1 a_2 \cdots a_n) = 0 \tag{4.1}$$

whenever $a_j \in \mathscr{A}_{i_j}$ with $\varphi(a_j) = 0$ and $i_j \neq i_{j+1}$ for $j = 1, \ldots, n-1$. Elements $a_i \in \mathscr{A}$ are called free if the unital subalgebras generated by a_i are free.

Existence of free cumulants with the properties of Definition 4.2 was proved already in [29]. A beautiful systematic theory of free cumulants was found by Speicher in his combinatorial approach to free probability via noncrossing partitions [24]; see [17] for an explanation why noncrossing partitions appear. The explicit computation involves generating functions as follows.

Theorem 4.4 (Voiculescu [30]). *Let $M(z) = 1 + \sum_{n=1}^{\infty} \mu_n z^n$ be the ordinary moment generating function and $C(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be the function implicitly defined by the relation $C(zM(z)) = M(z)$. Then the coefficients c_n satisfy the requirements of Definition* 4.2 *and are called the free or noncrossing cumulants and $C(z)$ is the cumulant generating function.*

The moments and the free cumulants are related explicitly by the following combinatorial formula (see [23,8] for the formulation in terms of Łukasiewicz language, and [24] for noncrossing partitions). Define a valuation on Łukasiewicz paths by putting the following weights on the steps:

$$\begin{aligned} v(y, y-k) &= c_{k+1}, \qquad k \geqslant 0, \\ v(y, y+1) &= 1, \end{aligned} \tag{4.2}$$

then

$$\mu_n = \sum_{\pi \in \mathscr{L}_n} v(\pi). \tag{4.3}$$

Another interpretation of this relation is the *Fock space model* of Voiculescu [30] (see [13] for a simpler proof). It involves Toeplitz operators as follows. Let S be the forward shift on $\ell_2(\mathbb{N}_0)$, i.e., $Se_n = e_{n+1}$. A (formal) *Toeplitz operator* is a linear combination of powers of S and of its adjoint, the backward shift $S^* e_n = e_{n-1}$. The linear functional $\omega(X) = \langle X e_0, e_0 \rangle$ is called the *vacuum expectation*. Let c_n be the free cumulants of the moment sequence (μ_n) as defined above and set

$$T = S^* + \sum_{n=0}^{\infty} c_{n+1} S^n. \tag{4.4}$$

Then, $\omega(T^n) = \mu_n$ and writing T in matrix form

$$T = \begin{bmatrix} c_1 & 1 & & & \\ c_2 & c_1 & 1 & & \\ c_3 & c_2 & c_1 & 1 & \\ \vdots & & \ddots & & \ddots \end{bmatrix},$$

we can expand the matrix product and obtain

$$\omega(T^n) = \langle T^n e_0, e_0 \rangle = \sum_{i_1,\dots,i_{n-1} \geqslant 0} T_{0,i_1} T_{i_1,i_2} \dots T_{i_{n-1},0},$$

where $T_{ij} \neq 0$ only for $j \geqslant i-1$. Again this can be interpreted as a sum over lattice paths which this time turn out to be Łukasiewicz paths with weights (4.2) coming from the matrix entries

$$T_{i,i+1} = 1, \quad T_{i,i-k} = c_{k+1}, \quad k \geqslant 0$$

and we obtain sum (4.3). There are multivariate generalizations of this Toeplitz model (see [21]).

4.2. Classical cumulants

Classical cumulants linearize convolution of measures and can be defined via the Fourier transform.

Definition 4.5. *Let* $F(z) = \mu(\mathrm{e}^{xz}) = \sum_{n=0}^{\infty} (\mu_n/n!) z^n$ *be the exponential moment generating function. Let* $K(z) = \log F(z) = \sum_{n=1}^{\infty} (\kappa_n/n!) z^n$ *be its formal logarithm. The coefficients* κ_n *are the classical cumulants of the functional* μ.

There is a model on bosonic Fock space for classical cumulants which is analogous to the Toeplitz model (4.4).

Let D and x be annihilation and creation operators which satisfy the *canonical commutation relations* (*CCR*): $[D,x] = 1$, i.e., on the Hilbert space with basis e_n and inner product $\langle e_m, e_n \rangle = n! \delta_{m,n}$, set $De_n = ne_{n-1}$, $xe_n = e_{n+1}$. Then D and x are adjoints of each other.

