

# The Lehmer matrix and its recursive analogue

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## Abstract

This paper considers the Lehmer matrix and its recursive analogue. The determinant of Lehmer matrix is derived explicitly by both its LU and Cholesky factorizations. We further define a generalized Lehmer matrix with  $(i, j)$  entries  $g_{ij} = \frac{\min\{u_{i+1}, u_{j+1}\}}{\max\{u_{i+1}, u_{j+1}\}}$  where  $u_n$  is the  $n$ th term of a binary sequence  $\{u_n\}$ . We derive both the LU and Cholesky factorizations of this analogous matrix and we precisely compute the determinant.

## 1 Introduction

D.H. Lehmer (see [2]) constructed an  $n \times n$  symmetric matrix  $A = (a_{ij})_{i,j}$  whose  $(i, j)$  entry is

$$a_{ij} = \frac{\min\{i, j\}}{\max\{i, j\}} = \begin{cases} i/j & j \geq i, \\ j/i & i > j. \end{cases}$$

Define the second order recurrence  $\{U_n(p, q)\}$  as follows:

$$U_n(p, q) = pU_{n-1}(p, q) - qU_{n-2}(p, q),$$

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where  $U_0(p, q) = 0$  and  $U_1(p, q) = 1$  for  $n > 1$ .

As an interesting example, we mention that the set of natural numbers can be obtained from the sequence  $\{U_n(p, q)\}$  by taking  $p = 2$ ,  $q = 1$ . Throughout this paper, we consider the case  $q = -1$  and we denote  $u_n = U_n(p, -1)$ .

We now define an  $n \times n$  generalized Lehmer matrix, namely  $\mathcal{F}_n = (g_{ij})_{1 \leq i, j \leq n}$  defined below:

$$g_{ij} = \frac{\min\{u_{i+1}, u_{j+1}\}}{\max\{u_{i+1}, u_{j+1}\}} = \begin{cases} \frac{u_{i+1}}{u_{j+1}} & \text{if } j \geq i, \\ \frac{u_{j+1}}{u_{i+1}} & \text{if } i > j. \end{cases}$$

where  $u_n$  is the  $n$ th term of the sequence  $\{u_n\}$ . In this paper, we obtain the general LU factorization and other explicit formulas for both the Lehmer matrix and its recursive analogue.

The Lehmer matrix is part of a family of matrices known as test matrices, which are used to evaluate the accuracy of matrix inversion programs since the exact inverses are known (see [1, 2]). It is hoped that our generalized Lehmer matrix will add to the literature of special matrices with known inverse.

## 2 The Lehmer Matrix

We start by obtaining the LU factorization of the Lehmer matrix  $A$ . Using the inverses of  $L$  and  $U$ , we obtain the explicit form for the inverse of  $A$ , whose inverse is well-known, thus obtaining another proof of this result.

We define the  $n \times n$  invertible lower triangular matrix  $L = (\ell_{ij})$  where  $\ell_{ij} = j/i$  for  $i \geq j$  and 0 otherwise. Next, we define the  $n \times n$  invertible upper triangular matrix  $U = (u_{ij})$  with  $u_{ij} = \frac{2i-1}{ij}$  for

$i \leq j$  and 0 otherwise. For example, when  $n = 5$ , we get

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{3}{4} & \frac{3}{6} & \frac{3}{8} & \frac{3}{10} \\ 0 & 0 & \frac{5}{9} & \frac{5}{12} & \frac{5}{15} \\ 0 & 0 & 0 & \frac{7}{16} & \frac{7}{20} \\ 0 & 0 & 0 & 0 & \frac{9}{25} \end{bmatrix}.$$

The following result holds.

**Theorem 1.** For  $n > 0$ , the LU factorization of Lehmer matrix is given by

$$A = LU$$

where  $L$  and  $U$  were defined previously.

*Proof.* We split the proof into three cases.

