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INVERSIONS RELATING STIRLING, TANH, LAH NUMBERS AND AN APPLICATION TO MATHEMATICAL STATISTICS

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ABSTRACT. Inversion formulas have been found, converting between *Stirling*, *tanh* and *Lah* numbers. *Tanh* and *Lah* polynomials, analogous to the Stirling polynomials, have been defined and their basic properties established. New identities for Stirling and tangent numbers and polynomials have been derived from the general inverse relations. In the second part of the paper, it has been shown that if shifted-gamma probability densities and negative binomial distributions are *matched* by equating their first three semi-invariants (cumulants), then the cumulants of the two distributions are related by a pair of reciprocal linear combinations equivalent to the inversion formulas established in the first part.

1. Introduction

The usual form of an *inversion formula* is

$$g(n) = \sum_i a(n,i)f(i) \leftrightarrow f(n) = \sum_i A(n,i)g(i),$$

where $\{a(n,m), A(n,m)\}$ is a pair of *inverse numbers* and $\{f(n)\}$, $\{g(n)\}$ are numerical sequences. An extensive study of inverse relations and related topics can be found in [7]. In this paper we will consider inversion formulas of a more general form:

$$g(n,m) = \sum_i a(n,i)f(i,m) \leftrightarrow f(n,m) = \sum_i A(n,i)g(i,m),$$

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with *double-sequences* $\{f(n,m)\}$, $\{g(n,m)\}$ and corresponding *number inverses* $\{F(n,m)\}$, $\{G(n,m)\}$; in other words, we are interested in inverse relations converting between *arrays*. Of course, from the general formula, one can get inverse relations for *sequences* by fixing one of the two indices. As we shall see, the case $m = 1$ is of particular interest because, in general, it is associated with identities involving important classical numbers. We used this approach to find relations connecting Stirling $\left\{ \left[{n \atop m} \right], \left\{ {n \atop m} \right\} \right\}$, $\{\arctan\, t(n,m), \text{tangent } T(n,m)\}$ and Lah $L(n,m)$ numbers. Then, in analogy to the Stirling polynomials $\sigma_k(x)$ case, we introduced *tangent* $\delta_k(x)$ and Lah $\lambda_k(x)$ polynomials and derived connecting relations. Most of the identities we obtained seem to be original. Finally, we discussed the above results in the light of a problem in Statistical Mathematics dealing with semi-invariants (cumulants) of *shifted-gamma* probability densities $g(\vartheta;\ a,b,c) = \Gamma(\vartheta + c;\ a,b)$ and *negative binomial distributions* $nb(\varpi; r, \lambda)$. We showed that if $g(\vartheta;\ a,b,c)$ and $nb(\varpi; r, \lambda)$ are *matched* by equating their first three cumulants, then the cumulants $\gamma(n)$ and $\eta(n)$ of the two distributions are related by reciprocal linear combinations equivalent to the *array* inversion formulas established previously.

2. Relations between Stirling, tanh and Lah numbers

We will use Stirling $\left[{n \atop m} \right]$, $\left\{ {n \atop m} \right\}$ [3], arctan $t(n,m)$ and tangent $T(n,m)$ ([1], see pp. 258-260), and Lah $L(n,m)$ ([8], see pp. 43-44) numbers, but with scale factors and appropriate notations:

$$\theta(n,m) = (-1)^{(n-m)/2} t(n,m), \quad \Theta(n,m) = (-1)^{(n-m)/2} T(n,m);$$
$$\overline{\begin{bmatrix} n \\ m \end{bmatrix}} = (-2)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}, \quad \overline{\begin{Bmatrix} n \\ m \end{Bmatrix}} = 2^{n-m} \begin{Bmatrix} n \\ m \end{Bmatrix};$$
$$\bar{l}(n,m) = (-1)^{n-m} \overline{L}(n,m) = (-1)^n L(n,m) = \frac{n!}{m!} \binom{n-1}{m-1}.$$

The orthogonal relations satisfied by $\{\theta(n,m), \Theta(n,m)\}$ and $\left\{ \overline{\left[{n \atop m} \right]}, \overline{\left\{ {n \atop m} \right\}} \right\}$ are of the form $\sum_i a(n,i) A(i,m) = \sum_i A(n,i) a(i,m) = [m = n]$ and it is easily verified by direct calculation that these relations are also valid for $\left\{ \bar{l}(n,m), \overline{L}(n,m) \right\}$.

