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q-SOBOLEV ORTHOGONALITY OF THE q-LAGUERRE POLYNOMIALS $\{L_n^{(-N)}(\cdot;q)\}_{n=0}^{\infty}$ FOR POSITIVE INTEGERS N

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Abstract. The family of q-Laguerre polynomials $\{L_n^{(\alpha)}(\cdot;q)\}_{n=0}^{\infty}$ is usually defined for $0 < q < 1$ and $\alpha > -1$. We extend this family to a new one in which arbitrary complex values of the parameter α are allowed. These so-called *generalized* q-Laguerre polynomials fulfil the same three term recurrence relation as the original ones, but when the parameter α is a negative integer, no orthogonality property can be deduced from Favard's theorem. In this work we introduce non-standard inner products involving q-derivatives with respect to which the generalized q-Laguerre polynomials $\{L_n^{(-N)}(\cdot;q)\}_{n=0}^{\infty}$, for positive integers N, become orthogonal.

1. Introduction

We begin with a brief revision of non-standard orthogonality results in the literature. Concretely, we will be concerned with those results relative to the Laguerre family and its several discrete and q-extensions.

Non-standard orthogonality can be understood as an orthogonality statement for a system of monic polynomials $\{P_n^{(\lambda_1,\ldots,\lambda_m)}\}_{n=0}^{\infty}$, with parameters $\lambda_1,\ldots,\lambda_m$, satisfying

$$\begin{aligned}&xP_n^{(\lambda_1,\ldots,\lambda_m)}(x)\\&=P_{n+1}^{(\lambda_1,\ldots,\lambda_m)}(x)+a_nP_n^{(\lambda_1,\ldots,\lambda_m)}(x)+b_nP_{n-1}^{(\lambda_1,\ldots,\lambda_m)}(x),\quad n=0,1,\ldots,\end{aligned}$$

(here $a_n = a_n^{(\lambda_1,\ldots,\lambda_m)}$ and $b_n = b_n^{(\lambda_1,\ldots,\lambda_m)}$ are complex numbers), where b_n vanishes for some $n \geq 1$. This topic has attracted great interest in recent years. In 1995 K. H. Kwon and L. L. Littlejohn state in [6] the orthogonality of the Laguerre polynomials $\{L_n^{(-N)}\}_{n=0}^{\infty}$ for positive integers N. A year later, T. E.

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Pérez and M. A. Piñar [16] give an elegant and unified approach to the orthogonality of the (generalized) Laguerre polynomials $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$ for every real value of the parameter α. An adaptation to the discrete case of the technique of Pérez and Piñar was used in [1] to develop the non-standard orthogonality for generalized Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n=0}^{\infty}$ where $0 < \mu < 1$ (as usual) and where $\gamma \in \mathbb{R}$ (we recall that $\gamma > 0$ in classical considerations). The special case $\{M_n^{(-N,\mu)}\}_{n=0}^{\infty}$ $(N \in \{0, 1, 2, \ldots\})$, in which Favard's theorem can not ensure any orthogonality condition, is treated with special emphasis. In [13] we have given non-standard orthogonality for another discrete extension of the Laguerre polynomials: concretely, and after extending the classical family of Meixner-Pollaczek polynomials $\{P_n^{(\lambda)}(\cdot;\phi)\}_{n=0}^{\infty}$ to arbitrary complex values of the parameter λ, we introduce a non-standard discrete-continuous inner product that fills up the "Favard's gap" in the orthogonality scenery. With respect to the q-analogues of the Laguerre polynomials, the non-standard orthogonality of the families of big, little and continuous q-Laguerre polynomials can be found in [12, 14, 15]. In the case of the q-Laguerre polynomials $\{L_n^{(\alpha)}(\cdot;q)\}_{n=0}^{\infty}$, classically defined for $0 < q < 1$ and $\alpha > -1$, they satisfy both an orthogonality relation with respect to a discrete measure (as big and little q-Laguerre families do) and also an orthogonality relation with respect to an absolutely continuous measure (similarly to the case of continuous q-Laguerre polynomials). Our objective is to accomplish two kinds of "q-Sobolev orthogonality" for the (generalized) q-Laguerre polynomials $\{L_n^{(-N)}(\cdot;q)\}_{n=0}^{\infty}$ with negative integer parameter (by q-Sobolev orthogonality we mean orthogonality with respect to either a discrete-continuous or a purely discrete inner product involving q-derivatives).

