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An application in stochastics of the Laguerre-type polynomials

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Abstract

We explain how an inner product derived from a perturbation of a weight function by the addition of a delta distribution is used in the orthogonalization procedure of a sequence of martingales related to a Lévy process. The orthogonalization is done by isometry. The resulting set of pairwise strongly orthogonal martingales involved are used as integrators in the so-called (extended) chaotic representation property. As example, we analyse a Lévy process which is a combination of Brownian motion and the Gamma process and encounter the Laguerre-type polynomials introduced by Littlejohn.

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1. Introduction

In stochastic modelling, Lévy processes (that is, stochastic processes with independent and stationary increments) play a prominent role. For example, they are used to model financial assets. A Lévy process consists of three basic stochastically independent parts: a deterministic part, a pure jump part, and Brownian motion. Combination of the last two components is crucial in the modelling of financial objects. Indeed, imagine that there exists something like a continuous change of stock price (modelled by Brownian motion), but that suddenly a jump (modelled by the jump part) shows up by a release of new information.

The *chaotic representation property* (CRP) has been studied in [10] for normal martingales X. CRP lies at the heart of the stochastic calculus. This property says that any square integrable random variable measurable with respect to X can be expressed as an orthogonal sum of multiple

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stochastic integrals with respect to X. It is known, see for example [4,5], that the only two normal martingales X, with the CRP, that are also Lévy processes are the Brownian motion and the compensated Poisson process. Recently, the CRP was extended under very weak conditions to a general Lévy process setting by allowing a sequence of orthogonalized martingales as stochastic integrators (See, [14]).

In the orthogonalization procedure of the martingales an inner product of the following form is employed:

$$\langle f,g\rangle = \int_{-\infty}^{+\infty} f(x)g(x)\,\mathrm{d}\mu(x) + \sigma^2 f(0)g(0),$$

where the measure μ comes from the pure jump part and the constant σ^2 comes from Brownian motion. The coefficients of the orthogonal polynomials involved especially come into play.

In this paper, we study the orthogonalization procedure for a sequence consisting of the compensated power jump martingales of our Lévy process, also called the Teugels martingales. In Section 2, we introduce these martingales. Section 3 is devoted to the orthogonalization procedure and the extended CRP is briefly discussed. Finally in Section 4, we discuss a particular example. In this context, the Laguerre-type polynomials introduced by Littlejohn [12] will turn up.

2. Teugels martingales

2.1. Lévy Processes

Let $X = \{X_t, t \geqslant 0\}$ be a real-valued stochastic process. For $0 \leqslant s < t$ the random variable $X_t - X_s$ is called the *increment* of the process X over the interval $[s,t]$. A stochastic process X is said to be a process with *independent increments* if the increments over non-overlapping intervals (common endpoints are allowed) are stochastically independent. A process X is called *stationary* if the distribution of the increment $X_{t+s} - X_t$ depends only on s, but is independent of t. A stationary process with $X_0 = 0$ and independent increments is called a *Lévy process*. For an up-to-date and comprehensive account of Lévy processes, we refer the reader to [2].

Let X be a Lévy process and denoted by

$$X_{t-} = \lim_{s\to t, s<t} X_s, \quad t > 0$$

the left limit process and by $\Delta X_t = X_t - X_{t-}$ the jump size at time t. We denote the characteristic function [13] of the distribution of $X_{t+s} - X_t$ by

$$\phi(\theta,s) = E[\exp(\mathrm{i}\theta(X_{t+s} - X_t))], \quad t,s \geqslant 0.$$

It is known that $\phi(\theta,s)$ is *infinitely divisible*; i.e., for every positive integer n, it is the nth power of some characteristic function, and that for $s \geqslant 0$,

$$\phi(\theta,s) = (\phi(\theta,1))^s.$$

If we have an infinitely divisible distribution with characteristic function $\phi(\theta)$, we can define a Lévy process X through the relations

$$E[\exp(\mathrm{i}\theta X_t)] = (\phi(\theta))^t, \quad t \geqslant 0.$$

We call the function $\psi(\theta) = \log\phi(\theta)$ the *characteristic exponent*.

