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# Legendre inversions and balanced hypergeometric series identities

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**Abstract**

By means of Legendre inverse series relations, we prove two terminating balanced hypergeometric series formulae. Their reversals and linear combinations yield several known and new hypergeometric series identities.


*MSC:* primary 05A19; secondary 33C20

*Keywords:* Legendre inversions; Binomial convolution; Hypergeometric series

## 1. Introduction and notation

For an indeterminate $c$ and a natural number $n$, denote the shifted-factorial by

$$(c)_0 = 1 \quad \text{and} \quad (c)_n = c(c+1)\cdots(c+n-1) \quad \text{for} \quad n = 1, 2, \ldots .$$

Following Bailey [3], the hypergeometric series is defined by

$$ {}_{1+r}F_s \left[ \begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \ldots (a_r)_k}{k!(b_1)_k \ldots (b_s)_k} z^k, $$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right-hand side. If one of numerator parameters $\{a_k\}$ is a negative integer, then the series becomes terminating, which reduces to a polynomial in $z$. In particular, if the sum of denominator parameters minus that of numerator parameters results in a natural number $m$, the series will be called $m$-balanced. When $m = 1$, we shall simply say that the hypergeometric series is balanced.

Inversion techniques have been shown to be powerful to deal with the terminating hypergeometric series identities (cf. Chu [4,5,8]). This paper will employ the Legendre inversions to derive balanced terminating hypergeometric series identities.

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For the subsequent applications, we reproduce two pairs of Legendre inversions (cf. Riordan [11, p. 68, Table 2.5]). Let $\lambda$ be a fixed real number. Then for all $n \in \mathbb{N}_0$, the following inverse series relations hold:

$$f(n)=\sum_{k=0}^{n}(-1)^k\binom{\lambda+2n}{n-k}\frac{\lambda+2k}{\lambda+2n}g(k), \tag{1.1a}$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\binom{\lambda+n+k-1}{n-k}f(k). \tag{1.1b}$$

One implication of this inverse pair is that for every identity of the form (1.1a) or (1.1b), there is a companion of the dual identity. To prove each is to prove both. We shall also use another pair of inversions:

$$F(n)=\sum_{k=0}^{n}(-1)^k\binom{\lambda+2n}{n-k}G(k), \tag{1.2a}$$

$$G(n)=\sum_{k=0}^{n}(-1)^k\binom{\lambda+n+k}{n-k}\frac{\lambda+2n}{\lambda+n+k}F(k). \tag{1.2b}$$

This follows directly from the replacements

$$F(k)=(\lambda+2k)f(k) \quad \text{and} \quad G(k)=(\lambda+2k)g(k)$$

in (1.1a) and (1.1b). In particular, when $\lambda=0$, one should be aware that the fraction $(\lambda+2n)/(\lambda+n+k)$ displayed in (1.2b) is set to be one for $n=k=0$ in view of the initial condition $F(0)=G(0)$ for both (1.2a) and (1.2b).

## 2. Balanced hypergeometric series identities

For the binomial coefficients, recall the Chu–Vandermonde convolution formula

$$\binom{x+y}{m}=\sum_{k=0}^{m}\binom{x}{k}\binom{y}{m-k}. \tag{2.1}$$

Then for $\varepsilon=1,2$, it is not difficult to show that

$$\frac{a-2c}{a+2n}\binom{a+2n}{\varepsilon+2n}=\sum_{k=0}^{\varepsilon+2n}\frac{\varepsilon+2n-2k}{\varepsilon+2n}\binom{c+n}{k}\binom{a-c+n}{\varepsilon+2n-k}.$$

Splitting the last sum into two parts and then performing replacements $k\to n-k$ and $k\to\varepsilon+n+k$, respectively, for the first and the second sum, we can manipulate the sum as follows:

$$\frac{a-2c}{a+2n}\binom{a+2n}{\varepsilon+2n}=\left\{\sum_{k=0}^{n}+\sum_{k=\varepsilon+n}^{\varepsilon+2n}\right\}\frac{\varepsilon+2n-2k}{\varepsilon+2n}\binom{c+n}{k}\binom{a-c+n}{\varepsilon+2n-k}$$
$$=\sum_{k=0}^{n}\frac{\varepsilon+2k}{\varepsilon+2n}\left\{\binom{c+n}{n-k}\binom{a-c+n}{\varepsilon+n+k}-\binom{c+n}{\varepsilon+n+k}\binom{a-c+n}{n-k}\right\}.$$