Recursions and egfs are simply obtained by introducing scales in the ordinary relations. Since the egfs of $\theta(n,m)$, $\Theta(n,m)$ egfs are of particular interest, we give the following explicit derivation. The known egfs [1] are

$$t_m(v) = \sum_n t(n,m)\frac{v^n}{n!} = \frac{1}{m!}\arctan^m v, \quad T_m(v) = \sum_n T(n,m)\frac{v^n}{n!} = \frac{1}{m!}\tan^m v.$$

Since $t_m(v)$, $T_m(v)$, as functions of v, and m have opposite parity, $t(n,m) = T(n,m) = 0$ when $n-m$ is *odd*. Thus, with $n-m$ is even, putting $v = \imath u$, $\imath^2 = -1$, $\arctan(\imath u) = \imath \operatorname{arg}\tanh u$ and $\tan(\imath u) = \imath \tanh u$, we get

$$\begin{aligned}
(2.1)\quad \theta_m(u) &= \textstyle\sum_n \imath^{n-m} t(n,m)\frac{u^n}{n!} = \sum_n (-1)^{\frac{n-m}{2}} t(n,m)\frac{u^n}{n!} = \frac{1}{m!}\operatorname{arg}\tanh^m u \\
&= \textstyle\frac{1}{m!}\left(\frac{1}{2}\ln\frac{1+u}{1-u}\right)^m = \frac{1}{m!}\left(\sum_j \theta_{2j+1}\frac{u^{2j+1}}{(2j+1)!}\right)^m;\ \theta_{2j+1} = (-1)^j t_{2j+1} = (2j)!; \\
\Theta_m(u)_m(u) &= \textstyle\sum_n \imath^{n-m} T(n,m)\frac{u^n}{n!} = \sum_n (-1)^{\frac{n-m}{2}} T(n,m)\frac{u^n}{n!} = \frac{1}{m!}\tanh^m u \\
&= \textstyle\frac{1}{m!}\left(\sum_j \Theta_{2j+1}\frac{u^{2j+1}}{(2j+1)!}\right)^m;\ \Theta_{2j+1} = (-1)^j T_{2j+1} = \frac{4^{j+1}(4^{j+1}-1)}{2j+2}B_{2j+2}.
\end{aligned}$$

Thus, scale factors $(-1)^{(n-m)/2}$ change *arctan* $t(n,m)$, *tangent* $T(n,m)$ numbers in *arctanh* $\theta(n,m)$, *tanh* $\Theta(n,m)$ numbers. For brevity, the pairs $\{\overline{\left[{n \atop m}\right]}, \overline{\left\{{n \atop m}\right\}}\}$, $\{\theta(n,m), \Theta(n,m)\}$ and $\{\bar{l}(n,m), \overline{L}(n,m)\}$ will be called *Stirling*, *tanh* and *Lah* numbers, respectively. Scaled numbers basic properties are listed in Table 1.

Theorem 2.1. *Numbers in each pair,* Stirling*,* tanh *and* Lah*, convert between numbers in the other two pairs.*

Let $\tanh u = \frac{v}{1+v}$, $v = \frac{e^{2u-1}}{2}$, then from the egfs listed in Table 1 we get:

$$\begin{aligned}
&\textstyle\sum_n \Theta(n,m)\frac{u^n}{n!} = \frac{1}{m!}\tanh^m u = \frac{1}{m!}\left(\frac{v}{1+v}\right)^m = \sum_i \overline{L}(i,m)\frac{v^i}{i!} \\
(2.2)\qquad &\textstyle= \sum_i \overline{L}(i,m)\frac{1}{i!}\left(\frac{e^{2u}-1}{2}\right)^i = \sum_i \overline{L}(i,m)\sum_n \overline{\left\{{n \atop i}\right\}}\frac{u^n}{n!} \\
&\textstyle= \sum_n \left(\sum_i \overline{L}(i,m)\overline{\left\{{n \atop i}\right\}}\right)\frac{u^n}{n!}.
\end{aligned}$$

These equations imply $\Theta(n,m) = \sum_{i=m}^n \overline{L}(i,m)\overline{\left\{{n \atop i}\right\}}$ and, by dualities and inversions (Table 1), $\theta(n,m) = \sum_{i=m}^n \bar{l}(i,m)\overline{\left[{n \atop i}\right]}$, $\overline{\left[{n \atop m}\right]} = \sum_i \overline{L}(n,i)\theta(i,m)$ and $\overline{\left\{{n \atop m}\right\}} = \sum_i \bar{l}(n,i)\Theta(i,m)$. Hence, Lah numbers convert between Stirling and tanh numbers. In Table 2 we listed the identities that are derived by use of inversions and/or dualities given in Table 1. Since in the third and fourth inversion formulas Stirling numbers convert between Lah and tanh numbers, and in the fifth and sixth tanh numbers convert between Lah and Stirling numbers, the proof is complete.

The basic structure connecting tanh and Stirling numbers is the following.