The structure of the paper is the following. In Section 2 we recall some basic facts of classical q-Laguerre polynomials $\{L_n^{(\alpha)}(\cdot;q)\}_{n=0}^{\infty}$ where $\alpha > -1$. Section 3 is devoted to give a reasonable extension of classical q-Laguerre polynomials, through its hypergeometric representation, to arbitrary complex values of the parameter α. These so-called *generalized* polynomials satisfy exactly the same three term recurrence relation that the classical family, and nothing about orthogonality can be deduced from Favard's theorem in case $-\alpha \in \{1, 2, \ldots\}$. As auxiliary results, in Section 3 we will show that for a positive integer N and for $n \geq N$, the point 0 is a "q-zero" of precise order N of the polynomials $L_n^{(-N)}(\cdot;q)$, which means that $\left(D_q^k L_n^{(-N)}\right)(0;q) = 0$ for $0 \leq k \leq N-1$, and we will also state that $D_q^N L_n^{(-N)}(x;q)$ equals $L_{n-N}^{(0)}(q^N x;q)$, up to a multiplicative constant. In Section 4, by means of a discrete-continuous bilinear form, we define a non-standard inner product which provides the orthogonality of the generalized family of q-Laguerre polynomials with negative integer parameters. Should it be of interest to the readers, we add an appendix in which we fix all the notations, conventions and terminologies used throughout the paper.

2. The classical q-Laguerre polynomials

For $\alpha > -1$, (classical continuous) monic Laguerre polynomials (briefly, Laguerre polynomials) are defined by (see [17, 5.1.8, 5.3.3] and [9, 1.11.1, 1.11.4])

$$\begin{aligned} L_n^{(\alpha)}(x) &= (-1)^n(\alpha+1)_n\, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \,\middle|\, x \right) \\ &= \sum_{k=0}^{n} (-1)^n \binom{n}{k} (\alpha+1+k)_{n-k}(-x)^k, \qquad n \in \mathbb{N}_0. \end{aligned} \tag{2.1}$$

Meixner and Meixner-Pollaczek polynomials can be considered as discrete extensions of Laguerre polynomials ([9, 2.9.1], [9, 2.7.1]). Also, there are a number of q-extensions of Laguerre polynomials, one of which the so-called set of *q-Laguerre* polynomials $L_n^{(\alpha)}(\cdot;q)$ (introduced by W. Hahn [4] and treated in detail by D. S. Moak [10] with a different normalization we will adopt in this paper).

Fixed $0 < q < 1$ and $\alpha > -1$, the explicit form of the nth degree monic q-Laguerre polynomial reads ([10, 2.3], [5, 2.1, 2.3])

$$L_n^{(\alpha)}(x;q) = \frac{(-1)^n (q^{\alpha+1};q)_n}{(1-q)^n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (1-q)^k q^{k(k+1)/2}}{(q^{\alpha+1};q)_k (q;q)_k q^{(n-k)(\alpha+n)}}\, x^k, \quad n \in \mathbb{N}_0.$$

Using in the previous expression that $\lim_{q\uparrow 1}((q^{\beta};q)_n/(1-q)^n) = (\beta)_n$ for all real β and all nonnegative integer n, and also that $(\alpha+1)_n/(\alpha+1)_k = (\alpha+1+k)_{n-k}$, we find the well known fact (see [9, Remarks in p. 48])

$$\lim_{q\uparrow 1} L_n^{(\alpha)}(x;q) = (-1)^n \sum_{k=0}^{n} \frac{(-n)_k}{k!} (\alpha+1+k)_{n-k} x^k = L_n^{(\alpha)}(x), \quad n \in \mathbb{N}_0,$$

(this is why $L_n^{(\alpha)}(\cdot;q)$ is called a q-analogue of $L_n^{(\alpha)}$).