Let us assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space. A stochastic process is *càdlàg* (which is the abbreviation of the French "continu à droite limite à gauche") if its sample paths are right continuous and have left-hand limits. If X is a Lévy process, then there exists a unique modification of it which is càdlàg and which remains a Lévy process [15, Theorem 30, p. 21]. We will henceforth *always assume* that we are using this unique càdlàg version of any given Lévy process. Let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N}$, where $\mathcal{G}_t = \sigma\{X_s; 0 \leqslant s \leqslant t\}$ be the natural filtration of X, and $\mathcal{N}$ are the P-null sets of $\mathcal{F}$, then $(\mathcal{F}_t)_{0 \leqslant t < \infty}$ is right continuous [15, Theorem 31, p. 22]. Further, let $L^2(\Omega, \mathcal{F}) = L^2(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant \infty}, P)$ denote the filtered and complete probability space of all square integrable random variables.

An important formula is the *Lévy–Khintchine formula* [2]; a function $\psi : R \to C$ is the characteristic exponent of an infinitely divisible distribution if and only if there are constants $a \in R$, $\sigma^2 \geqslant 0$ and a measure ν on $R \setminus \{0\}$ with $\int_{-\infty}^{+\infty} (1 \wedge x^2)\nu(\mathrm{d}x) < \infty$ such that

$$\psi(\theta) = \mathrm{i}a\theta - \frac{\sigma^2}{2}\theta^2 + \int_{-\infty}^{+\infty} (\exp(\mathrm{i}\theta x) - 1 - \mathrm{i}\theta x 1_{(|x|<1)})\nu(\mathrm{d}x)$$

for every θ and where $1_{(A)} = 1$ if A is true and zero otherwise.

The measure ν is called the *Lévy measure*. A Lévy process is completely determined by the triplet $[a, \sigma^2, \nu(\mathrm{d}x)]$. The constant a reflects a deterministic linear part, the constant σ^2 comes from a Brownian motion part (i.e., a continuous stochastic part) and, finally, the measure $\nu(\mathrm{d}x)$ comes from a stochastic pure jump part, which is stochastically independent of the Brownian motion part. For an interpretation of the measure $\nu(\mathrm{d}x)$ in terms of the distribution of possible jumps, we refer to [2]. The three most-known examples are *Brownian motion*, *the Poisson process*, and *the Gamma process*, respectively, given by the triplets $[0, \sigma^2, 0]$, $[0, 0, \delta(1)]$, and $[0, 0, \mathrm{e}^{-x}x^{-1}1_{(x>0)}\mathrm{d}x]$. Here $\delta(x)$ is the dirac measure, placing mass 1 at the point x.

We *suppose* that the Lévy measure satisfies for every $\varepsilon > 0$,

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\nu(\mathrm{d}x) < \infty \quad \text{for some } \lambda > 0.$$

This implies that the Lévy measure satisfies

$$\int_{-\infty}^{+\infty} |x|^i \nu(\mathrm{d}x) < \infty, \quad i = 2, 3, \ldots, \tag{1}$$

and that the characteristic function $E[\exp(\mathrm{i}uX_t)]$ is analytic in the neighborhood of 0. As such, X_t has moments of all order and the polynomials will be dense in $L^2(R, \mathrm{d}\varphi_t(x))$, where $\varphi_t(x) = P(X_t \leqslant x)$.

2.2. Power jump processes and Teugels martingales

We consider the following transformations of the Lévy process that will play an important role in the subsequent analysis. We write

$$X_t^{(i)} = \sum_{0<s\leqslant t} (\Delta X_s)^i, \qquad i = 2, 3, \ldots .$$

We also put $X_t^{(1)} = X_t$, but note that this does not necessarily mean that $X_t^{(1)} = \sum_{0<s\leqslant t} \Delta X_t$. The processes $X^{(i)} = \{X_t^{(i)}, t \geqslant 0\}$, $i = 1, 2, \ldots$, are again Lévy processes and are called the *power jump*

processes. Their sample paths show, jumps at the same time points as the sample paths of the original Lévy process X. However, now the jump-sizes are powers of the original jump size.

The power jump processes $\{X^{(i)},\ i=1,2,\dots,\}$ have finite means. Indeed, $E[X_t]=E[X_t^{(1)}]=tm_1<\infty$ and by [15, p. 29],

$$E[X_t^{(i)}]=E\left[\sum_{0<s\leqslant t}(\Delta X_s)^i\right]=t\int_{-\infty}^{\infty}x^i\nu(\mathrm{d}x)=m_it<\infty,\quad i=2,3,\dots.$$

Therefore, we can compensate the power jump processes by

$$Y_t^{(i)}:=X_t^{(i)}-E[X_t^{(i)}]=X_t^{(i)}-m_it,\quad i=1,2,3,\dots.$$

The compensated power jump process $Y^{(i)}$ of order i is a normal martingale, which was also called the *Teugels martingale of order i*.

