

# LAH NUMBERS FOR R-POLYNOMIALS

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## 1. INTRODUCTION

According to [1], [2], and [3], given two sequences of polynomials,  $P_1(x, n)$  and  $P_2(x, n)$ ,  $n = 0, 1, 2, \dots$ .

$$(1) \quad P_k(x, n) = \sum_{m=0}^n C_{k,n}^m x^m, \quad k = 1, 2,$$

$$(1a) \quad C_{k,n}^m = 0, \quad \text{for } n < m, \quad m < 0, \quad n < 0,$$

and the inverse expansion

$$(2) \quad x^n = \sum_{m=0}^n D_{k,n}^m P_k(x, m) \quad k = 1, 2,$$

$$(2a) \quad D_{k,n}^m = 0, \quad \text{for } n < m, \quad m < 0, \quad n < 0,$$

the coefficients  $C_{k,n}^m$  and  $D_{k,n}^m$  are called respectively Generalized Stirling Numbers of First and Second Kind of the polynomials  $P_k(x, n)$ . Examples of such numbers can be found in [3], [4], and [5].

Let then

$$(3) \quad P_k(x, n) = \sum_{m=0}^n L_{k,h,n}^m P_h(x, m), \quad k, h = 1, 2, \quad k \neq h, \quad n = 0, 1, 2, \dots$$

$$(3a) \quad L_{k,h,n}^m = 0, \quad \text{for } n < m, \quad m < 0, \quad n < 0.$$

More explicitly,

$$\begin{aligned}
 P_k(x, n) &= \sum_{s=0}^n C_{k,n}^s x^s = \sum_{s=0}^n C_{k,n}^s \sum_{m=0}^s D_{h,s}^m P_h(x, m) , \\
 &= \sum_{s=0}^n C_{k,n}^s \sum_{m=0}^n D_{h,s}^m P_h(x, m) \\
 &= \sum_{m=0}^n P_h(x, m) \sum_{s=m}^n C_{k,n}^s D_{h,s}^m ,
 \end{aligned}$$

so that

$$(3b) \quad L_{k,h,n}^m = \sum_{s=m}^n C_{k,n}^s D_{h,s}^m .$$

The coefficients  $L_{k,h,n}^m$  are called Generalized Lah Numbers for the two sequences of polynomials  $P_k$  and  $P_h$ ,  $k \neq h$ ,  $k, h = 1, 2$ .

## 2. QUASI-ORTHOGONALITY

Under the conditions stated, the generalized Stirling numbers of first and second kind for a given sequence of polynomials  $P_k(x, n)$  are said to quasi-orthogonal to each other (cf. [3] if

$$(4) \quad \sum_{m=s}^n D_{k,n}^m C_{k,m}^s = \delta_n^s .$$

This result is proved in [3] for both the Q- and R-polynomials, but since the proof does not use the structure of the polynomials it is true for any sequence of polynomials as defined by (1).

Similarly the generalized Lah numbers for two sequences of polynomials  $P_k$  and  $P_h$  are quasi-orthogonal to the generalized Lah numbers for the sequences of polynomials  $P_h$  and  $P_k$ , i. e.,

$$(5) \quad \sum_{m=s}^n L_{k,h,n}^m L_{h,k,m}^s = \delta_n^s .$$

This result is proved in [2] for the Q-polynomials, but here again the proof does not use the structure of the polynomials, thus is valid for any two sequences of polynomials as defined by (1).

### 3. RECALL ABOUT R-POLYNOMIALS

In [2] we have studied the generalized Lah numbers for two sequences of Q-polynomials. We shall now study the same for R-polynomials as defined in [3], i. e.,

$$(6) \quad R(x, n) = \sum_{m=0}^n C_n^m x^m$$

$$(7) \quad R(x, n+1) = [K(n+1) + L(n+1)x]R(x, n) + \sum_{m=0}^n [M(m+1) + N(m+2)x]C_n^m x^m ,$$

$$(8) \quad R(x, 0) = K(0)$$

$$(9) \quad x^n = \sum_{m=0}^n D_n^m R(x, m) .$$

In order to simplify the results in [3] it was assumed that  $L = 1$ . Letting  $N(n+1) + 1 = P(n)$  it was proved that the numbers  $C_n^m$  and  $D_n^m$  satisfy the recurrence relations

$$(10) \quad C_n^m = [K(n) + M(m+1)]C_{n-1}^m + P(m)C_{n-1}^{m-1}$$

$$(11) \quad D_n^m = -[K(m+1) + M(n)]D_{n-1}^m / P(n) + D_{n-1}^{m-1} / P(n).$$

In the following we shall consider two sets of R-polynomials  $R_1(x, n)$  and  $R_2(x, n)$  and the corresponding generalized Stirling numbers  $\{C_{1,n}^m, C_{2,n}^m\}$  and  $\{D_{1,n}^m, D_{2,n}^m\}$  which all satisfy the conditions given in sections 1 and 2. The generalized Lah-numbers for the two sequences are  $L_{1,2,n}^m$  and  $L_{2,1,n}^m$ . They satisfy conditions (3a), (3b), and (5). We shall assume that  $L_1(n) = L_2(n) = 1$ .