*Case 1:*  $i = j$ . By  $\sum_{k=1}^i (2k - 1) = i^2$ , then

$$a_{ii} = \sum_{k=1}^n \ell_{ik} u_{ki} = \sum_{k=1}^i \ell_{ik} u_{ki} = \sum_{k=1}^i \frac{k(2k-1)}{i \cdot k \cdot i} = \sum_{k=1}^i \frac{2k-1}{i^2} = 1.$$

*Case 2:*  $i > j$ . Thus

$$\begin{aligned} a_{ij} &= \sum_{k=1}^n \ell_{ik} u_{kj} = \sum_{k=1}^j \ell_{ik} u_{kj} = \sum_{k=1}^j \frac{k(2k-1)}{i \cdot k \cdot j} \\ &= \sum_{k=1}^j \frac{2k-1}{ij} = \frac{1}{ij} \sum_{k=1}^j 2k-1 = \frac{j}{i}. \end{aligned}$$

*Case 3:*  $j > i$ . Then

$$\begin{aligned} a_{ij} &= \sum_{k=1}^n \ell_{ik} u_{kj} = \sum_{k=1}^i \ell_{ik} u_{kj} = \sum_{k=1}^i \frac{k(2k-1)}{i \cdot k \cdot j} \\ &= \sum_{k=1}^i \frac{2k-1}{ij} = \frac{1}{ij} \sum_{k=1}^i 2k-1 = \frac{i}{j}, \end{aligned}$$

which completes the proof.  $\square$

We display an example below:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & 1 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{3}{4} & \frac{3}{5} \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{3}{4} & \frac{3}{6} & \frac{3}{8} & \frac{3}{10} \\ 0 & 0 & \frac{5}{9} & \frac{5}{12} & \frac{5}{15} \\ 0 & 0 & 0 & \frac{7}{16} & \frac{7}{20} \\ 0 & 0 & 0 & 0 & \frac{9}{25} \end{bmatrix}.$$

As a consequence of Theorem 1, we obtain an explicit value of the determinant of the Lehmer matrix in the following corollary.

**Corollary 1.** For  $n > 0$ ,

$$\det A = \frac{(2n)!}{2^n (n!)^3}$$

*Proof.* The proof follows from the LU factorization of matrix  $A$  by considering  $\det A = \det U = \prod_{i=1}^n \frac{2i-1}{i^2}$ .  $\square$

The  $n$ th Catalan number is given in terms of binomial coefficients by

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.$$

Thus we may note that

$$\det A = \frac{(n+1)}{2^n n!} C_n.$$

We continue our analysis by determining the  $L_1 L_1^T$  (named after Cholesky) factorization of the Lehmer matrix, where  $L_1$  is a lower triangular matrix. The Cholesky factorization was obtained for a different kind of matrix defined using binary sequences by the second author in [3].

**Theorem 2.** *The Cholesky factorization of the Lehmer matrix is given by*

$$A = L_1 L_1^T$$

where  $L_1 = (f_{ij})$  is a lower triangular matrix with  $f_{ij} = \frac{\sqrt{2j-1}}{i}$  for all  $i \geq j$ .

*Proof.* If  $i > j$ , then

$$\begin{aligned} a_{ij} &= \sum_{r=1}^n f_{ir} f_{jr} = \sum_{r=1}^j f_{ir} f_{jr} = \sum_{r=1}^j \frac{\sqrt{2r-1}}{i} \frac{\sqrt{2r-1}}{j} \\ &= \frac{1}{ij} \sum_{r=1}^j (2r-1) = \frac{j}{i}. \end{aligned}$$

If  $i = j$ , then

$$\begin{aligned} a_{ii} &= \sum_{r=1}^n f_{ir}^2 = \sum_{r=1}^i f_{ir}^2 = \sum_{r=1}^i \left( \frac{\sqrt{2r-1}}{i} \right)^2 \\ &= \frac{1}{i^2} \sum_{r=1}^i (2r-1) = \frac{i^2}{i^2} = 1. \end{aligned}$$

Finally, if  $i < j$ , then

$$a_{ij} = \sum_{r=1}^n f_{ir} f_{jr} = \sum_{r=1}^i f_{ir} f_{jr} = \frac{1}{ij} \sum_{r=1}^i (2r-1) = \frac{i}{j},$$

which proves the theorem.  $\square$

As an example, for  $n = 5$  and  $p = 1$  (the Fibonacci sequence case), we have

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & 1 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{3}{4} & \frac{3}{5} \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{5}}{3} & 0 & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{7}}{4} & 0 \\ \frac{1}{5} & \frac{\sqrt{3}}{5} & \frac{\sqrt{5}}{5} & \frac{\sqrt{7}}{5} & \frac{\sqrt{9}}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{5} \\ 0 & 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{5} \\ 0 & 0 & 0 & \frac{\sqrt{7}}{4} & \frac{\sqrt{7}}{5} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{9}}{5} \end{bmatrix}.$$

By Theorem 2, we find that, since  $A = L_1 L_1^T$ , we have that  $\det(A) = \prod_{i=1}^n f_{ii}^2 = \prod_{t=1}^n \frac{2i-1}{i^2} = \frac{(2n)!}{2^n (n!)^3}$ , that is, Corollary 1.