Denote again by ω the vacuum expectation $\omega(T) = \langle Te_0, e_0 \rangle$, i.e. $\omega(x^k D^n) = 0 \ \forall n \neq 0$ and $\omega(f(x)) = f(0)$. Then the Fourier–Laplace transform of

$$T = D + \sum_{n=0}^{\infty} \frac{\kappa_{n+1}}{n!} x^n$$

is

$$\omega(\mathrm{e}^{zT}) = \mathrm{e}^{\sum_{n=1}^{\infty} (\kappa_n/n!) z^n}.$$

In the basis $\{e_n\}$, T has matrix representation:

$$\hat{T} = \begin{bmatrix} \kappa_1 & 1 & & & & \\ \kappa_2 & \kappa_1 & 2 & & & \\ \frac{\kappa_3}{2!} & \kappa_2 & \kappa_1 & 3 & & \\ \frac{\kappa_4}{3!} & \frac{\kappa_3}{2!} & \kappa_2 & \kappa_1 & 4 & \\ \vdots & & & \ddots & & \ddots \end{bmatrix},$$

e.g. for the standard Gaussian distribution, we have $\kappa_2 = 1$ and higher cumulants vanish, so the model is tridiagonal:

$$\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 2 & & \\ & 1 & 0 & 3 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

and this is the well-known Jacobi operator matrix for the Hermite polynomials. Similar to the formula above we get the following combinatorial sum for the moments of T:

$$\mu_n = \sum_{\pi \in \mathscr{L}_n} v(\pi)$$

with valuation

$$v(y, y-k) = \frac{\kappa_{k+1}}{k!}, \quad k \geqslant 0,$$
$$v(y, y+1) = y + 1 \tag{4.5}$$

4.3. Boolean cumulants

Boolean cumulants linearize *Boolean convolution* which was introduced in [25] (compare also [33,34]) and is a special case of *conditional independence* ([5,4]). In the theory of random walks Boolean cumulants arise as *first return probabilities*.

Definition 4.6. Given a noncommutative probability space $(\mathscr{A}, \varphi)$, the subalgebras $\mathscr{A}_i \subseteq \mathscr{A}$ are called *boolean independent* if

$$\varphi(a_1 a_2 \dots a_n) = \varphi(a_1)\ \varphi(a_2) \cdots \varphi(a_n) \tag{4.6}$$

whenever $a_j \in \mathscr{A}_{i_j}$ with $i_j \neq i_{j+1}$ for $j = 1, \dots, n-1$.

With this notion of independence, convolution of measures is well defined and the appropriate cumulants can be calculated as follows [25]. Let $M(z) = \sum_{n=0}^{\infty} \mu_n z^n$ be the ordinary moment generating function. Then,

$$M(z) = \frac{1}{1 - H(z)}, \tag{4.7}$$

where

$$H(z) = \sum_{n=1}^{\infty} h_n z^n, \tag{4.8}$$

where h_n are the *Boolean cumulants* of the distribution with moment sequence (μ_n). Expanding (4.7) in a geometric series we find that the moments can be expressed as Cauchy convolution

$$\mu_n = \sum_{r=1}^{n} \sum_{\substack{i_1+\cdots+i_r=n \\ i_j \geqslant 1}} h_{i_1} h_{i_2} \cdots h_{i_r},$$

and from Theorem 2.3 and the fact that every Motzkin path has a unique factorization into irreducible paths, it follows that

$$h_n = \sum_{\pi \in \mathscr{M}_n^{\mathrm{irr}}} v(\pi), \tag{4.9}$$

where v is valuation (2.2). This can also be seen by comparing the continued fraction expansion (2.3) with (4.7). In terms of the Jacobi parameters the Boolean cumulant generating function has the continued fraction expansion

$$H(z) = \alpha_0 z + \cfrac{\lambda_1 z^2}{1 - \alpha_1 z - \cfrac{\lambda_2 z^2}{1 - \alpha_2 z - \cfrac{\lambda_3 z^2}{\ddots}}}.$$

Similarly, we can express the boolean cumulants in terms of free cumulants (resp. classical cumulants):

$$h_n = \sum_{\pi \in \mathscr{L}_n^{\mathrm{irr}}} v(\pi), \tag{4.10}$$

using valuations (4.2) (resp. (4.5)).

5. A formula for free cumulants

Theorem 5.1. *For $n \geqslant 2$, we have the following formula for the free cumulant c_n in terms of the Jacobi parameters*:

$$c_n = \sum_{\pi \in \mathscr{M}_n} \frac{(-1)^{|\pi|_0 - 1}}{n-1} \binom{n-1}{|\pi|_0} v(\pi). \tag{5.1}$$

Here $|\pi|_0$ is the number of returns to zero of the path and v is valuation (2.2) from Flajolet's formula. Note that the path consisting of n horizontal steps does not contribute.