#### 4. RECURRENCE RELATIONS

According to relations (6) through (9) we can write

$$\begin{aligned} R_2(x, n+1) &= \sum_{s=0}^{n+1} C_{2,n+1}^m x^m \\ &= [K_2(n+1) + x]R_2(x, n) + \sum_{s=0}^n [M_2(s+1) + N_2(s+2)x]C_{2,n}^s x^s, \end{aligned}$$

and, according to the definition of the generalized Lah-numbers,

$$R_2(x, n+1) = \sum_{m=0}^{n+1} L_{2,1,n+1}^m R_1(x, m),$$

so that

$$\begin{aligned} (12) \quad \sum_{m=0}^{n+1} L_{2,1,n+1}^m R_1(x, m) &= K_2(n+1) \sum_{m=0}^n L_{2,1,n}^m R_1(x, m) \\ &\quad + \sum_{m=0}^n L_{2,1,n}^m x R_1(x, m) \\ &\quad + \sum_{s=0}^n M_2(s+1) C_{2,n}^s \sum_{m=0}^s D_{1,s}^m R_1(x, m) \\ &\quad + \sum_{s=0}^n N_2(s+2) C_{2,n}^s \sum_{m=0}^{s+1} D_{1,s+1}^m R_1(x, m). \end{aligned}$$

On the other hand we have

$$R_1(x, m+1) = [K_1(m+1) + x]R_1(x, m) + \sum_{p=0}^m [M_1(p+1) + N_1(p+2)x]C_{1,m}^p x^p$$

thus

$$(13) \quad xR_1(x, m) = R_1(x, m+1) - K_1(m+1)R_1(x, m) - \sum_{p=0}^m [M_1(p+1) + N_1(p+2)x]C_{1,m}^p x^p.$$

Substituting (13) into (12), and reorganizing the last two terms with the help of (1a) and (2a), we obtain

$$\begin{aligned} \sum_{m=0}^{n+1} L_{2,1,n+1}^m R_1(x, m) &= K_2(n+1) \sum_{m=0}^n L_{2,1,n}^m R_1(x, m) \\ &+ \sum_{m=0}^n L_{2,1,n}^m \left[ R_1(x, m+1) - K_1(m+1)R_1(x, m) - \sum_{p=0}^m [M_1(p+1) + N_1(p+2)x]C_{1,m}^p x^p \right] \\ &+ \sum_{m=0}^n R_1(x, m) \sum_{s=m}^n M_2(s+1)C_{2,n}^s D_{1,s}^m \\ &+ \sum_{m=0}^{n+1} R_1(x, m) \sum_{s=m-1}^n N_2(s+2)C_{2,n}^s D_{1,s+1}^m, \end{aligned}$$

or, interchanging the indices  $m$  and  $s$ ,

$$\begin{aligned}
 (14) \quad & \sum_{m=0}^{n+1} L_{2,1,n+1}^m R_1(x, m) = K_2(n+1) \sum_{m=0}^n L_{2,1,n}^m R_1(x, m) \\
 & + \sum_{m=0}^n L_{2,1,n}^m R_1(x, m+1) - \sum_{m=0}^n L_{2,1,n}^m K_1(m+1) R_1(x, m) \\
 & - \sum_{s=0}^n L_{2,1,n}^s \sum_{p=0}^s M_1(p+1) C_{1,s}^p \sum_{m=0}^p D_{1,p}^m R_1(x, m) \\
 & - \sum_{s=0}^n L_{2,1,n}^s \sum_{p=0}^s N_1(p+2) C_{1,s}^p \sum_{m=0}^{p+1} D_{1,p+1}^m R_1(x, m) \\
 & + \sum_{m=0}^n R_1(x, m) \sum_{s=m}^n M_2(s+1) C_{2,n}^s D_{1,s}^m + \sum_{m=0}^{n+1} R_1(x, m) \sum_{s=m-1}^n N_2(s+2) C_{2,n}^s D_{1,s+1}^m
 \end{aligned}$$