### 3 The Inverse of the Lehmer Matrix

Now we find an explicit formula for the inverse of the Lehmer matrix. For this purpose, we use its LU factorization as  $A^{-1} = U^{-1} L^{-1}$ . We first derive the inverses of the matrices  $L$  and  $U$ .

**Lemma 1.** *Let  $L^{-1} = (t_{ij})$  denote the inverse of  $L$ . Then*

$$t_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\frac{j}{i} & \text{if } i = j + 1, \\ 0 & \text{otherwise,} \end{cases}$$

*Proof.* The proof can be easily checked from the product  $L^{-1} L$ .  $\square$

**Lemma 2.** *Let  $U^{-1} = (w_{ij})$  denote the inverse of  $U$ . Then*

$$w_{ij} = \begin{cases} \frac{i^2}{2i-1} & \text{if } i = j \\ -\frac{i(i+1)}{2i+1} & \text{if } i + 1 = j, \\ 0 & \text{otherwise,} \end{cases}$$

*Proof.* The proof follows from the product  $U^{-1} U$ .  $\square$

The inverse of the Lehmer matrix is found in the following theorem.

**Theorem 3.** *For  $n > 0$ , let  $A^{-1} = (b_{ij})$ , then*

$$b_{ij} = \begin{cases} \frac{4i^3}{4i^2-1} & \text{if } i = j < n \\ \frac{n^2}{2n-1} & \text{if } i = j = n, \\ -\frac{i(i+1)}{2i+1} & \text{if } |i - j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

*Proof.* Since  $A^{-1} = U^{-1}L^{-1}$ , using the previous two lemmas, we obtain for  $1 \leq i \leq n-1$ ,

$$\begin{aligned} b_{ii} &= \sum_{k=1}^n w_{ik} t_{ki} = w_{ii} + w_{i,i+1} t_{i+1,i} \\ &= \frac{i^2}{2i-1} + \frac{i(i+1)}{2i+1} \frac{i}{i+1} = \frac{i^2}{2i-1} + \frac{i^2}{2i+1} = \frac{4i^3}{4i^2-1}. \end{aligned}$$

When  $i = j = n$ , it is easy to see that  $b_{nn} = w_{nn} = \frac{n^2}{2n-1}$ . If  $i = j+1$ , then

$$\begin{aligned} b_{i+1,i} &= \sum_{k=1}^n w_{i+1,k} t_{ki} = w_{i+1,i+1} t_{i+1,i} \\ &= \frac{(i+1)^2}{2i+1} \left( \frac{-i}{i+1} \right) = -\frac{i(i+1)}{2i+1}. \end{aligned}$$

The last case  $j = i+1$  can be similarly done, and the proof is complete.  $\square$

Therefore we recover the known fact that the inverse of the Lehmer matrix is a symmetric tridiagonal matrix.

We give the following example as a consequence of the above theorem: for  $n = 4$ ,

$$\begin{aligned} A^{-1} &= \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & \frac{32}{15} & -\frac{6}{5} & 0 \\ 0 & -\frac{6}{5} & \frac{108}{35} & -\frac{12}{7} \\ 0 & 0 & -\frac{12}{7} & \frac{16}{7} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{2}{3} & 0 & 0 \\ 0 & \frac{4}{3} & -\frac{6}{5} & 0 \\ 0 & 0 & \frac{9}{5} & -\frac{12}{7} \\ 0 & 0 & 0 & \frac{16}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \end{aligned}$$

We also give a relation between the terms of inverse of the Lehmer matrix and triangular numbers. Recall that the  $n$ th triangular number  $T_n$  is defined as the sum of the first  $n$  natural numbers, that is,  $T_n = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$ . We can re-write  $A^{-1} = (b_{ij})$  as  $b_{ij} = -\frac{2T_i}{2i+1}$  for  $|i - j| = 1$ , and  $b_{ii} = \frac{4i^3}{4i^2-1}$ .