Remark 1. In the case of a Poisson process, all power jump processes will be the same, and equal to the original Poisson process. In the case of a Brownian motion, all power jump processes of order strictly greater than one will be equal to zero.

3. Orthogonalization of the Teugels martingales

An important question is the orthogonalization of the set $\{Y^{(i)},i=1,2,\dots\}$ of martingales. Let $\mathscr{M}^2$ be the space of square integrable martingales M such that $\sup_t E(M_t^2)<\infty$, and $M_0=0$ a.s. Notice that if $M\in\mathscr{M}^2$, then $\lim_{t\to\infty}E(M_t^2)=E(M_\infty^2)<\infty$ and that $M_t=E[M_\infty|\mathscr{F}_t]$. Thus, each $M\in\mathscr{M}^2$ can be identified with its terminal value M_∞. As discussed in [15, p. 148], we say that two martingales $M,N\in\mathscr{M}^2$ are *strongly orthogonal*, written as $M\times N$, if and only if their product MN is a uniformly integrable martingale. Two random variables $X,Y\in L^2(\Omega,\mathscr{F})$ are called *weakly orthogonal*, $X\perp Y$, if $E[XY]=0$. Clearly, strong orthogonality implies weak orthogonality.

We are looking for a set of pairwise strongly orthogonal martingales $\{H^{(i)},i=1,2,\dots\}$, such that each $H^{(i)}$ is a linear combination of the $Y^{(j)}$, $j=1,2,\dots,i$. In [14] it is shown that the orthogonalization of $\{Y^{(j)},\ j=1,2,\dots\}$ can be achieved through an isometry. Indeed, one considers two spaces. The first space S_1 is the space of all real polynomials on the positive real line endowed with a scalar product $\langle .,.\rangle_1$, given by

$$\langle P(x),Q(x)\rangle_1=\int_{-\infty}^{+\infty}P(x)Q(x)x^2\nu(\mathrm{d}x)+\sigma^2P(0)Q(0).$$

Note that

$$\langle x^{i-1},x^{j-1}\rangle_1=\int_{-\infty}^{+\infty}x^{i+j}\nu(\mathrm{d}x)=m_{i+j}+\sigma^2 1_{(i=j=1)}<\infty,\quad i,j=1,2,\dots.$$

The other space S_2 is the space of all linear transformations of the Teugels martingales of the Lévy process, i.e.

$$S_2=\{a_1Y^{(1)}+a_2Y^{(2)}+\cdots+a_nY^{(n)};n\in\{1,2,\dots\},a_i\in\mathbb{R},\ i=1,\dots,n\}.$$

This space is endowed with the scalar product $\langle .,. \rangle_2$, given by

$$\langle Y^{(i)}, Y^{(j)} \rangle_2 = m_{i+j} + \sigma^2 1_{(i=j=1)}, \quad i,j = 1,2,\dots .$$

In [14], it was shown that $H^{(i)}H^{(j)}$, $i,j=1,2,\dots$ is a martingale if and only if $\langle H^{(i)}, H^{(j)} \rangle_2 = 0$. So, one clearly sees that $x^{i-1} \leftrightarrow Y^{(i)}$ is an isometry between S_1 and S_2. An orthogonalization of $\{1, x, x^2, \dots\}$ in S_1 gives an orthogonalization of $\{Y^{(1)}, Y^{(2)}, Y^{(3)}, \dots\}$. Other martingale relations between orthogonal polynomials and Lévy processes can be found in [16,17].