Recurrence relations

$$\overline{\left[{n+1 \atop m}\right]} = \overline{\left[{n \atop m-1}\right]} - 2n\overline{\left[{n \atop m}\right]}, \quad \overline{\left\{{n+1 \atop m}\right\}} = \overline{\left\{{n \atop m-1}\right\}} + 2m\overline{\left\{{n \atop m}\right\}};$$

$$\overline{\left[{0 \atop m}\right]} = \overline{\left\{{0 \atop m}\right\}} = [m=0], \quad \overline{\left[{n \atop 0}\right]} = \overline{\left\{{n \atop 0}\right\}} = [n=0].$$

$$\theta(n+1,m) = \theta(n,m-1) + n(n-1)\theta(n-1,m),$$

$$\Theta(n+1,m) = \Theta(n,m-1) - m(m+1)\Theta(n,m+1);$$

$$\theta(0,m) = \Theta(0,m) = [m=0];\ \theta(n,0) = \Theta(n,0) = [n=0];\ \theta(1,0) = \Theta(1,0) = 0.$$

$$\overline{l}(n+1,m) = (n+m)\overline{l}(n,m) + \overline{l}(n,m-1),$$

$$\overline{L}(n+1,m) = -(n+m)\overline{L}(n,m) + \overline{L}(n,m-1);$$

$$\overline{l}(n,0) = \overline{L}(n,0) \equiv 0; \quad (-1)^m\overline{l}(0,m) = \overline{L}(0,m) = -\tfrac{1}{m!} + [m=0].$$

Duality laws and Orthogonal relations

$$\overline{\left[{n \atop m}\right]} = (-1)^{n-m}\overline{\left\{{-m \atop -n}\right\}};\ \theta(n,m) = \Theta(-m,-n);\ \overline{l}(n,m) = (-1)^{n-m}\overline{L}(-m,-n).$$

$$\textstyle\sum_i a(n,i)A(i,m) = \sum_i A(n,i)a(i,m) = [m=n].$$

Exponential generating functions

$$\textstyle\sum_n \overline{\left[{n \atop m}\right]}\frac{u^n}{n!} = \frac{1}{m!}\left(\frac{1}{2}\ln(1+2u)\right)^m, \quad \sum_n \overline{\left\{{n \atop m}\right\}}\frac{u^n}{n!} = \frac{1}{m!}\left(\frac{e^{2u}-1}{2}\right)^m$$

$$\textstyle\theta_m(u) = \sum_n \theta(n,m)\frac{u^n}{n!} = \frac{1}{m!}\operatorname{arg\,tanh}^m u = \frac{1}{m!}\left(\frac{1}{2}\ln\frac{1+u}{1-u}\right)^m$$

$$\textstyle\Theta_m(u) = \sum_n \Theta(n,m)\frac{u^n}{n!} = \frac{1}{m!}\tanh^m u$$

$$\textstyle\sum_n \overline{l}(n,m)\frac{u^n}{n!} = \frac{1}{m!}\left(\frac{u}{1-u}\right)^m, \quad \sum_n \overline{L}(n,m)\frac{u^n}{n!} = \frac{1}{m!}\left(\frac{u}{1+u}\right)^m.$$

TABLE 1. Basic properties of scaled Stirling, tanh and Lah numbers

Corollary 2.2. *Tanh numbers are finite sums of multiples of Stirling numbers, and inversely*

$$\begin{aligned}
&\textstyle\theta(n,n-k) = \sum_{i=0}^k i!\binom{n}{i}\binom{n-1}{i}\overline{\left[{n-i \atop n-k}\right]},\\
&\textstyle\overline{\left[{n \atop n-k}\right]} = \sum_{i=0}^k (-1)^i i!\binom{n}{i}\binom{n-1}{i}\theta(n-i,n-k),\\
&\textstyle\Theta(n+k,n) = \sum_{i=0}^k (-1)^{k-i} i!\binom{n+i}{i}\binom{n+i-1}{i}\overline{\left\{{n+k \atop n+i}\right\}},\\
&\textstyle\overline{\left\{{n+k \atop n}\right\}} = \sum_{i=0}^k (-1)^{k-i} i!\binom{n+i}{i}\binom{n+i-1}{i}\Theta(n+k,n+i).
\end{aligned}$$

These relations are obtained from the first two inverse pairs in Table 2 with $k = n-m$, i replaced by $n-i$ and the use of Lah numbers explicit expressions. As we know ([4], see p. 418), Stirling numbers $\overline{\left[{x \atop x-k}\right]}$, $\overline{\left\{{x+k \atop x}\right\}}$ can be viewed as polynomial in x. Thus, Corollary (2.2) implies that $\theta(x,x-k)$, $\Theta(x+k,x)$ can also be treated as polynomials; these have the following properties.

Proposition 2.3. *If $k = 2j \geq 0$, then $\theta(x,x-k)$, $\Theta(x+k,x)$ are polynomials in x having degree $\frac{3k}{2} = 3j$ and leading coefficient $\frac{1}{3^j \times j!}$, $(-1)^j\frac{1}{3^j \times j!}$, respectively.*