## 4 Recursive Analogue of the Lehmer Matrix

In this section we investigate the same questions for our generalized recursive analogue of the Lehmer matrix  $\mathcal{F}_n$  defined in the first section, namely,  $\mathcal{F}_n = (g_{ij})$ :

$$g_{ij} = \frac{\min\{u_{i+1}, u_{j+1}\}}{\max\{u_{i+1}, u_{j+1}\}} = \begin{cases} \frac{u_{i+1}}{u_{j+1}} & \text{if } j \geq i, \\ \frac{u_{j+1}}{u_{i+1}} & \text{if } i > j. \end{cases}$$

where  $u_n$  is the  $n$ th term of the sequence  $\{u_n\}$ .

For example, when  $n = 5$  and  $p = 1$ , the matrix  $\mathcal{F}_5$  takes the following form:

$$\mathcal{F}_5 = \begin{bmatrix} 1 & \frac{u_2}{u_3} & \frac{u_2}{u_4} & \frac{u_2}{u_5} & \frac{u_2}{u_6} \\ \frac{u_2}{u_3} & 1 & \frac{u_3}{u_4} & \frac{u_3}{u_5} & \frac{u_3}{u_6} \\ \frac{u_2}{u_4} & \frac{u_3}{u_4} & 1 & \frac{u_4}{u_5} & \frac{u_4}{u_6} \\ \frac{u_2}{u_5} & \frac{u_3}{u_5} & \frac{u_4}{u_5} & 1 & \frac{u_5}{u_6} \\ \frac{u_2}{u_6} & \frac{u_3}{u_6} & \frac{u_4}{u_6} & \frac{u_5}{u_6} & 1 \end{bmatrix}.$$

In order to give the LU factorization of the matrix  $\mathcal{F}_n$ , we define two triangular matrices.

Define the  $n \times n$  unit lower triangular matrix  $L_2 = (c_{ij})$  with  $c_{ij} = \frac{u_{j+1}}{u_{i+1}}$  for all  $i \geq j$  and  $u_{ij} = 0$  for all  $i < j$ .

For example, when  $n = 5$ , the matrix takes the form:

$$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{u_2}{u_3} & 1 & 0 & 0 & 0 \\ \frac{u_2}{u_4} & \frac{u_3}{u_4} & 1 & 0 & 0 \\ \frac{u_2}{u_5} & \frac{u_3}{u_5} & \frac{u_4}{u_5} & 1 & 0 \\ \frac{u_2}{u_6} & \frac{u_3}{u_6} & \frac{u_4}{u_6} & \frac{u_5}{u_6} & 1 \end{bmatrix}.$$

Before defining an upper triangular matrix for the LU factorization of the matrix  $\mathcal{F}_n$ , we need to introduce a new sequence  $\{t_n\}$  by the following relation:

$$t_n = (p-1)u_n + u_{n-1}, \text{ that is, } t_n = u_{n+1} - u_n, \quad n > 1,$$

where  $u_n$  is defined as before.

Define the  $n \times n$  upper triangular matrix  $U_2 = (d_{ij})$  with  $d_{1j} = \frac{u_2}{u_{j+1}}$  for  $1 \leq j \leq n$ ,  $d_{ij} = \frac{(u_i + u_{i+1})t_i}{u_{i+1}u_{j+1}}$  for  $1 < i \leq j \leq n$ .

From the definition of the sequence  $\{t_n\}$ , we rewrite the matrix  $U_2$  with  $d_{1j} = \frac{u_2}{u_{j+1}}$  for  $1 \leq j \leq n$ ,  $d_{ij} = \frac{u_{i+1}^2 - u_i^2}{u_{i+1}u_{j+1}}$  for  $1 < i \leq j \leq n$ .