Monic q-Laguerre polynomials verify the three term recurrence relation ([10, 3.2], [5, 1.3, 2.4, 2.5])

$$L_{n+1}^{(\alpha)}(x;q) = \left(x - a_{q,n}^{(\alpha)}\right) L_n^{(\alpha)}(x;q) - b_{q,n}^{(\alpha)} L_{n-1}^{(\alpha)}(x;q), \qquad n \geq 0, \tag{2.2}$$

where

$$a_{q,n}^{(\alpha)} = \frac{1+q-q^{n+1}-q^{\alpha+1+n}}{(1-q)q^{\alpha+1+2n}}, \qquad b_{q,n}^{(\alpha)} = \frac{(1-q^n)(1-q^{\alpha+n})}{(1-q)^2 q^{2\alpha-1+4n}}. \tag{2.3}$$

One orthogonality relation is ([10, Theorem 1])

$$\begin{aligned} &\int_0^\infty L_m^{(\alpha)}(x;q) L_n^{(\alpha)}(x;q) \frac{x^\alpha}{(-(1-q)x;q)_\infty}\, dx \\ &\quad = \frac{\Gamma(\alpha+1)\Gamma(-\alpha)}{\Gamma_q(-\alpha)} \frac{(q^{\alpha+1};q)_n}{(1-q)^n} \frac{(q;q)_n}{(1-q)^n} q^{-2n(\alpha+n+\frac{1}{2})} \delta_{mn}, \quad m,n \in \mathbb{N}_0, \end{aligned} \tag{2.4}$$

where in the case of $\alpha = N \in \mathbb{N}_0$, $\Gamma(N+1)\Gamma(-N)/\Gamma_q(-N)$ must be understood as its analytic continuation ([8, p. 60])

$$\begin{aligned}\frac{\Gamma(N+1)\Gamma(-N)}{\Gamma_q(-N)} &= \lim_{\alpha\to N}\frac{\Gamma(\alpha+1)(-\alpha+N)\Gamma(-\alpha)}{(-\alpha+N)\Gamma_q(-\alpha)}\\ &= \frac{(-1)^N}{N!}\frac{\Gamma(N+1)(q^{-N};q)_N(-\ln q)}{(1-q)^{N+1}}\\ &= \frac{(q;q)_N(-\ln q)}{q^{N(N+1)/2}(1-q)^{N+1}}. \end{aligned}\tag{2.5}$$

We remark that $1/(-(1-q)x;q)_\infty = e_q(-(1-q)x) \to e^{-x}$ as $q \uparrow 1$ and also that $\Gamma_q(x) \to \Gamma(x)$ as $q \uparrow 1$, so the relation (2.4) can be interpreted as a q-analogue of the orthogonality condition for monic Laguerre polynomials (see [17, 5.1.1, 5.1.8] and [9, 1.11.2, 1.11.4])

$$\begin{aligned}\int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)x^\alpha e^{-x}\,dx &= \Gamma(\alpha+1)(\alpha+1)_n(1)_n\delta_{mn}\\ &= \Gamma(\alpha+1+n)n!\delta_{mn}, \qquad m,n\in\mathbb{N}_0.\end{aligned}$$

As shown by Moak, q-Laguerre polynomials not only fulfil an orthogonality relation for an absolutely continuous measure but also verifies an orthogonality relation for a discrete measure [10, Theorem 2]. Concretely, fixed $c > 0$,

$$\begin{aligned}&\sum_{k=-\infty}^{\infty} L_m^{(\alpha)}(cq^k;q)L_n^{(\alpha)}(cq^k;q)\frac{q^{k(\alpha+1)}}{(-c(1-q)q^k;q)_\infty C}\\ &= \frac{(q^{\alpha+1};q)_n}{(1-q)^n}\frac{(q;q)_n}{(1-q)^n}q^{-2n(\alpha+n+\frac{1}{2})}\delta_{mn}, \quad m,n\in\mathbb{N}_0,\end{aligned}\tag{2.6}$$

where the choice

$$C = \sum_{k=-\infty}^{\infty}\frac{q^{k(\alpha+1)}}{(-c(1-q)q^k;q)_\infty},$$

provides that the measure has total mass one.