In order to identify the exact coefficients in the orthogonalization procedure, we proceed as follows: Let $\{P_n(x), n \geqslant 0\}$ be a system of orthogonal polynomials with respect to the inner product without the jump of σ^2 in zero, i.e.

$$\langle f, g \rangle_3 = \int_{-\infty}^{+\infty} f(x) g(x) x^2 \nu(\mathrm{d}x).$$

If we write

$$K_n(x) = \sum_{i=0}^{n} \frac{P_i(x) P_i(0)}{\langle P_i, P_i \rangle_3}, \qquad n = 0,1,\dots,$$

for the so-called *Kernel polynomials* of $\{P_n(x), n \geqslant 0\}$ [3]. Then following [1], a system of orthogonal polynomials $\{P_n^{\sigma^2}(x), n \geqslant 0\}$ with respect to $\langle .,. \rangle_1$, is given by

$$P_n^{\sigma^2}(x) = (1 + \sigma^2 K_{n-1}(0)) P_n(x) - \sigma^2 P_n(0) K_{n-1}(x), \quad n = 0,1,2,\dots .$$

A consequence is that the $P_n^{\sigma^2}(x)$ can be obtained by using connection coefficients. Indeed, in [1] it is shown that one has

$$P_n^{\sigma^2}(x) = (1 + \sigma^2 q_{n,n}) P_n(x) - \sigma^2 \sum_{k=0}^{n-1} q_{n,k} P_k(x), \qquad n = 0,1,2,\dots,$$

where

$$q_{n,k} = \frac{P_n(0) P_k(0)}{\langle P_k, P_k \rangle_3} \quad \text{and} \quad q_{n,n} = K_{n-1}(0) = \sum_{i=0}^{n-1} \frac{(P_i(0))^2}{\langle P_i, P_i \rangle_3}, \quad 0 \leqslant k \leqslant n.$$

Another relevant fact is that the Kernel polynomials $\{K_n(x), n \geqslant 0\}$ are orthogonal with respect to the measure $x^3 \nu(\mathrm{d}x)$ [3].

The orthogonalized set of martingales $H^{(i)}$, $i = 1,2,\dots$ is now used in the chaotic representation property as integrators. More precisely, one can show, as in [14], that every random variable F in $L^2(\Omega, \mathscr{F})$ has a representation of the form

$$F = E[F] + \sum_{j=1}^{\infty} \sum_{(i_1,\dots,i_j) \in N^j} \int_0^{\infty} \int_0^{t_1-} \dots \int_0^{t_{j-1}-} f_{(i_1,\dots,i_j)}(t_1,\dots,t_j)\, \mathrm{d}H_{t_j}^{(i_j)} \dots \mathrm{d}H_{t_2}^{(i_2)}\, \mathrm{d}H_{t_1}^{(i_1)}$$

where the $f_{(i_1,\dots,i_j)}$'s are real deterministic functions and $N = \{1,2,3,\dots\}$. A direct consequence is the weaker *predictable representation property* (PRP), saying that every random variable F in $L^2(\Omega, \mathscr{F})$ has a representation of the form

$$F = E[F] + \sum_{i=1}^{\infty} \int_0^{\infty} \phi_s^{(i)}\, \mathrm{d}H_s^{(i)},$$

where $\phi_s^{(i)}$ is predictable. Furthermore, because we can identify every martingale $M \in \mathscr{M}^2$ with is terminal value $M_\infty \in L^2(\Omega, \mathscr{F})$ and since $M_t = E[M_\infty | \mathscr{F}_t]$, we have the predictable representation

$$M_t = \sum_{i=1}^{\infty} \int_0^t \phi_s^{(i)} \, \mathrm{d}H_s^{(i)},$$

which is a sum of strongly orthogonal martingales.

4. Example

Consider as Lévy process $X = \{X_t, t \geqslant 0\}$, the one given by the triplet $[0, \sigma^2, 1_{(x>0)} \mathrm{e}^{-x} x^{-1} \, \mathrm{d}x]$. This process has no deterministic part and the stochastic part consists of a Brownian motion, $\{B_t, t \geqslant 0\}$ with parameter σ^2 and an independent pure jump part, $\{G_t, t \geqslant 0\}$ which is called a Gamma process. The law of G_t is indeed a gamma distribution with mean t and scale parameter equal to one. The Gamma process is used in insurance and mathematical finance models [6–9]. In the orthogonalization of the Teugels martingales of this process, we employ the space described above. The first space S_1 is the space of all real polynomials on the positive real line endowed with a scalar product $\langle .,. \rangle_1$, given by

$$\langle P(x), Q(x) \rangle_1 = \int_0^\infty P(x) Q(x) x \mathrm{e}^{-x} \, \mathrm{d}x + \sigma^2 P(0) Q(0).$$