Proof. The proof consists in comparing the expressions for the moments and free cumulants in terms of the Boolean cumulants (4.8), which themselves represent the sum over irreducible Motzkin paths (4.9). We have then by the multinomial formula

$$
\begin{aligned}
\mu_n &= [z^n]M(z) \\
&= [z^n]\sum_{m=0}^{n} H(z)^m \\
&= [z^n]\sum_{m=0}^{n}\sum_{k_1+k_2+\cdots+k_n=m}\binom{m}{k_1,\ldots,k_n} h_1^{k_1}\cdots h_n^{k_n} z^{k_1+2k_2+\cdots+nk_n} \\
&= \sum_{k_1+2k_2+\cdots+nk_n=n}\binom{k_1+\cdots+k_n}{k_1,\cdots,k_n} h_1^{k_1}\cdots h_n^{k_n}.
\end{aligned}
$$

The cumulants can be expressed in terms of the moments with the help of Lagrange's inversion formula [7]

$$
\begin{aligned}
c_n &= -\frac{1}{n-1}[z^n]\frac{1}{M(z)^{n-1}} \\
&= -\frac{1}{n-1}[z^n](1-H(z))^{n-1} \\
&= -\frac{1}{n-1}[z^n](1-h_1 z-\cdots-h_n z^n)^{n-1} \\
&= -\frac{1}{n-1}[z^n]\sum_{k_0+k_1+\cdots+k_n=n-1}\frac{(n-1)!}{k_0!\cdots k_n!}(-h_1)^{k_1}\cdots(-h_n)^{k_n} z^{k_1+2k_2+\cdots+nk_n} \\
&= -\frac{1}{n-1}\sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_1<n}}(-1)^{k_1+\cdots+k_n}\binom{n-1}{n-1-k_1-\cdots-k_n,k_1,\ldots,k_n} \\
&\quad \times h_1^{k_1}\cdots h_n^{k_n}
\end{aligned}
$$

$$= \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_1<n}} \frac{(-1)^{k_1+\cdots+k_n-1}}{n-1} \binom{n-1}{k_1+\cdots+k_n} \binom{k_1+\cdots+k_n}{k_1,\ldots,k_n} h_1^{k_1}\cdots h_n^{k_n}$$

$$= \sum_{r=1}^{n-1} \sum_{\substack{i_1+\cdots+i_r=n \\ i_j\geqslant 1}} \frac{(-1)^{r-1}}{n-1} \binom{n-1}{r} h_{i_1}h_{i_2}\cdots h_{i_r}.$$

Comparison with the formula for the moments yields the result.

Remark 5.2. Actually by the same argument we can express free cumulants in terms of classical cumulants by simply replacing Motzkin paths by Łukasiewicz paths with valuation (4.5). However, there are cancellations in the sum. There are also lots of cancellations when expressing free cumulants in terms of free cumulants themselves using Łukasiewicz paths with valuation (4.2) (only one term survives). Note, however, that there are no cancellations in sum (5.1), since $|\pi|_0$ is equal to the number of α_0's and λ_1's and therefore every monomial appears always with the same sign.

6. Hankel determinants

In this section we survey some formulas expressing the Jacobi parameters a_n and λ_n in terms of cumulants.

It is well known (see e.g. [2]) and can be readily deduced from (3.1) that the Jacobi parameters can be expressed in terms of Hankel determinants and Hankel minors of the moments as follows:

$$\lambda_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}$$

and

$$a_n = \frac{\tilde{\Delta}_n}{\Delta_n} - \frac{\tilde{\Delta}_{n-1}}{\Delta_{n-1}},$$

where

$$\tilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \mu_1 & \mu_2 & & \mu_n & \mu_{n+2} \\ \vdots & & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & \mu_{2n+1} \end{vmatrix}.$$

We are therefore interested to express the minors of the infinite Hankel matrix $\mathscr{H}_\mu = [\mu_{i+j}]_{i,j\geqslant 0}$ in terms of cumulants. Namely, given finite sequences of indices

$i_1, i_2, \dots, i_p$ and $j_1, j_2, \dots, j_p$ we will denote

$$H\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} = \begin{vmatrix} \mu_{i_1+j_1} & \mu_{i_1+j_2} & \cdots & \mu_{i_1+j_p} \\ \mu_{i_2+j_1} & \mu_{i_2+j_2} & \cdots & \mu_{i_2+j_p} \\ \cdots & & & \\ \mu_{i_p+j_1} & \mu_{i_p+j_2} & \cdots & \mu_{i_p+j_p} \end{vmatrix}. \tag{6.1}$$

For example,

$$\Delta_n = H\begin{pmatrix} 0 & 1 & \cdots & n \\ 0 & 1 & \cdots & n \end{pmatrix}, \quad \tilde{\Delta}_n = H\begin{pmatrix} 0 & 1 & \cdots & n-1 & n \\ 0 & 1 & \cdots & n-1 & n+1 \end{pmatrix}.$$

The main tool is the so-called *Gessel–Viennot theory*, see [27,12,15,18]. Let Γ be a weighted graph, that is a graph with vertices V and edges E together with a valuation $v : E \to R$ on the edges, where R is some (commutative) ring. The valuation of a path $\omega = (s_0, \dots, s_m)$ is the product of the valuations of the steps $v(\omega) = v(s_0, s_1)\, v(s_1, s_2) \cdots v(s_{m-1}, s_m)$.