3. Generalized q-Laguerre polynomials

For $0 < q < 1$ and $\alpha > -1$, the nth degree monic q-Laguerre polynomial $L_n^{(\alpha)}(\cdot;q)$ can be defined in terms of the q-hypergeometric series ${}_1\phi_1$ by means of

$$L_n^{(\alpha)}(x;q) = \frac{(-1)^n(q^{\alpha+1};q)_n}{q^{n(\alpha+n)}(1-q)^n}\,{}_1\phi_1\left(\begin{matrix} q^{-n}\\ q^{\alpha+1}\end{matrix}\,\middle|\, q; -q^{\alpha+1+n}(1-q)x\right),\tag{3.1}$$

where $n \in \mathbb{N}_0$. This definition coincides, up to a normalization constant, with that given in [10, 2.3] and in [3, p. 210], is exactly the same as in [5, 2.3], and replacing in our definition x by $(1-q)^{-1}x$ we have the one in [9, 3.21.1, 3.21.5]. Observe that the above representation fails to work when $-\alpha \in \mathbb{N}$. Our intention is to accomplish the extension for all $\alpha \in \mathbb{C}$.

Starting with (3.1), and using (5.1), we get

$$\begin{aligned} & L_n^{(\alpha)}(x;q) \\ = & \sum_{k=0}^{\infty} \frac{(-1)^n (q^{\alpha+1};q)_n}{q^{n(\alpha+n)}(1-q)^n} \frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k} (-1)^k q^{\binom{k}{2}} \frac{(-q^{\alpha+1+n}(1-q)x)^k}{(q;q)_k} \\ = & \sum_{k=0}^{\infty} (-1)^n \left((-1)^k q^{k(2n-k+1)/2} \frac{(q^{-n};q)_k}{(q;q)_k} \right) \left(\frac{(q^{\alpha+1};q)_n}{(q^{\alpha+1};q)_k (1-q)^{n-k}} \right) \\ & \times q^{(k-n)(\alpha+k+n)} (-x)^k. \end{aligned}$$

Taking into account that for $0 \le k \le n$

$$\begin{aligned} (-1)^k q^{k(2n-k+1)/2} \frac{(q^{-n};q)_k}{(q;q)_k} &= \begin{bmatrix} n \\ k \end{bmatrix}_q, \\ \frac{(q^{\alpha+1};q)_n}{(q^{\alpha+1};q)_k} &= (q^{\alpha+1+k};q)_{n-k}, \end{aligned}$$

and that $(q^{-n};q)_k$ vanishes for $k > n$, we get:

Definition 3.1. Let $\alpha \in \mathbb{C}$. For each $n \in \mathbb{N}_0$ we define the nth degree monic generalized q-Laguerre polynomial $L_n^{(\alpha)}(\cdot;q)$ by

$$L_n^{(\alpha)}(x;q) = \sum_{k=0}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{\alpha+1+k};q)_{n-k}}{(1-q)^{n-k}} \frac{1}{q^{(n-k)(\alpha+n+k)}} (-x)^k. \tag{3.2}$$

In a similar way, monic Laguerre polynomials can be extended from its hypergeometric representation to arbitrary complex values of the parameter α, yielding to the expression (2.1), now working for all complex values of the parameter α. Therefore, the relation (2.1) can be used to define the *generalized Laguerre polynomials* with arbitrary complex parameter.