Applying two binomial relations

$$\binom{x+n}{n-k}=\frac{(1+x)_n}{(n-k)!(1+x)_k} \tag{2.2}$$

and

$$\binom{x+n}{\varepsilon+n+k}=(-1)^{\varepsilon+k}\frac{(1+x)_n(-x)_{k+\varepsilon}}{(\varepsilon+n+k)!}, \tag{2.3}$$

we can further reformulate the binomial identity as follows:

$$\frac{(1-a)_{\varepsilon-1}(2c-a)(a)_{2n}}{(1+c)_n(1+a-c)_n}=\sum_{k=0}^{n}(-1)^k\binom{\varepsilon+2n}{n-k}\frac{\varepsilon+2k}{\varepsilon+2n}\left\{\frac{(c-a)_{k+\varepsilon}}{(1+c)_k}-\frac{(-c)_{k+\varepsilon}}{(1+a-c)_k}\right\}.$$

The last identity matches (1.1a) exactly with $\lambda=\varepsilon$ and

$$f(n):=\frac{(1-a)_{\varepsilon-1}(2c-a)(a)_{2n}}{(1+c)_n(1+a-c)_n},$$

$$g(k):=\frac{(c-a)_{k+\varepsilon}}{(1+c)_k}-\frac{(-c)_{k+\varepsilon}}{(1+a-c)_k}.$$

The dual relation corresponding to (1.1b) gives us the following identity:

$$\left\{\frac{(c-a)_{n+\varepsilon}}{(1+c)_n}-\frac{(-c)_{n+\varepsilon}}{(1+a-c)_n}\right\}=\sum_{k=0}^{n}(-1)^k\binom{\varepsilon+n+k-1}{n-k}\frac{(1-a)_{\varepsilon-1}(2c-a)(a)_{2k}}{(1+c)_k(1+a-c)_k}.$$

Rewriting the binomial coefficient in terms of factorial fractions

$$\binom{\varepsilon+n+k-1}{n-k}=(-1)^k\frac{(\varepsilon)_n}{n!}\times\frac{(-n)_k(\varepsilon+n)_k}{(\varepsilon)_{2k}}, \tag{2.4}$$

we establish the following hypergeometric series identity.

**Theorem 1.** *For $\delta=0,1$, there holds the terminating balanced series identity*:

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}-n, & 1+\delta+n, & \frac{a}{2}, & \frac{1+a}{2}\\ & \delta+1/2, & 1+c, & 1+a-c\end{matrix}\,\middle|\,1\right]\\ &\quad=\frac{n!}{(1+\delta)_n(1-a)_\delta(2c-a)}\left\{\frac{(c-a)_{1+n+\delta}}{(1+c)_n}-\frac{(-c)_{1+n+\delta}}{(1+a-c)_n}\right\}.\end{aligned}$$

Reversing the summation order of the last ${}_4F_3$-series and then performing the parameter replacements $\delta+2n\to m$, $a\to -c-2n$ and $c\to -a-n$, we recover the following identity due to Andrews [1, p. 4].

**Proposition 2** (*Terminating balanced hypergeometric series identity*).

$${}_4F_3\left[\begin{matrix}-\frac{m}{2}, & \frac{1-m}{2}, & a, & c-a\\ & -m, & \frac{1+c}{2}, & \frac{2+c}{2}\end{matrix}\,\middle|\,1\right]=\frac{(a)_{m+1}-(c-a)_{m+1}}{(2a-c)(1+c)_m}.$$

## 3. Two-Balanced hypergeometric series identities

For the binomial coefficients, recall again the Chu–Vandermonde convolution formula (2.1). Then for $\delta=0,1$, we can check, without difficulty, that

$$\binom{a+2n-1}{\delta+2n}=\sum_{k=0}^{\delta+2n}\frac{a-2c+2n-2k+\delta}{a-2c}\binom{c+n}{\delta+2n-k}\binom{a-c+n}{k}.$$