For example, when  $n = 4$ , the matrix takes the form:

$$U_2 = \begin{bmatrix} 1 & \frac{u_2}{u_3} & \frac{u_2}{u_4} & \frac{u_2}{u_5} & \frac{u_2}{u_6} \\ 0 & \frac{u_3^2 - u_2^2}{u_3^2} & \frac{u_3^2 - u_2^2}{u_3 u_4} & \frac{u_3^2 - u_2^2}{u_3 u_5} & \frac{u_3^2 - u_2^2}{u_3 u_6} \\ 0 & 0 & \frac{u_4^2 - u_3^2}{u_4^2} & \frac{u_4^2 - u_3^2}{u_4 u_5} & \frac{u_4^2 - u_3^2}{u_4 u_6} \\ 0 & 0 & 0 & \frac{u_5^2 - u_4^2}{u_5^2} & \frac{u_5^2 - u_4^2}{u_5 u_6} \\ 0 & 0 & 0 & 0 & \frac{u_6^2 - u_5^2}{u_6^2} \end{bmatrix}.$$

**Theorem 4.** For  $n > 0$ , the factorization of matrix  $\mathcal{F}_n = (g_{ij})$  is given by

$$\mathcal{F}_n = L_2 U_2,$$

where  $U_2$  and  $L_2$  were defined previously.

*Proof.* Let  $L_2 U_2 = (h_{ij})$ . We consider two cases,  $i > j$  and  $i \leq j$ . For the first case, we write

$$\begin{aligned}
h_{ij} &= \sum_{m=1}^n c_{im} d_{mj} = \sum_{m=1}^j c_{im} d_{mj} \\
&= c_{i1} d_{1j} + \sum_{m=2}^j \left( \frac{u_{m+1}}{u_{i+1}} \frac{(u_{m+1}^2 - u_m^2)}{u_{m+1} u_{j+1}} \right) \\
&= \frac{u_2^2}{u_{i+1} u_{j+1}} + \frac{1}{u_{i+1} u_{j+1}} \sum_{m=2}^j (u_{m+1}^2 - u_m^2) \\
&= \frac{u_2^2}{u_{i+1} u_{j+1}} + \frac{1}{u_{i+1} u_{j+1}} (u_{j+1}^2 - u_2^2) = \frac{u_{j+1}}{u_{i+1}} = g_{ij}.
\end{aligned}$$

If  $i \leq j$ , then similarly

$$\begin{aligned}
h_{ij} &= \sum_{m=1}^n c_{im} d_{mj} = \sum_{m=1}^i c_{im} d_{mj} \\
&= c_{i1} d_{1j} + \sum_{m=2}^i \left( \frac{u_{m+1}}{u_{i+1}} \frac{(u_{m+1}^2 - u_m^2)}{u_{m+1} u_{j+1}} \right) \\
&= \frac{u_2^2}{u_{i+1} u_{j+1}} + \frac{1}{u_{i+1} u_{j+1}} \sum_{m=2}^i (u_{m+1}^2 - u_m^2) \\
&= \frac{u_{i+1}}{u_{j+1}} = g_{ij},
\end{aligned}$$

and the claim is shown.  $\square$

Now we can find the value of  $\det(\mathcal{F}_n)$  by considering its  $LU$  factorization.

**Corollary 2.** For  $n > 0$ ,

$$\det(\mathcal{F}_n) = \prod_{i=2}^n \left( \frac{u_{i+1}^2 - u_i^2}{u_{i+1}^2} \right).$$

As a special cases of the matrix  $\mathcal{F}_n$ , we take the matrix  $\mathcal{F}_n^0$  obtained using the Fibonacci sequence, that is,  $F_{n+1} = F_n + F_{n-1}$ ,  $F_0 = 0$ ,  $F_1 = 1$ . The determinant of this matrix becomes

$$\det(\mathcal{F}_n^0) = \frac{F_{n-1}!F_{n+2}!}{2(F_{n+1}!)^2},$$

where  $F_n!$  is the Fibonomial factorial, that is,  $F_n! = F_1 F_2 \cdots F_n$ .

Next we give the Cholesky factorization of the generalized Lehmer matrix  $\mathcal{F}_n$ . For this purpose we define a lower triangular matrix  $L_3 = (m_{ij})$  with  $m_{i,1} = \frac{u_2}{u_{i+1}}$  for  $1 \leq i \leq n$ ,  $m_{ij} = \frac{1}{u_{i+1}} \sqrt{u_{j+1}^2 - u_j^2}$  for  $1 < j \leq i \leq n$  and 0 otherwise.