From (3.2), and using

$$\lim_{q \uparrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}, \qquad \lim_{q \uparrow 1} \frac{(q^{\alpha+1+k};q)_{n-k}}{(1-q)^{n-k}} = (\alpha+1+k)_{n-k},$$

we obtain

$$\lim_{q \uparrow 1} L_n^{(\alpha)}(x;q) = L_n^{(\alpha)}(x), \qquad \alpha \in \mathbb{C}, \qquad n \in \mathbb{N}_0,$$

so generalized q-Laguerre polynomials can be interpreted as q-analogues of generalized Laguerre polynomials.

The remainder of this section is devoted to state that (new) generalized q-Laguerre polynomials share with the (original) "hypergeometric" q-Laguerre polynomials the same three term recurrence relation (2.2), (2.3) and also the same behavior with respect to the q-derivative operator. In relation with the roots, we will also state a radical newness that generalized q-Laguerre polynomials enjoy when the parameter α is a negative integer.

Proposition 3.1. *Let $\alpha \in \mathbb{C}$. The generalized monic q-Laguerre polynomials fulfil the three term recurrence relation*

$$L_{n+1}^{(\alpha)}(x;q) = \left(x - a_{q,n}^{(\alpha)}\right) L_n^{(\alpha)}(x;q) - b_{q,n}^{(\alpha)} L_{n-1}^{(\alpha)}(x;q), \qquad n \geq 0, \tag{3.3}$$

where

$$a_{q,n}^{(\alpha)} = \frac{1+q-q^{n+1}-q^{\alpha+1+n}}{(1-q)q^{\alpha+1+2n}}, \qquad b_{q,n}^{(\alpha)} = \frac{(1-q^n)(1-q^{\alpha+n})}{(1-q)^2 q^{2\alpha-1+4n}}. \tag{3.4}$$

Proof. Just introduce (3.2) and (3.4) in (3.3), and after some simplifications we establish the desired conclusion. □

The choice $\alpha \in \{-1,-2,-3,\ldots\}$ is the only one for which the coefficient $b_{q,n}^{(\alpha)}$ vanishes ($b_{q,-\alpha}^{(\alpha)} = 0$). Then, from Favard's theorem we know that the generalized family of q-Laguerre polynomials is orthogonal with respect to a quasi-definite moment functional if and only if $-\alpha \in \mathbb{C} \setminus \mathbb{N}$. The main result of this paper consists precisely in an orthogonality statement for these exceptional values of the parameter α.

The following result extends a similar well known one for classical q-Laguerre polynomials (confer [10, 4.16]).

Proposition 3.2. *Monic generalized q-Laguerre polynomials verify the q-difference relation*

$$D_q L_n^{(\alpha)}(x;q) = \frac{(1-q^n)}{(1-q)q^{n-1}} L_{n-1}^{(\alpha+1)}(qx;q), \qquad \alpha \in \mathbb{C}, \qquad n \geq 0. \tag{3.5}$$

Proof. For $n=0$, (3.5) trivially holds (recall that $P_{-1}=0$). If $n \geq 1$, then

$$\begin{aligned}
&D_q L_n^{(\alpha)}(x;q) \\
&= \sum_{k=0}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{\alpha+1+k};q)_{n-k}}{(1-q)^{n-k}} \frac{1}{q^{(n-k)(\alpha+k+n)}} \frac{(-x)^k - (-qx)^k}{(1-q)x} \\
&= \sum_{k=1}^{n} (-1)^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{\alpha+1+k};q)_{n-k}}{(1-q)^{n-k}} \frac{1}{q^{(n-k)(\alpha+k+n)}} \frac{(1-q^k)}{(1-q)} (-x)^{k-1} \\
&= \sum_{k=0}^{n-1} (-1)^{n-1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \frac{(q^{\alpha+2+k};q)_{n-k-1}}{(1-q)^{n-k}} \frac{(1-q^{k+1})}{q^{(n-k-1)(\alpha+k+1+n)}} (-x)^k.
\end{aligned}$$