$\overline{{n \brack m}} = (-2)^{n-m}{n \brack m}$,		$\overline{{n \brace m}} = 2^{n-m}{n \brace m}$
$\theta(n,m) = (-1)^{(n-m)/2}t(n,m)$,		$\Theta(n,m) = (-1)^{(n-m)/2}T(n,m)$
$\bar{l}(n,m) = (-1)^n L(n,m)$,		$\overline{L}(n,m) = (-1)^m L(n,m)$
$\Theta(n,m) = \sum_i \overline{L}(i,m)\overline{{n \brace i}}$	$\leftrightarrow$	$\overline{{n \brace m}} = \sum_i \bar{l}(i,m)\Theta(n,i)$
$\theta(n,m) = \sum_i \bar{l}(n,i)\overline{{i \brack m}}$	$\leftrightarrow$	$\overline{{n \brack m}} = \sum_i \overline{L}(n,i)\theta(i,m)$
$\overline{L}(n,m) = \sum_i \overline{{n \brack i}}\Theta(i,m)$	$\leftrightarrow$	$\Theta(n,m) = \sum_i \overline{{n \brace i}}\overline{L}(i,m)$
$\bar{l}(n,m) = \sum_i \overline{{i \brace m}}\theta(n,i)$	$\leftrightarrow$	$\theta(n,m) = \sum_i \overline{{i \brack m}}\bar{l}(n,i)$
$\bar{l}(n,m) = \sum_i \theta(n,i)\overline{{i \brace m}}$	$\leftrightarrow$	$\overline{{n \brace m}} = \sum_i \Theta(n,i)\bar{l}(i,m)$
$\overline{L}(n,m) = \sum_i \Theta(i,m)\overline{{n \brack i}}$	$\leftrightarrow$	$\overline{{n \brack m}} = \sum_i \theta(i,m)\overline{L}(n,i)$

TABLE 2. Conversions between Stirling, tanh, and Lah numbers

The proof is by induction on k applied to tanh numbers recurrence relations (Table 1) written with $k = n - m$ notations. Details of the proof are omitted because they are the same as those used by Gessel and Stanley ([2], see p. 25) in their study on Stirling numbers structure.

The first few cases are the following.

$$\theta(x,x-2) = -\Theta(x,x-2) = \tfrac{3!}{3\times 1!}\tbinom{x}{3}$$

$$\theta(x,x-4) = \tfrac{6!}{3^2\times 2!}\tbinom{x+1}{6} - 2^4\tbinom{x}{5}; \quad \Theta(x,x-4) = \tfrac{6!}{3^2\times 2!}\tbinom{x+1}{6} - 2^3\times 3\tbinom{x}{5}.$$

As pointed out in the Introduction, the general inverse relations in Table 2 yield interesting results in the case of $m = 1$, essentially because

$$\overline{{n \brack 1}} = (-2)^{n-m}(n-1)!; \quad \overline{{n \brace 1}} = 2^{n-m}.$$

$$\theta(2n+1,1) = (2n)!; \quad \Theta(2n+1,1) = (-1)^n T_{2n+1}; \quad \theta(2n,1) = \Theta(2n,1) \equiv 0.$$

The first general pair of inverse relations gives Stirling numbers identities:

$$\textstyle\sum_{i=1}^{n}(-1)^{i-1}i!\overline{{n \brace i}} = \begin{cases} 0, \text{ even } n \\ \Theta(n,1), \text{ odd } n \end{cases} \quad \leftrightarrow \quad 2^{n-1} = \sum_{i=1}^{n} i!\Theta(n,i). \tag{2.3}$$

The identity when n is even was given for the first time by Lengyel ([6], see p. 7), whereas the identity for n odd is new. The third inverse pair yields:

$$2^{n-1} = \sum_{i=1}^{n}\Theta(n,i)i! \quad \leftrightarrow \quad n! = \sum_{i=1}^{n}\theta(n,i)2^{i-1}, \tag{2.4}$$

showing that tanh numbers convert between factorials and powers of 2. From the fourth we get an inverse pair:

$$\theta(n,1)=\sum_{i=1}^{n}\bar{l}(n,i)(-2)^{i-1}(i-1)!\leftrightarrow(-2)^{n-1}(n-1)!=\sum_{i=1}^{n}\overline{L}(n,i)\theta(i,1),$$

extending an inversion formula $n!=\sum_{i=1}^{n}\overline{L}(n,i)2^{i-1}i!\leftrightarrow 2^{n-1}n!=\sum_{i=1}^{n}\bar{l}(n,i)i!$ given by Lah ([5], see p. 207). Finally, the fifth and the sixth pairs disclose original identities involving two out of the Stirling, tanh and Lah numbers:

$$\Theta(n,1)=\textstyle\sum_{i=1}^{n}\overline{\left\{{n \atop i}\right\}}(-1)^{i-1}i! \quad\leftrightarrow\quad (-1)^{n-1}n!=\sum_{i=1}^{n}\overline{\left[{n \atop i}\right]}\Theta(i,1)$$

$$(-1)^{n-1}n!=\textstyle\sum_{i=1}^{n}\Theta(i,1)\overline{\left[{n \atop i}\right]} \ \leftrightarrow\ (-2)^{n-1}(n-1)!=\sum_{i=1}^{n}\theta(i,1)\overline{L}(n,i).$$

3. Stirling, tanh and Lah polynomials

Since $\theta(x,x-k)=\sum_{i=0}^{k}i!\binom{x}{i}\binom{x-1}{i}\overline{\left[{x-i \atop x-k}\right]}$ and $\bar{l}(x,x-k)=k!\binom{x}{k}\binom{x-1}{k}$ vanish for $x=0,1,\ldots,k$, *tanh polynomials* $\delta_k(x)$ and *Lah polynomials* $\lambda_k(x)$ can be defined by rules similar to $\overline{\sigma}_k(x)=\overline{\left[{x \atop x-k}\right]}\Big/x^{\underline{k+1}}$ [3] used for Stirling polynomials. For clarity, we recall that $x^{\underline{k+1}}=x(x-1)\ldots,(x-k)=(-1)^{k+1}(k-x)^{\underline{k+1}}$.