The fourth and fifth quantities on the right-hand side of (14) can be written as follows:

$$\begin{aligned}
 (15) \quad & \sum_{s=0}^n L_{2,1,n}^s \sum_{p=0}^s M_1(p+1) C_{1,s}^p \sum_{m=0}^p D_{1,p}^m R_1(x, m) \\
 & = \sum_{m=0}^n R_1(x, m) \sum_{s=0}^n L_{2,1,n}^s \sum_{p=m}^s M_1(p+1) C_{1,s}^p D_{1,p}^m,
 \end{aligned}$$

$$\begin{aligned}
 (16) \quad & \sum_{s=0}^n L_{2,1,n}^s \sum_{p=0}^s N_1(p+2) C_{1,s}^p \sum_{m=0}^{p+1} D_{1,p+1}^m R_1(x, m) \\
 & = \sum_{m=0}^{n+1} R_1(x, m) \sum_{s=0}^n L_{2,1,n}^s \sum_{p=m-1}^s N_1(p+2) C_{1,s}^p D_{1,p+1}^m
 \end{aligned}$$

Substituting (15) and (16) into (14) we obtain by equating the coefficients of  $R_1(x, m)$

$$\begin{aligned}
(17) \quad L_{2,1,n+1}^m &= K_2(n+1) - K_1(m+1) L_{2,1,n}^m + L_{2,1,n}^{m-1} \\
&- \sum_{s=m-1}^n L_{2,1,n}^s \sum_{p=m}^s M_1(p+1) C_{1,s}^p D_{1,p}^m + \sum_{p=m-1}^s N_1(p+2) C_{1,s}^p D_{1,p+1}^m \\
&+ \sum_{s=m}^n M_2(s+1) C_{2,n}^s D_{1,s}^m + \sum_{s=m-1}^n N_2(s+2) C_{1,n}^s D_{1,s+1}^m,
\end{aligned}$$

or, changing  $n$  into  $n-1$ ,

$$\begin{aligned}
(18) \quad L_{2,1,n}^m &= K_2(n) - K_1(m+1) L_{2,1,n-1}^m + L_{2,1,n-1}^m \\
&- \sum_{s=m-1}^{n-1} L_{2,1,n-1}^s \sum_{p=m}^s M_1(p+1) C_{1,s}^p D_{1,p}^m + \sum_{p=m-1}^s N_1(p+2) C_{1,s}^p D_{1,p+1}^m \\
&+ \sum_{s=m}^{n-1} M_2(s+1) C_{2,n-1}^s D_{1,s}^m + \sum_{s=m-1}^{n-1} N_2(s+2) C_{2,n-1}^s D_{1,s+1}^m
\end{aligned}$$

Relation (18) is the recurrence relation for the generalized numbers  $L_{2,1,n}^m$ . A similar relation for the Lah-numbers  $L_{1,2,n}^m$  will be obtained by interchanging the indices 1 and 2 in (18).

## 5. EXAMPLE

We illustrate by the following example based on examples I and II of section 5 of [3]. Thus:

$$\begin{aligned}
K_1(\alpha) &= \alpha + 1, \quad M_1(\alpha) = (\alpha - 1)^2, \quad N_1(\alpha) = 0, \quad K_2(\alpha) = \alpha, \quad M_2(\alpha) = \alpha, \\
N_2(\alpha) &= \alpha,
\end{aligned}$$

where the index 1 corresponds to example I and the index 2 to example II of section 5 of [3]. The numerical values of  $C_{1,n}^m$  are those of  $C_n^m$ , of  $D_{1,n}^m$  those of  $D_n^m$  of example I, while  $C_{2,n}^m$  and  $D_{2,n}^m$  those of  $C_n^m$  and  $D_n^m$  of example II. Under these conditions we obtain the following for  $L_{2,1,n}^m$ :

$$\begin{aligned}
 (19) \quad L_{2,1,n}^m &= (n - m - 2)L_{2,1,n-1}^m + L_{2,1,n-1}^{m-1} \\
 &- \sum_{s=m-1}^{n-1} L_{2,1,n-1}^s \left[ \sum_{p=m}^s p^2 C_{1,s}^p D_{1,p}^m \right] + \sum_{s=m}^{n-1} (s+1) C_{2,n-1}^s D_{1,s}^m \\
 &+ \sum_{s=m-1}^{n-1} (s+2) C_{2,n-1}^s D_{1,s+1}^m .
 \end{aligned}$$

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m=	0	1	2	3	4
n:					
0	1				
1	-4	3			
2	42	-54	12		
3	-1488	2124	-696	60	
4	99680	-170640	67440	-8880	360

## REFERENCES

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