We now apply

$$\begin{bmatrix} n \\ k+1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{(1-q^n)}{1-q^{k+1}}, \quad n \geq 1, \quad 0 \leq k \leq n-1,$$

to replace in the last expression of $D_q L_n^{(\alpha)}(x;q)$, to get finally

$$D_q L_n^{(\alpha)}(x;q) = \frac{(1-q^n)}{(1-q)q^{n-1}} \sum_{k=0}^{n-1} (-1)^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{(q^{(\alpha+1)+1+k};q)_{(n-1)-k}}{(1-q)^{(n-1)-k}}$$

$$\times \frac{1}{q^{((n-1)-k)((\alpha+1)+k+(n-1))}}(-qx)^k$$
$$= \frac{(1-q^n)}{(1-q)q^{n-1}} L_{n-1}^{(\alpha+1)}(qx;q). \qquad \square$$

It is suggestive the fact that relation $D_q L_n^{(\alpha)}(x;q) = [n]_q q^{1-n} L_{n-1}^{(\alpha+1)}(qx;q)$ is the q-analogue of its continuous version $DL_n^{(\alpha)}(x) = nL_{n-1}^{(\alpha+1)}(x)$, where $D = d/dx$ is the derivative operator (see [17, 5.1.14, 5.1.8] and [9, 1.11.6, 1.11.4]).

Iterating expression (3.5) it follows that:

Proposition 3.3. *Let $\alpha \in \mathbb{C}$. For a nonnegative integer n*

$$D_q^k L_n^{(\alpha)}(x;q) = \frac{(q^{n-k+1};q)_k}{(1-q)^k} q^{k(k-n)} L_{n-k}^{(\alpha+k)}(q^k x;q), \quad 0 \le k \le n+1.$$

When $\alpha > -1$, the family $\{L_n^{(\alpha)}(\cdot;q)\}_{n=0}^{\infty}$ is orthogonal with respect to a positive measure, so for a positive integer N, the roots of $L_n^{(N)}(\cdot;q)$ are real, simple and strictly positive. The behavior of the polynomials $L_n^{(-N)}(\cdot;q)$ are quite different: We will show that for $n \ge N$ the point 0 is a zero (also, a "q-zero") of precise order N of the polynomials $L_n^{(-N)}(\cdot;q)$.

The following result, stated without proof, can be found in [2, p. 980]. In fact, the reading of this relation gives us the key to adapt our technique in [11] to the case of q-Laguerre polynomials.

Proposition 3.4. *Let $N \in \mathbb{N}$. For each $n \ge N$,*

$$L_n^{(-N)}(x;q) = x^N L_{n-N}^{(N)}(x;q).$$

Proof. Fix $N \in \mathbb{N}$. From (3.2) we find, for each $n \in \mathbb{N}_0$,

$$L_n^{(-N)}(x;q) = \sum_{k=0}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{-N+1+k};q)_{n-k}}{(1-q)^{n-k}} \frac{1}{q^{(n-k)(-N+k+n)}}(-x)^k.$$

Since $(q^{-N+1+k};q)_{n-k} = (1-q^{-N+1+k})(1-q^{-N+2+k})\cdots(1-q^{-N+n})$, if $n \ge N$ (which implies that the exponent in the last factor in the q-shifted factorial is a non-negative integer), then for each $k \le N-1$ (which implies the non-positiveness of the exponent in the first factor) we have $(q^{-N+1+k};q)_{n-k} = 0$. Consequently, for $n \ge N$

$$\begin{aligned} & L_n^{(-N)}(x;q) \\ = & \sum_{k=N}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{-N+1+k};q)_{n-k}}{(1-q)^{n-k}} \frac{1}{q^{(n-k)(-N+n+k)}}(-x)^k \\ = & \sum_{k=0}^{n-N} (-1)^n \begin{bmatrix} n \\ N+k \end{bmatrix}_q \frac{(q^{k+1};q)_{n-N-k}}{(1-q)^{n-N-k}} \frac{1}{q^{(n-N-k)(n+k)}}(-x)^{N+k}. \end{aligned}$$