Definition 3.1.

$$\delta_k(x)=\tfrac{\theta(x,x-k)}{x^{\underline{k+1}}}\ \sim\ \delta_k(k-x)=(-1)^{k+1}\tfrac{\Theta(x,x-k)}{x^{\underline{k+1}}};$$

$$\lambda_k(x)=\tfrac{\bar{l}(x,x-k)}{x^{\underline{k+1}}}\ \sim\ \lambda_k(k-x)=-\tfrac{\overline{L}(x,x-k)}{x^{\underline{k+1}}}=(-1)^{k+1}\lambda_k(x).$$

In the case of integers n,m, the above definitions assume the form

$$\theta(n,m)=\frac{n!}{(m-1)!}\delta_{n-m}(n);\quad \Theta(n,m)=-\frac{n!}{(m-1)!}\delta_{n-m}(-m). \tag{3.1}$$

Corollary (2.2) with x instead of n and a factor $x^{\underline{k+1}}$ divided out, yields:

Proposition 3.2. *Tanh polynomials $\delta_k(x)$ are finite sums of multiples of Stirling polynomials $\overline{\sigma}_k(x)$, and inversely*

$$\delta_k(x)=\textstyle\sum_{i=0}^{k}\binom{x-1}{i}\overline{\sigma}_{k-i}(x-i)\ \sim\ \delta_k(k-x)=\sum_{i=0}^{k}\binom{k-x-1}{k-i}\overline{\sigma}_i(i-x);$$

$$\overline{\sigma}_k(x)=\textstyle\sum_{i=0}^{k}\binom{x-1}{i}\delta_{k-i}(x-i)\ \sim\ \overline{\sigma}_k(k-x)=\sum_{i=0}^{k}\binom{k-x-1}{k-i}\delta_i(i-x).$$

Proposition 3.3. *The generating functions of $x\delta_k(x)$ and $x\delta_k(k+x)$ are:*

$$\textstyle\sum_k x\delta_k(x)u^k=(u\coth u)^x=\left(\sum_j 2^{2j}\frac{B_{2j}}{(2j)!}u^{2j}\right)^x,\ B_{2j}\text{ Bernoulli numbers;}$$

$$\textstyle\sum_k x\delta_k(k+x)u^k=\left(\frac{1}{u}\operatorname{arg}\tanh u\right)^x=\left(\frac{1}{2u}\ln\frac{1+u}{1-u}\right)^x=\left(\sum_{j\geq 0}\frac{1}{2j+1}u^{2j}\right)^x.$$

The egf of $\theta(n+k,n)$ (2.1) and definitions (3.1) yield

$$\sum_k \theta(n+k,n)\frac{u^{n+k}}{(n+k)!} = \sum_k \frac{(n+k)!}{(n-1)!}\delta_k(n+k)\frac{u^{n+k}}{(n+k)!} = \frac{1}{n!}\arg\tanh^n u,$$
$$\sum_k x\delta_k(k+x)u^k = \left(\frac{1}{u}\arg\tanh u\right)^x$$

where, on the basis of the "polynomial" argument, we replaced n with x. Using the egf of $\Theta(n+k,n)$ and proceeding as above, we obtain $\sum_k x\delta_k(x)u^k = (u\coth u)^x$.

Proposition 3.4. *Tanh polynomials $\delta_k(x)$ satisfy the recurrence relation*

$$(x+1)\delta_k(x+1) = (x-k)\delta_k(x) + (x-1)\delta_{k-2}(x-1); \quad x\delta_0(x) \equiv 1.$$
$$\delta_k(x),\ k=2j>0,\ \text{has degree } k/2-1 \text{ and } \delta_k(x)\equiv 0,\ k=2j+1.$$