When  $n = 4$ , the matrix  $L_3$  takes the form:

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{u_2}{u_3} & \frac{1}{u_3} \sqrt{u_3^2 - u_2^2} & 0 & 0 \\ \frac{u_2}{u_4} & \frac{1}{u_4} \sqrt{u_3^2 - u_2^2} & \frac{1}{u_4} \sqrt{u_4^2 - u_3^2} & 0 \\ \frac{u_2}{u_5} & \frac{1}{u_5} \sqrt{u_3^2 - u_2^2} & \frac{1}{u_5} \sqrt{u_4^2 - u_3^2} & \frac{1}{u_5} \sqrt{u_5^2 - u_4^2} \end{bmatrix}.$$

The proof of the next theorem is analogous to the proof of Theorem 4, so it will be omitted.

**Theorem 5.** *The Cholesky factorization of the recursive analogue of the Lehmer matrix is given by*

$$\mathcal{F}_n = L_3 L_3^T$$

where  $L_3$  is the lower triangular matrix defined previously.

## 5 The Inverse of the Generalized Lehmer Matrix

Here we give the inverse of the recursive analogue of the Lehmer matrix  $\mathcal{F}_n^{-1}$  by considering its LU factorization. Before this, we give the inverses of the matrices  $L_2$  and  $U_2$  in the following lemmas, stated without proofs, as they are immediate.

**Lemma 3.** Let  $U_2^{-1} = (\hat{w}_{ij})$  denote the inverse of  $U_2$ . Then

$$\hat{w}_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \\ -\frac{u_{i+1}^2}{u_i^2 - u_{i+1}^2} & \text{if } 1 < i = j, \\ \frac{u_{i+1}u_{i+2}}{u_{i+1}^2 - u_{i+2}^2} & \text{if } i + 1 = j, \\ 0 & \text{otherwise,} \end{cases}$$

**Lemma 4.** Let  $L_2^{-1} = (\hat{t}_{ij})$  denote the inverse of  $L$ . Then

$$\hat{t}_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\frac{u_i}{u_{i+1}} & \text{if } i = j + 1, \\ 0 & \text{otherwise,} \end{cases}$$

Thus the inverse of the matrix  $\mathcal{F}_n$  is found in the following theorem.

**Theorem 6.** For  $n > 0$ , let  $\mathcal{F}_n^{-1} = (q_{ij})$ , then  $q_{11} = \frac{u_3^2}{u_3^2 - u_2^2}$ ,  $q_{nn} = \frac{u_{n+1}^2}{u_{n+1}^2 - u_n^2}$ ,  $q_{i,i+1} = q_{i+1,i} = \frac{u_{i+1}u_{i+2}}{u_{i+1}^2 - u_{i+2}^2}$  for  $1 \leq i \leq n - 1$ ,  $q_{ii} = \frac{u_{i+1}^2(u_{i+2}^2 - u_i^2)}{(u_{i+1}^2 - u_i^2)(u_{i+2}^2 - u_{i+1}^2)}$  for  $2 \leq i \leq n - 1$  and 0 otherwise.

*Proof.* Since  $\mathcal{F}_n^{-1} = U_2^{-1}L_2^{-1}$ , the proof follows from the previous two lemmas and from matrix multiplication.  $\square$

For example, for  $n = 4$ ,

$$\mathcal{F}_5^{-1} = \begin{bmatrix} \frac{u_3^2}{u_3^2 - u_2^2} & \frac{u_2u_3}{u_2^2 - u_3^2} & 0 & 0 \\ \frac{u_2u_3}{u_2^2 - u_3^2} & \left(\frac{u_3^2}{u_3^2 - u_2^2}\right)\left(\frac{u_4^2 - u_2^2}{u_4^2 - u_3^2}\right) & \frac{u_3u_4}{u_3^2 - u_4^2} & 0 \\ 0 & \frac{u_3u_4}{u_3^2 - u_4^2} & \left(\frac{u_4^2}{u_4^2 - u_3^2}\right)\left(\frac{u_5^2 - u_3^2}{u_5^2 - u_4^2}\right) & \frac{u_4u_5}{u_4^2 - u_5^2} \\ 0 & 0 & \frac{u_4u_5}{u_4^2 - u_5^2} & \frac{u_5^2}{u_5^2 - u_4^2} \end{bmatrix}.$$

## 6 Further comment

With a bit more care, one can certainly remove the constraint  $q = -1$  on the sequence  $U_n$ , and prove similar results like in the present paper for the corresponding generalized Lehmer matrix.

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## References

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