Using that

$$\begin{bmatrix} n \\ N+k \end{bmatrix}_q (q^{k+1};q)_{n-N-k} = \begin{bmatrix} n-N \\ k \end{bmatrix}_q (q^{N+1+k};q)_{n-N-k},$$

we simplify the above equation to

$$\begin{aligned} &L_n^{(-N)}(x;q) \\ &= x^N \sum_{k=0}^{n-N} (-1)^{n-N} \begin{bmatrix} n-N \\ k \end{bmatrix}_q \frac{(q^{N+1+k};q)_{n-N-k}}{(1-q)^{n-N-k}} \frac{1}{q^{(n-N-k)(N+n-N+k)}} (-x)^k \\ &= x^N L_{n-N}^{(N)}(x;q). \end{aligned}$$

□

The previous result implies that

$$\left.\frac{d^k}{dx^k} L_n^{(-N)}(x;q)\right|_{x=0} = 0, \quad 0 \le k \le N-1, \quad n \ge N.$$

In view of the fact that

$$(D_q^k f)(0) = \frac{(q;q)_k}{(1-q)^k} \frac{f^{(k)}(0)}{k!}, \quad k \ge 0, \tag{3.6}$$

for a function f analytic in a neighborhood of 0 (see [10, 6.8] and [7]), we also have:

Proposition 3.5. *Let N be a positive integer. For each $n \ge N$,*

$$\left(D_q^k L_n^{(-N)}\right)(0;q) = 0, \quad 0 \le k \le N-1. \tag{3.7}$$

4. q-Sobolev orthogonality of q-Laguerre polynomials with negative integer parameters

As in the previous sections, we will consider a fixed $q \in (0,1)$.

As we have stated in the previous section, no orthogonality results can be deduced from Favard's theorem for the family of generalized q-Laguerre polynomials $\{L_n^{(\alpha)}(\cdot;q)\}_{n=0}^\infty$ when the parameter α is a negative integer. By considering a suitable modification of our previous result [11, Theorem 3], adapted to the case of generalized q-Laguerre polynomials, we are going to give an orthogonality result for the family $\{L_n^{(-N)}(\cdot;q)\}_{n=0}^\infty$, where $N \in \mathbb{N}$. To achieve this aim we will use that for a positive integer N and for $n \ge N$, the point 0 is a "q-zero" of precise order N of the polynomials $L_n^{(-N)}(\cdot;q)$ (Proposition 3.5), and also that the Nth q-derivative of $L_n^{(-N)}(x;q)$ equals, up to a multiplicative constant, $L_{n-N}^{(0)}(q^N x;q)$ (Proposition 3.3).

Due to the fact that orthogonality for $\{L_n^{(-N)}(\cdot;q)\}_{n=N}^\infty$ will strongly depend on the orthogonality of $\{L_n^{(0)}(q^N\cdot;q)\}_{n=0}^\infty$, to the fact that the family $\{L_n^{(0)}(q^N\cdot;q)\}_{n=0}^\infty$ is orthogonal with respect to a positive measure supported on $[0,\infty)$, and to the fact that $L_n^{(-N)}(x;q)$ is real for real x, the natural choice is to describe our orthogonality result in the real setting.

Theorem 4.1. *For each positive integer N, there exists a symmetric and positive definite matrix A of order N such that the family of generalized q-Laguerre polynomials $\{L_n^{(-N)}(\cdot;q)\}_{n=0}^{\infty}$ is orthogonal with respect to the inner product $(\cdot,\cdot)_q^{(N;A)}$ defined by*

$$(4.1)\qquad (p_1,p_2)_q^{(N;A)} = \left(\left(D_q^k p_1(0)\right)_{k=0}^{N-1}\right) A \left(\left(D_q^k p_2(0)\right)_{k=0}^{N-1}\right)^t + \int_0^{\infty} \left(D_q^N p_1(x)\right)\left(D_q^N p_2(x)\right) \frac{q^N}{(-(1-q)q^N x;q)_\infty}\,dx, \quad p_1,p_2\in\mathbb{P}.$$