The recurrence relation is obtained by dividing out a common factor $x^{\underline{k}}$ in the recurrence relation of $\theta(n,m)$ (Table 1), written with $n=x$ and $m=x+1-k$. (Compare with $(x+1)\sigma_k(x+1) = (x-k)\sigma_k(x) + x\sigma_{k-1}(x)$, ([3] Exercise 18, Chapter 6)). $\delta_k(x),\ k=2j>0$ has degree $k/2-1$ follows from Proposition (2.3) and $\delta_k(x)\equiv 0,\ k=2j+1$, because $\theta(x,x-k)=\Theta(x+k,x)\equiv 0,\ k=2j+1$. (Recall that $\overline{\sigma}_k(x),\ k>0$, has degree $k-1$, [3]). The first few cases are the following:

$$\delta_k(1) = 2^k\tfrac{B_k}{k!} + [k=1]; \quad k\delta_k(0) = 2^k(2^k-2)\tfrac{B_k}{k!}, \quad k>0;$$
$$\delta_k(-1) = -\tfrac{\Theta_{k+1}}{(k+1)!}; \quad \delta_{2j}(2j+1) = \tfrac{1}{2j+1}.$$
$$\delta_2(x)\equiv\tfrac{1}{3};\ \delta_4(x) = \tfrac{1!}{3^2\times 2!}\tbinom{x-1}{1} - \tfrac{1}{3^2\times 5};\ \delta_6(x) = \tfrac{2!}{3^3\times 3!}\tbinom{x}{2} - \tfrac{2^3}{3^4\times 5}\tbinom{x-1}{1} + \tfrac{2}{3^3\times 7\times 5}.$$

A companion expression of (2.3), derived from Proposition (3.2), is

$$\textstyle\sum_{i=1}^k \binom{x-1}{i}\overline{\sigma}_{k-i}(x-i) = \begin{cases} 0, & \text{odd } k \\ \delta_k(x), & \text{even } k \end{cases}; \quad \overline{\sigma}_k(x) = \displaystyle\sum_{i=1}^k \binom{x-1}{i}\delta_{k-i}(x-i).$$

$\sum_{i=1}^{2j+1}\binom{x-1}{i}\overline{\sigma}_{2j+1-i}(x-i) = 0$ is a new Stirling polynomials identity.

The properties of $x\lambda_k(x)$ follow at ounce from $x\lambda_k(x) = \binom{x}{k}$.

Proposition 3.5. *Polynomials $x\lambda_k(x) = \binom{x}{k}$ have degree k. Their generating function and recurrence relation are*

$$\textstyle\sum_{k\ge 0} x\lambda_k(x)u^k = (1+u)^x, \quad \sum_{k\ge 0} x\lambda_k(k+x)u^k = \frac{1}{(1-u)^x},$$
$$(x+1)\lambda_k(x+1) = \tbinom{x+1}{k} = \tbinom{x}{k-1} + \tbinom{x}{k} = x\left[\lambda_{k-1}(x) + \lambda_k(x)\right], \quad x\lambda_0(x)\equiv 1.$$

4. An application to a problem in Mathematical Statistics

We now show that the above results have an application in a problem of Statistical Mathematics dealing with semi-invariants (cumulants) of *shifted-gamma* densities $g(\vartheta;\ a,b,c) = \Gamma(\vartheta + c;\ a,b)$ and *negative binomial distributions* $nb(\varpi; r, \lambda)$.

For the sake of completeness, we first recall some standard definitions.

$$g(\vartheta;\ a,b,c) = \Gamma(\vartheta + c;\ a,b) = \begin{cases} \frac{1}{b^a\Gamma(a)}(\vartheta+c)^{a-1}\exp[-\frac{(\vartheta+c)}{b}], & \vartheta > -c, \\ 0 & \text{otherwise} \end{cases}$$

$$nb(\varpi;\ r,\lambda) = \tfrac{\Gamma(r+\varpi)}{\Gamma(r)\Gamma(\varpi+1)}\left(\tfrac{1}{1+\lambda}\right)^r\left(\tfrac{\lambda}{1+\lambda}\right)^\varpi, \quad \lambda \geq 0$$

From the *moment* egfs $M_{sg}(t) = e^{-ct}/(1-bt)^a$, $M_{nb}(t) = 1/[1-\lambda(e^t-1)]^r$ and the *cumulant* egfs $\ln M_{sg}(t)$, $\ln M_{nb}(t)$:

$$\begin{aligned} &\ln M_{sg}(t) = \textstyle\sum_{n>0}\gamma(n)\frac{t^n}{n!} = (-c+ab)t + \sum_{n>1} a\frac{b^n}{n}t^n, \\ &\ln M_{nb}(t) = \textstyle\sum_{n>0}\eta(n)\frac{t^n}{n!} = r\sum_{m>0}\frac{\lambda^m}{m}(e^t-1)^m \\ &= \textstyle\sum_{n>0}\left[\sum_{m>0} r(m-1)!\left\{{n \atop m}\right\}\lambda^m\right]\frac{t^n}{n!}\ \left[\text{with } (e^t-1)^m = m!\sum_{n\geq 0}\left\{{n \atop m}\right\}\frac{t^n}{n!}\right], \end{aligned}$$

we get the nth cumulants $\gamma(n)$, $\eta(n)$ of $g(\vartheta;\ a,b,c)$, $nb(\varpi;\ r,\lambda)$:

$$\gamma(n) = \begin{cases} -c+ab, & n=1, \\ (n-1)!\ ab^n, & n>1; \end{cases} \tag{4.1}$$

$$\eta(n) = \textstyle\sum_{m=1}^n r(m-1)!2^{m-n}\overline{\left\{{n \atop m}\right\}}\lambda^m, \quad n>0. \tag{4.2}$$

Definition 4.1. Two distributions $g(\vartheta;\ a,b,c)$ and $nb(\varpi;\ r,\lambda)$ are said to be *matched* if their first three cumulants are equal.