Splitting the last sum into two parts and then performing replacements $k \to n-k$ and $k \to n+k+\delta$, respectively, for the first and the second sum, we can manipulate the binomial sum as follows:

$$\begin{aligned}
&\binom{a+2n-1}{\delta+2n} + (1-\delta)\binom{c+n}{n}\binom{a-c+n}{n} \\
&= \left\{\sum_{k=0}^{n} + \sum_{k=\delta+n}^{\delta+2n}\right\} \frac{a-2c+2n-2k+\delta}{a-2c}\binom{c+n}{\delta+2n-k}\binom{a-c+n}{k} \\
&= \sum_{k=0}^{n}\left\{\frac{a-2c+2k+\delta}{a-2c}\binom{c+n}{\delta+n+k}\binom{a-c+n}{n-k} + \frac{a-2c-2k-\delta}{a-2c}\binom{c+n}{n-k}\binom{a-c+n}{\delta+n+k}\right\}.
\end{aligned}$$

By means of (2.2) and (2.3), the last binomial identity can be reformulated as

$$\begin{aligned}
&\frac{(1-a)_\delta (a)_{2n}}{(1+c)_n(1+a-c)_n} + (1-\delta)\binom{2n}{n} \\
&= \sum_{k=0}^{n}(-1)^k\binom{\delta+2n}{n-k}\left\{\frac{(a-2c+2k+\delta)(-c)_{k+\delta}}{(a-2c)(1+a-c)_k} + \frac{(a-2c-2k-\delta)(c-a)_{k+\delta}}{(a-2c)(1+c)_k}\right\}.
\end{aligned}$$

This is the case $\lambda = \delta$ of (1.2a) under specification

$$F(n) := (1-\delta)\binom{2n}{n} + \frac{(1-a)_\delta (a)_{2n}}{(1+c)_n(1+a-c)_n},$$

$$G(k) := \frac{(a-2c+2k+\delta)(-c)_{k+\delta}}{(a-2c)(1+a-c)_k} + \frac{(a-2c-2k-\delta)(c-a)_{k+\delta}}{(a-2c)(1+c)_k}.$$

The dual relation corresponding to (1.2b) leads to the following identity:

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{\delta+n+k}{n-k}\frac{\delta+2n}{\delta+n+k}\left\{(1-\delta)\binom{2k}{k} + \frac{(1-a)_\delta (a)_{2k}}{(1+c)_k(1+a-c)_k}\right\} \\
&= \frac{(a-2c+2n+\delta)(-c)_{n+\delta}}{(a-2c)(1+a-c)_n} + \frac{(a-2c-2n-\delta)(c-a)_{n+\delta}}{(a-2c)(1+c)_n}.
\end{aligned}$$

Now we are going to show that for $n > 0$, the following expression just displayed

$$(1-\delta)\sum_{k=0}^{n}(-1)^k\binom{\delta+n+k}{n-k}\frac{\delta+2n}{\delta+n+k}\binom{2k}{k}$$

results in zero. This is obviously true for $\delta = 1$. Then for $\delta = 0$, it is verified as follows:

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n+k}{n-k}\frac{2n}{n+k}\binom{2k}{k} \\
&= 2\sum_{k=0}^{n}(-1)^k\binom{n+k-1}{k}\binom{n}{k} \\
&= 2\sum_{k=0}^{n}\binom{-n}{k}\binom{n}{k} = 2\binom{0}{n} = 0.
\end{aligned}$$

Hence we find the following hypergeometric identity.

**Theorem 3** (*Terminating 2-balanced hypergeometric series identity*).

$$
{}_4F_3\left[\begin{matrix}-n, & \delta+n, & \frac{a}{2}, & \frac{1+a}{2} \\ & \delta+1/2, & 1+c, & 1+a-c\end{matrix}\,\middle|\,1\right]
= \begin{cases} \dfrac{(a-2c+2n)(-c)_n}{2(a-2c)(1+a-c)_n} + \dfrac{(a-2c-2n)(c-a)_n}{2(a-2c)(1+c)_n}, & \delta=0; \\[2ex] \dfrac{1}{(1+2n)(1-a)}\left\{\dfrac{(a-2c+2n+1)(-c)_{n+1}}{(a-2c)(1+a-c)_n} + \dfrac{(a-2c-2n-1)(c-a)_{n+1}}{(a-2c)(1+c)_n}\right\}, & \delta=1. \end{cases}
$$

Similarly reversing the summation order for the last ${}_4F_3$-series and then making the parameter replacements $\delta+2n \to m$, $a \to -c-2n$ and $c \to -a-n$, we get the following identity.