Select two sets of distinct vertices A_i, $B_i \in V$, $i = 1, \dots, n$ and suppose that the set of paths

$$\Omega_{ij} = \{\omega = (s_0, \dots, s_m) \colon s_0 = A_i, s_m = B_j, v(\omega) \neq 0\}$$

is finite for every pair of indices (i, j). We will say that two paths *intersect* if they have a vertex in common. We will not care about *crossings* of paths, that is, crossings of edges in a graphical representation.

Define a matrix

$$a_{ij} = \sum_{\omega \in \Omega_{ij}} v(\omega), \qquad 1 \leqslant i, j \leqslant n.$$

Then the determinant of this matrix has the following combinatorial interpretation.

Proposition 6.1 (Viennot [27, Proposition IV.2]).

$$\det[a_{ij}]_{n \times n} = \sum_{(\sigma; \omega_1, \dots, \omega_n)} \operatorname{sign}(\sigma) v(\omega_1) \cdots v(\omega_n), \tag{6.2}$$

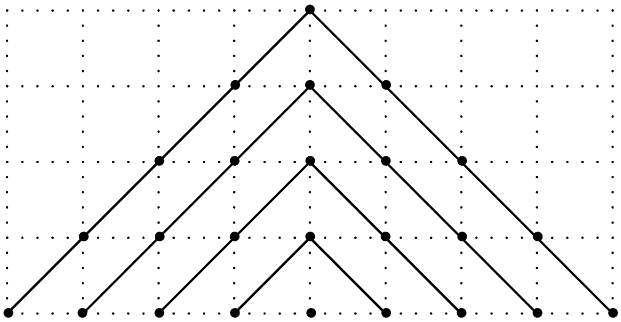
where the sum is over all permutations $\sigma \in \mathfrak{S}_n$, $\omega_i \in \Omega_{i,\sigma_i}$ and all nonintersecting (but possibly crossing) paths $\omega_i \in \Omega_{i,\sigma(i)}$.

We have seen various examples above where the nth moment is equal to the sum of the valuations of paths of length n. All our valuations are translation independent and we will interpret the (i, j) entry of $\mathscr{H}_\mu$ as a sum $\mu_{i+j} = \sum v(\omega)$ over the set of paths starting at $(-i, 0)$ and ending at $(j, 0)$. Minor (6.1) is therefore equal to

$$H\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} = \sum_{\sigma \in \mathfrak{S}_p} \sum_{\omega_1, \omega_2, \dots, \omega_p} (-1)^{\operatorname{sign}(\sigma)} v(\omega_1) v(\omega_2) \cdots v(\omega_p),$$

where the sum extends over all permutations σ of the indices and all nonintersecting configurations of paths ω_k starting at $(-i_k, 0)$ and ending at $(j_{\sigma(k)}, 0)$. Because the vertices are ordered, the sign is $(-1)^{\operatorname{sign}(\pi)} = (-1)^K$, where K is the number of crossings of the configuration corresponding to π.

For example, for the full Hankel determinant from (3.2), where the moments are interpreted as sum over Motzkin paths as in Theorem 2.3, there is only one nonintersecting configuration whose picture is (for $n = 4$)

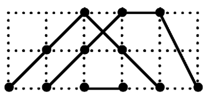
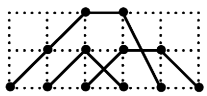


$$H\begin{pmatrix} 0,1,2,3,4 \\ 0,1,2,3,4 \end{pmatrix}$$

and therefore $\Delta_n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_{n-1}^2 \lambda_n$.

Note that in this example the point $(0,0)$ is considered as a path of length zero and is not allowed to lie on any of the other paths.

In contrast, in the case of Łukasiewicz paths with valuations (4.2) and (4.5), there may occur crossings of paths and there are many contributing terms with different signs. It is also possible that cancellations occur in the sum, e.g.

$v\Bigg($  $\Bigg) = -c_1^2 c_2^2 c_3 = -v\Bigg($  $\Bigg)$

are cancelling contributions to

$$H\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}.$$

Therefore, and because the sums are complicated, the formulae are of rather limited value.

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