By equating the first three cumulants

$1st$ *cumulant* (mean) $= \ \mu = -c+ab = r\lambda$

$2nd$ *cumulant* (variance) $= \sigma^2 = ab^2 = r\lambda(1+\lambda)$

$3rd$ *cumulant* $= \gamma(3) = 2!\ ab^3 = \eta(3) = 2!\ r\lambda(1+\lambda)(1/2+\lambda)$

we get matching conditions

$$\begin{aligned} &a = r\tfrac{\lambda(1+\lambda)}{(1/2+\lambda)^2}, \quad b = 1/2+\lambda, \quad c = \tfrac{ab}{1+2b} = \tfrac{r\lambda}{1+2\lambda}; \\ &r = \tfrac{ab^2}{b^2-1/4}, \qquad \lambda = b - 1/2. \end{aligned} \tag{4.3}$$

For convenience, we will use scaled cumulants $\overline{\eta(n)} = \frac{2^n}{r}\eta(n)$ and $\overline{\gamma(n)} = \frac{2^n}{r}\gamma(n)$.

Lemma 4.2. *In a matched pair* $\{g(\vartheta;\ a,b,c), nb(\varpi;\ r,\lambda)\}$*, cumulants* $\overline{\gamma}(n)$ *and* $\overline{\eta}(n)$ *are polynomials in* λ *having degree* n

$$\overline{\gamma}(n) = \textstyle\sum_{m=1}^{n} \gamma(n,m)\lambda^m, \quad n>0, \quad \overline{\eta}(n) = \sum_{m=1}^{n} \eta(n,m)\lambda^m;$$

$$\gamma(n,m) = 2^m(m-1)!\left[\bar{l}(n-1,m-1) + 2m\bar{l}(n-1,m)\right], \tag{4.4}$$

$$\eta(n,m) = 2^m(m-1)!\overline{\left\{ {n \atop m} \right\}}. \tag{4.5}$$

The lemma holds for $\overline{\eta}(n)$ as we already know (4.2). It is also true for $\overline{\gamma}(n)$ because if $\{a,b,c\}$ are replaced by the matching values (4.3), then from (4.1):

$$\overline{\gamma}(n) = \begin{cases} 2\lambda, & n=1 \\ 2^n(n-1)!\lambda(1+\lambda)(1/2+\lambda)^{n-2} = \sum_{m=1}^{n}\gamma(n,m)\lambda^m, & n>1, \end{cases}$$

$$\lambda(1+\lambda)(1/2+\lambda)^{n-2} = \textstyle\sum_{m=1}^{n}\left[\binom{n-2}{n-m}\frac{1}{2^{n-m}} + \binom{n-2}{n-m-1}\frac{1}{2^{n-m-1}}\right]\lambda^n$$

$$\text{and } \gamma(n,m) = 2^m(m-1)!\left[\bar{l}(n-1,m-1) + 2m\bar{l}(n-1,m)\right], \quad n \geq 1.$$

Theorem 4.3. *In a matched pair* $\{g(\vartheta;\ a,b,c), nb(\varpi;\ r,\lambda)\}$*, cumulants* $\overline{\gamma}(n)$ *and* $\overline{\eta}(n)$ *of the two distributions are related by reciprocal linear combinations*

$$\overline{\gamma}(n+1) = \sum_{i=0}^{n}\theta(n,i)\overline{\eta}(i+1) \quad \leftrightarrow \quad \overline{\eta}(n+1) = \sum_{i=0}^{n}\Theta(n,i)\overline{\gamma}(i+1), \quad n \geq 0,$$

$$\text{where } \theta(n,m) = (-1)^{(n-m)/2}t(n,m) \text{ and } \Theta(n,m) = (-1)^{(n-m)/2}T(n,m).$$

Since cumulants are polynomials in λ, a solution $\theta(n,m)$ and $\Theta(n,m)$ exists if like powers of λ on both sides are equal:

$$\eta(n+1,m+1) \equiv [\lambda^{m+1}]\sum_i \Theta(n,i)\overline{\gamma}(i+1) = \sum_i \Theta(n,i)\gamma(i+1,m+1), \tag{4.6}$$

$$\gamma(n+1,m+1) \equiv [\lambda^{m+1}]\sum_i \theta(n,i)\overline{\eta}(i+1) = \sum_i \theta(n,i)\eta(i+1,m+1).$$