Note that

$$\langle x^{i-1}, x^{j-1} \rangle_1 = \int_0^\infty x^{i+j-1} \mathrm{e}^{-x} \, \mathrm{d}x = (i+j-1)! + \sigma^2 1_{(i=j=1)}, \quad i,j = 1,2,3,\ldots \, .$$

The other space S_2 is the space of all linear transformations of the Teugels martingales $\{Y_t^{(i)}, t \geqslant 0\}$ of our Lévy process X; i.e.

$$S_2 = \{a_1 Y^{(1)} + a_2 Y^{(2)} + \cdots + a_n Y^{(n)}; n \in \{1,2,\ldots\}, a_i \in \mathbb{R}, i = 1,\ldots,n\}$$

and is endowed with the scalar product $\langle .,. \rangle_2$, given by

$$\langle Y^{(i)}, Y^{(j)} \rangle_2 = (i+j-1)! + \sigma^2 1_{(i=j=1)}, \quad i,j = 1,2,3,\ldots \, .$$

One clearly sees that $x^{i-1} \leftrightarrow Y^{(i)}$ is an isometry between S_1 and S_2. An orthogonalization of $\{1, x, x^2, \ldots\}$ in S_1 gives the *Laguerre-type polynomials* $L_n^{1,\sigma^2}(x)$ introduced in [12]; by isometry we also find an orthogonalization of $\{Y^{(1)}, Y^{(2)}, Y^{(3)}, \ldots\}$.

Next, we explicitly calculate the coefficients $\{a_{ij}, 1 \leqslant i \leqslant j\}$, such that

$$\{H^{(j)} = a_{1j} Y^{(1)} + a_{2j} Y^{(2)} + \cdots + a_{jj} Y^{(j)}, \quad j = 1,2,\ldots\}$$

is a strongly pairwise orthogonal sequence of martingales. We will use the *Laguerre polynomials* [11] $\{L_n^{(\alpha)}(x), n = 0,1,\ldots\}$, defined for every $\alpha > -1$ by

$$L_n^{(\alpha)}(x) = \frac{1}{n!} \sum_{k=0}^{n} (-n)_k (\alpha + k + 1)_{n-k} \frac{x^k}{k!}, \quad n = 0,1,\ldots \tag{2}$$

and their Kernel polynomials $\{K_n(x), n \geqslant 0\}$. The Laguerre polynomials are orthogonal with respect to the measure $1_{(x>0)} \mathrm{e}^{-x} x^\alpha \, \mathrm{d}x$.

The polynomials orthogonal with respect to $\langle .,. \rangle_3$ (or equivalently with respect to the measure $x^2 \nu(\mathrm{d}x) = 1_{(x>0)} x \mathrm{e}^{-x}$) are therefore the Laguerre polynomials $L_n^{(1)}(x)$. As mentioned above, the Kernel polynomials will be orthogonal with respect to the measure $x^3 \nu(\mathrm{d}x) = 1_{(x>0)} x^2 \mathrm{e}^{-x}$. Consequently, they are up to a constant the Laguerre polynomials $L_n^{(2)}(x)$. Now, a straightforward calculation gives

$$\begin{aligned} L_n^{1,\sigma^2}(x) &= b_{n,n} x^n + b_{n-1,n} x^{n-1} + \cdots + b_{1,n} x + b_{0,n} \\ &= L_n^{(1)}(x) + \sigma^2 L_n^{(1)}(x) L_{n-1}^{(2)}(0) - \sigma^2 L_n^{(1)}(0) L_{n-1}^{(2)}(x) \\ &= \left(1 + \sigma^2 \frac{n(n+1)}{2}\right) L_n^{(1)}(x) - \sigma^2 (n+1) L_{n-1}^{(2)}(x). \end{aligned}$$

Using (2), we conclude that

$$a_{n,n} = b_{n-1,n-1} = \left(1 + \sigma^2 \frac{(n-1)n}{2}\right) \frac{(-1)^{n-1}}{(n-1)!}$$

and that for $k = 1, \ldots, j-1$

$$\begin{aligned} a_{n,k} &= b_{n-1,k-1} \\ &= \left(1 + \sigma^2 \frac{n(n-1)}{2}\right) \frac{(-n+1)_{k-1}(k+1)_{n-k}}{(n-1)!(k-1)!} - \frac{\sigma^2 n(-n+2)_{k-1}(k+2)_{n-k}}{(n-2)!(k-1)!}. \end{aligned}$$

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