**Proposition 4.** *For a natural number m, there holds the terminating* 2*-balanced hypergeometric series identity*:

$$
{}_4F_3\left[\begin{matrix}-\frac{m}{2}, & \frac{1-m}{2}, & a, & c-a \\ & 1-m, & \frac{1+c}{2}, & \frac{2+c}{2}\end{matrix}\,\middle|\,1\right] = \frac{(2a-c+m)(a)_m + (2a-c-m)(c-a)_m}{(2a-c)(1+c)_m}.
$$

## 4. Linear combinations and related hypergeometric formulae

By means of the linear combination of the identities displayed in Theorem 1 and Theorem 3 multiplied, respectively, by $\delta+n$ and $\gamma-\delta-n$, we establish the following identity with an extra free parameter $\gamma$.

**Theorem 5** (*Terminating balanced hypergeometric series identity*).

$$
{}_5F_4\left[\begin{matrix}-n, & \delta+n, & 1+\gamma, & \frac{a}{2}, & \frac{1+a}{2} \\ & \delta+1/2, & \gamma, & 1+c, & 1+a-c\end{matrix}\,\middle|\,1\right]
= \begin{cases} \dfrac{(n+\gamma)a-2\gamma(c+n)}{2\gamma(a-2c)}\dfrac{(c-a)_n}{(1+c)_n} - \dfrac{(n-\gamma)a+2\gamma(c-n)}{2\gamma(a-2c)}\dfrac{(-c)_n}{(1+a-c)_n}, & \delta=0; \\[2ex] \dfrac{\gamma(1+2n+2c-a)-na-c}{(1+2n)\gamma(a-1)(a-2c)}\dfrac{(c-a)_{1+n}}{(1+c)_n} + \dfrac{\gamma(-1-2n+2c-a)+(1+n)a-c}{(1+2n)\gamma(a-1)(a-2c)}\dfrac{(-c)_{1+n}}{(1+a-c)_n}, & \delta=1. \end{cases}
$$

Taking $\gamma=c$ and $\gamma=(n+2\delta)/3$, respectively, in the last theorem, we get the following two balanced series identities.

**Corollary 6** (*Terminating balanced hypergeometric series identity*).

$$
{}_4F_3\left[\begin{matrix}-n, & \delta+n, & \frac{a}{2}, & \frac{1+a}{2} \\ & \delta+1/2, & c, & 1+a-c\end{matrix}\,\middle|\,1\right]
= \begin{cases} \dfrac{(1-c)_n}{2(1+a-c)_n} + \dfrac{(c-a)_n}{2(c)_n}, & \delta=0; \\[2ex] \dfrac{1}{(1+2n)(1-a)}\left\{\dfrac{(c-a)_{n+1}}{(c)_n} + \dfrac{(1-c)_{n+1}}{(1+a-c)_n}\right\}, & \delta=1. \end{cases}
$$

**Corollary 7** (*Terminating balanced hypergeometric series identity*).

$$
{}_5F_4\left[\begin{matrix}-n, & \delta+n, & 1+\frac{n+2\delta}{3}, & \frac{a}{2}, & \frac{1+a}{2} \\ & \delta+1/2, & \frac{n+2\delta}{3}, & 1+c, & 1+a-c\end{matrix}\,\middle|\,1\right]
= \begin{cases} \dfrac{2a-c-n}{a-2c}\dfrac{(c-a)_n}{(1+c)_n} - \dfrac{a+c-n}{a-2c}\dfrac{(-c)_n}{(1+a-c)_n}, & \delta=0; \\[2ex] \dfrac{2-2a+c+n}{(a-1)(a-2c)(2+n)}\dfrac{(c-a)_{1+n}}{(1+c)_n} + \dfrac{a+c-n-2}{(a-1)(a-2c)(2+n)}\dfrac{(-c)_{1+n}}{(1+a-c)_n}, & \delta=1. \end{cases}
$$

Analogously, the combination of the identities displayed in Propositions 2 and 4 multiplied, respectively, by $-m$ and $\gamma+m$ results in another identity.

**Proposition 8.** *For a natural number m, there holds the terminating balanced hypergeometric series identity*:

$$
{}_5F_4\left[\begin{matrix} -\frac{m}{2}, & \frac{1-m}{2}, & 1+\gamma, & a, & c-a \\ & 1-m, & \gamma, & \frac{1+c}{2}, & \frac{2+c}{2} \end{matrix}\middle| 1\right] = \frac{\{a(m+2\gamma)-c(m+\gamma)+m\gamma\}(a)_m}{(2a-c)\gamma(1+c)_m} + \frac{\{a(m+2\gamma)-(c+m)\gamma\}(c-a)_m}{(2a-c)\gamma(1+c)_m}.
$$

Letting $\gamma=c/2$ and $\gamma=-2m/3$, respectively, in the last proposition, we recover two identities due to Krattenthaler [10, Lemmas A3 and A4].