From (4.5) and Stirling recurrence relation, and (4.4), we get

$$\begin{aligned} \eta(n+1,m+1) &= 2^{m+1}m!\overline{\left\{ {n+1 \atop m+1} \right\}} = 2^{m+1}m!\left[\overline{\left\{ {n \atop m} \right\}} + 2(m+1)\overline{\left\{ {n \atop m+1} \right\}}\right], \\ \gamma(i+1,m+1) &= 2^{m+1}m!\left[\bar{l}(i,m) + 2(m+1)\bar{l}(i,m+1)\right]. \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6), rearranging the terms and dividing by $2^m m!$, we obtain a recursion in m which automatically terminates after $n-m$ recursive steps

$$\overline{\left\{ {n \atop m} \right\}} - \textstyle\sum_{i=m}^{n}\Theta(n,i)\bar{l}(i,m) = 2(m+1)\left[\overline{\left\{ {n \atop m+1} \right\}} - \sum_{i=m+1}^{n}\Theta(n,i)\bar{l}(i,m+1)\right]$$

$$= 2^2(m+2)(m+1)\left[\overline{\left\{ {n \atop m+2} \right\}} - \textstyle\sum_{i=m+1}^{n}\Theta(n,i)\bar{l}(i,m+2)\right] =, \ldots, = 0,$$

Thus $\overline{\left\{ {n \atop m} \right\}} - \sum_i \Theta(n,i)\bar{l}(i,m) = 0$, and $\Theta(n,m) = (-1)^{(n-m)/2}T(n,m)$ from Theorem (2.1). Since $\gamma(n+1) = \sum_{i=0}^{n} \theta(n,i)\eta(i+1)$ is the reciprocal of $\overline{\eta}(n+1) = \sum_{i=0}^{n} \Theta(n,i)\overline{\gamma}(i+1)$, $\theta(n,m)$ and $\Theta(n,m)$ are necessarily *number inverses*, therefore $\theta(n,m) = (-1)^{(n-m)/2}t(n,m)$. This completes the proof. Conversely,

$$\eta(n+1,m+1) = \sum_i \Theta(n,i)\gamma(i+1,m+1) \leftrightarrow \gamma(n+1,m+1) = \sum_i \theta(n,i)\eta(i+1,m+1)$$

implies Theorem (2.1) and the inversion relations in Table 2, hence, *the problem in Theorem* (4.3) *of converting between cumulants is equivalent to the problem in Theorem* (2.1) *of converting between Stirling, tanh and Lah numbers.*

Finally, let us show that the inverse pair (2.4): $n! = \sum_{i=0}^{n} \theta(n,i)2^{i-1} \leftrightarrow 2^{n-1} = \sum_{i=0}^{n} \Theta(n,i)i!$, corresponds to an interesting particular case of Theorem (4.3). In fact, let $\lambda \to 0$ while $r\lambda = \alpha$ remains constant, then, as we know, the negative binomial distribution $nb(\varpi; r, \lambda)$ tends towards the Poisson distribution $p(\varpi; \alpha)$ with cumulants all equal to α. The matched shifted-gamma density tends towards $g(\vartheta; 4\alpha, \frac{1}{2}, \alpha)$, since, according to the matching conditions (4.3), $a \to 4\alpha$, $b \to \frac{1}{2}$, $c \to \alpha$, and its cumulants are obtained from (4.1). Using the cumulant limiting values, one verifies that the reciprocal relations in Theorem (4.3) reduce to (2.4).

5. Main results and Conclusion

From general inverse relations converting between Stirling, tanh and Lah numbers, we obtained a certain number of new identities by fixing $m = 1$ in the double-sequences involved. The same approach was used to study connections between $\sigma_k(x)$ Stirling, $\delta_k(x)$ tanh and $\lambda_(x)$ Lah polynomials. Finally, we showed that the cumulants of a shifted-gamma probability density and a negative binomial distribution can be related by reciprocal linear combinations (Theorem 4.3) which turned out to be an instance of the tanh numbers inversion formula, hence, that this problem and our problem (Theorem 2.1) on number arrays, are equivalent.

References

1. L. Comtet, *Advanced combinatorics*, Riedel, Dordrecht, 1974.
2. I. Gessel and Stanley R.P., *Stirling polynomials*, Journal of Combinatorial Theory **A24** (1978), 24–33.
3. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics: A foundation for computer sciences*, 1990 ed., Addison-Wesley, 1989.

4. D. E. Knuth, *Two notes on notation*, Amer. Math. Monthly **99** (1992), no. 5, 403–422.

5. I. Lah, *Eine neue art von zahlen, ihre eigenschaften und anwendung in der mathematischen statistik*, Mitteilungsbl.Math.Statis. **7** (1955), 203–216.

6. T. Lengyel, *On some properties of the series* $\sum_{k=0}^{\infty} k^n x^k$ *and the Stirling numbers of the second kind*, Discrete Math. **150** (1996), no. 1-3, 281–292, Selected papers in honour of Paul Erdös on the occasion of his 80th birthday.

7. J. Riordan, *Combinatorial identities*, 1979 ed., Krieger, 1968.

8. ———, *An introduction to combinatorial theory*, Princeton University Press, 1980.

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