**Corollary 9.** *For a natural number m, there holds the terminating balanced hypergeometric series identity*:

$$
{}_4F_3\left[\begin{matrix} -\frac{m}{2}, & \frac{1-m}{2}, & a, & c-a \\ & 1-m, & \frac{c}{2}, & \frac{1+c}{2} \end{matrix}\middle| 1\right] = \frac{(c-a)_m}{(c)_m} + \frac{(a)_m}{(c)_m}.
$$

**Corollary 10.** *For a natural number m, there holds the terminating balanced hypergeometric series identity*:

$$
{}_5F_4\left[\begin{matrix} -\frac{m}{2}, & \frac{1-m}{2}, & 1-\frac{2m}{3}, & a, & c-a \\ & 1-m, & -\frac{2m}{3}, & \frac{1+c}{2}, & \frac{2+c}{2} \end{matrix}\middle| 1\right]
= \frac{(a+c+2m)(a)_m}{2(2a-c)(1+c)_m} + \frac{(a-2c-2m)(c-a)_m}{2(2a-c)(1+c)_m}.
$$

These identities have played an important role in the determinant evaluation for plane partition enumeration (cf. Krattenthaler [10]).

## 5. Polynomial argument and further hypergeometric identities

For two polynomials of degree $\leqslant n$, if they have the same values at $n+1$ distinct points, then they are identical. Based on the hypergeometric series identities established in the previous sections, we shall use this polynomial argument to prove further hypergeometric series identities as well as the corresponding binomial convolution formulae.

### 5.1. Polynomial argument

Let $a=-m$ with $m\in\mathbb{N}_0$ in Corollary 6. Then we have

$$
{}_4F_3\left[\begin{matrix} -n, & \delta+n, & -\frac{m}{2}, & \frac{1-m}{2} \\ & \delta+1/2, & c, & 1-c-m \end{matrix}\middle| 1\right]
= \begin{cases} \dfrac{(c-n)_m+(c+n)_m}{2(c)_m}, & \delta=0; \\[2ex] \dfrac{1}{(1+2n)(1+m)}\left\{\dfrac{(c+n)_{m+1}}{(c)_m} - \dfrac{(c-1-n)_{m+1}}{(c)_m}\right\}, & \delta=1. \end{cases}
$$

Multiplying both sides by $(1+2n)$, we see that the last equality is valid for all $n \geqslant m$. Therefore it is a polynomial identity in $n$. Relabeling $n$ by $a$, we get the following companion of Corollary 6.

$$ {}_4F_3\left[\begin{matrix} -\frac{m}{2}, & \frac{1-m}{2}, & -a, & \delta+a \\ & \delta+1/2, & c, & 1-c-m \end{matrix}\middle| 1\right] \tag{5.5a}$$

$$ = \begin{cases} \dfrac{(c+a)_m+(c-a)_m}{2(c)_m}, & \delta=0; \\[2ex] \dfrac{(c+a)_{m+1}-(c-a-1)_{m+1}}{(1+m)(1+2a)(c)_m}, & \delta=1. \end{cases} \tag{5.5b}$$

From the identities displayed in Theorems 1, 3, and Corollary 7, we can analogously prove the following three hypergeometric series identities.

$$ {}_4F_3\left[\begin{matrix} -\frac{m}{2}, & \frac{1-m}{2}, & -a, & 1+\delta+a \\ & \delta+1/2, & 1+c, & 1-c-m \end{matrix}\middle| 1\right] \tag{5.2a}$$

$$ = \begin{cases} \dfrac{c}{2c+m}\dfrac{(c+a+1)_m+(c-a)_m}{(c)_m}, & \delta=0; \\[2ex] \dfrac{c}{2c+m}\dfrac{(c+a+1)_{m+1}-(c-a-1)_{m+1}}{(1+m)(1+a)(c)_m}, & \delta=1. \end{cases} \tag{5.2b}$$

$$ {}_4F_3\left[\begin{matrix} -\frac{m}{2}, & \frac{1-m}{2}, & -a, & \delta+a \\ & \delta+1/2, & 1+c, & 1-c-m \end{matrix}\middle| 1\right] \tag{5.3a}$$

$$ = \begin{cases} \dfrac{c(2c+2a+m)}{2(c+a)(2c+m)}\dfrac{(c+a)_m}{(c)_m}+\dfrac{c(2c-2a+m)}{2(c-a)(2c+m)}\dfrac{(c-a)_m}{(c)_m}, & \delta=0; \\[2ex] \dfrac{c(2c+2a+m+1)}{(1+m)(2c+m)(2a+1)}\dfrac{(c+a+1)_m}{(c)_m}-\dfrac{c(2c+m-2a-1)}{(1+m)(2c+m)(2a+1)}\dfrac{(c-a)_m}{(c)_m}, & \delta=1. \end{cases} \tag{5.3b}$$

$$ {}_5F_4\left[\begin{matrix} -\frac{m}{2}, & \frac{1-m}{2}, & 1+\frac{a+2\delta}{3}, & -a, & \delta+a, \\ & \delta+1/2, & \frac{a+2\delta}{3}, & 1+c, & 1-c-m \end{matrix}\middle| 1\right] \tag{5.4a}$$

$$ = \begin{cases} \dfrac{c(c+a+2m)}{(c+a)(2c+m)}\dfrac{(c+a)_m}{(c)_m}+\dfrac{c(c-a-m)}{(c-a)(2c+m)}\dfrac{(c-a)_m}{(c)_m}, & \delta=0; \\[2ex] \dfrac{c(2+2m+a+c)}{(1+m)(2c+m)(2+a)}\dfrac{(c+a+1)_m}{(c)_m}+\dfrac{c(2+m+a-c)}{(1+m)(2c+m)(2+a)}\dfrac{(c-a)_m}{(c)_m}, & \delta=1. \end{cases} \tag{5.4b}$$

### 5.2. Binomial convolution formulae

Corresponding to the cases $\delta=0$ and 1, the identities displayed in (5.1a–5.1b) are equivalent to the following two binomial convolution formulae:

$$ \sum_{k\geqslant 0}\frac{2a}{a+k}\binom{a+k}{2k}\binom{c-k}{n-2k}=\binom{a+c}{n}+\binom{c-a}{n}, \tag{5.5a}$$

$$ \sum_{k\geqslant 0}\frac{2a-1}{a+k}\binom{a+k}{1+2k}\binom{c-k}{n-2k}=\binom{a+c}{1+n}-\binom{1+c-a}{1+n}. \tag{5.5b}$$

Both identities appeared for the first time in Andrews and Burge [1, Eqs. 3.1–3.2], who discovered them through hypergeometric transformations in their work on plane partition enumerations. Their generalizations can be found in Chu [6,7], where the formal power series method has been employed.

Similarly for the identities displayed in (5.2a–5.2b), (5.3a–5.3b) and (5.4a–5.4b), we have the following corresponding binomial convolution formulae:

$$\sum_{k\geqslant 0}\binom{a+k}{2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k}=\binom{a+c}{n}+\binom{c-a-1}{n}, \tag{5.6a}$$

$$\sum_{k\geqslant 0}\binom{a+k}{1+2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k}=\binom{c+a}{1+n}-\binom{c-a}{1+n}; \tag{5.6b}$$

$$\sum_{k\geqslant 0}\frac{2a}{a+k}\binom{a+k}{2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k}=\frac{2c+2a-n}{c+a}\binom{a+c}{n}+\frac{2c-2a-n}{c-a}\binom{c-a}{n}, \tag{5.7a}$$

$$\sum_{k\geqslant 0}\frac{2a-1}{a+k}\binom{a+k}{1+2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k}$$

$$=\frac{2a+2c-n-1}{a+c}\binom{a+c}{1+n}+\frac{n+2a-2c-1}{1+c-a}\binom{1+c-a}{1+n}; \tag{5.7b}$$

$$\sum_{k\geqslant 0}\frac{a+3k}{a+k}\binom{a+k}{2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k}=\frac{c+a+n}{c+a}\binom{a+c}{n}+\frac{c-a-2n}{c-a}\binom{c-a}{n}, \tag{5.8a}$$

$$\sum_{k\geqslant 0}\frac{a+3k+1}{a+k}\binom{a+k}{1+2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k}$$

$$=\frac{1+n+a+c}{a+c}\binom{a+c}{1+n}+\frac{1+2n+a-c}{1+c-a}\binom{1+c-a}{1+n}. \tag{5.8b}$$

Recall the Hagen–Rothe identities (cf. Gould [9]) on binomial convolutions:

$$\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\binom{c-bk}{n-k}=\binom{a+c}{n}; \tag{5.9a}$$

$$\sum_{k=0}^{n}\binom{a+bk}{k}\frac{c-bn}{c-bk}\binom{c-bk}{n-k}=\binom{a+c}{n}; \tag{5.9b}$$

$$\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\frac{c-bn}{c-bk}\binom{c-bk}{n-k}=\frac{a+c-bn}{a+c}\binom{a+c}{n}. \tag{5.9c}$$

It is curious to observe that when $b=\frac{1}{2}$, the last three sums with even summation indices become those displayed in (5.5a), (5.6a) and (5.7a), respectively. Similarly, these sums with odd indices correspond to the binomial sums displayed in (5.5b), (5.6b) and (5.7b), respectively.

### 5.3. Further hypergeometric series identities

For the identities displayed in Propositions 2, 4 and Corollaries 9,10, the same polynomial argument used in proving (5.1a–5.1b) will lead us further to the following identities:

$$ {}_4F_3\left[\begin{matrix}-n, & \frac{a}{2}, & \frac{1+a}{2}, & c+n\\ & \frac{c}{2}, & \frac{1+c}{2}, & 1+a\end{matrix}\middle|\,1\right]=\frac{(c-a)_n}{(c)_n}, \tag{5.10}$$

$$ {}_4F_3\left[\begin{matrix}-n, & \frac{1+a}{2}, & \frac{2+a}{2}, & c+n\\ & \frac{1+c}{2}, & \frac{2+c}{2}, & 1+a\end{matrix}\middle|\,1\right]=\frac{c}{c+2n}\frac{(c-a)_n}{(c)_n}, \tag{5.11}$$

$$ {}_4F_3\left[\begin{matrix}-n, & \frac{a}{2}, & \frac{1+a}{2}, & c+n\\ & \frac{1+c}{2}, & \frac{2+c}{2}, & 1+a\end{matrix}\middle|\,1\right]=\frac{c(c-a+2n)}{(c-a)(c+2n)}\frac{(c-a)_n}{(c)_n}, \tag{5.12}$$

$$ {}_5F_4\left[\begin{matrix}-n, c+n, 1+\frac{2a}{3}, \frac{a}{2}, \frac{1+a}{2}\\ 1+a, \frac{2a}{3}, \frac{1+c}{2}, \frac{2+c}{2}\end{matrix}\middle|\,1\right]=\frac{c(2c-2a+n)}{2(c-a)(c+2n)}\frac{(c-a)_n}{(c)_n}. \tag{5.13}$$

Among them, the first two identities have previously appeared in Bailey [2, p. 512] and Andrews [1, p. 2], respectively. Refer to Chu [4, Eqs. 1.8a–c] for proofs via inversion technique.

For the sake of brevity, we present only the proof of (5.11). The interested reader may do the same for the other three identities. Specify $a=-n$ with $n\in\mathbb{N}_0$ for the identity displayed in Proposition 2. When $m\geqslant n$, the resulting identity reads as

$$ {}_4F_3\left[\begin{matrix}-\frac{m}{2}, & \frac{1-m}{2}, & -n, & c+n\\ & -m, & \frac{1+c}{2}, & \frac{2+c}{2}\end{matrix}\middle|\,1\right]=\frac{c}{c+2n}\frac{(1+c+m)_n}{(c)_n}.$$

Multiplying both sides by $(-m)_n$, we obtain a polynomial identity in $m$. Performing the parameter replacement $m\to -1-a$ in the last identity, we have

$$ {}_4F_3\left[\begin{matrix}-n, & c+n, & \frac{1+a}{2}, & \frac{2+a}{2}\\ & 1+a, & \frac{1+c}{2}, & \frac{2+c}{2}\end{matrix}\middle|\,1\right]=\frac{c}{c+2n}\frac{(c-a)_n}{(c)_n}.$$

This proves the identity displayed in (5.11).

In addition, one can consider linear combinations of (5.1a) with (5.1b) and (5.2a) with (5.2b) to establish further ${}_5F_4$-series identities as done for Theorem 5 and Proposition 8. Due to space limitations, we will not give further details.

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