Proof. Let $\{l_j(\cdot;q)\}_{j=0}^{N-1}\subset\mathbb{P}_{N-1}$ be the set of polynomials defined by

$$l_j(x;q)=\frac{(1-q)^j}{(q;q)_j}x^j,\quad 0\le j\le N-1.$$

Since both $\{l_j(\cdot;q)\}_{j=0}^{N-1}$ and $\{L_j^{(-N)}(\cdot;q)\}_{j=0}^{N-1}$ are bases of $\mathbb{P}_{N-1}$, and taking into account that $(D_q^k l_j)(0;q)=\delta_{kj}$ (see (3.6)), we have

$$L_j^{(-N)}(x;q)=\sum_{k=0}^{N-1}(D_q^k L_j^{(-N)})(0;q)l_k(x;q),\quad 0\le j\le N-1.$$

Therefore, we can assure that the matrix $C=((D_q^k L_j^{(-N)})(0;q))_{j,k=0}^{N-1}$ is nonsingular. Also, if $B=(\kappa_j\delta_{jk})_{j,k=0}^{N-1}$ is a nonsingular diagonal matrix of order N ($\kappa_j\in\mathbb{R}\setminus\{0\}$), then the symmetric matrix $A=C^{-1}B^2(C^{-1})^t=(C^{-1}B)(C^{-1}B)^t$ is positive definite.

In order to state the orthogonality we will consider three cases:

i) In the case $0\le m,n\le N-1$, since

$$D_q^N L_m^{(-N)}(x;q)=D_q^N L_n^{(-N)}(x;q)=0,$$

we get

$$\begin{aligned}&(L_m^{(-N)}(\cdot;q),L_n^{(-N)}(\cdot;q))_q^{(N;A)}\\ &=\left(\left((D_q^k L_m^{(-N)})(0;q)\right)_{k=0}^{N-1}C^{-1}\right)B^2\left(\left((D_q^k L_n^{(-N)})(0;q)\right)_{k=0}^{N-1}C^{-1}\right)^t\\ &=((\delta_{mk})_{k=0}^{N-1})B^2((\delta_{nk})_{k=0}^{N-1})^t=\kappa_n^2\delta_{mn},\end{aligned}$$

where κ_j^2 is the (positive) (j,j) entry of the matrix B^2.

ii) If $0\le m\le N-1$ and $n\ge N$, then

$$D_q^N L_m^{(-N)}(x;q)=0,$$

and also (Proposition 3.5)

$$(D_q^k L_n^{(-N)})(0;q) = 0, \quad 0 \le k \le N-1,$$

so, clearly, we have

$$(L_m^{(-N)}(\cdot;q), L_n^{(-N)}(\cdot;q))_q^{(N;A)} = 0.$$

iii) Finally, when $m, n \ge N$, and using

$$(D_q^k L_m^{(-N)})(0;q) = (D_q^k L_n^{(-N)})(0;q) = 0, \quad 0 \le k \le N-1,$$

Proposition 3.3 yields

$$\begin{aligned}
&(L_m^{(-N)}(\cdot;q), L_n^{(-N)}(\cdot;q))_q^{(N;A)} \\
= \;& \int_0^\infty \left((D_q^N L_m^{(-N)})(x;q)\right)\left((D_q^N L_n^{(-N)})(x;q)\right) \frac{q^N}{(-(1-q)q^N x;q)_\infty}\,dx \\
= \;& c_{mn} \int_0^\infty L_{m-N}^{(0)}(q^N x;q) L_{n-N}^{(0)}(q^N x;q) \frac{q^N}{(-(1-q)q^N x;q)_\infty}\,dx \\
= \;& c_{mn} \int_0^\infty L_{m-N}^{(0)}(t;q) L_{n-N}^{(0)}(t;q) \frac{1}{(-(1-q)t;q)_\infty}\,dt,
\end{aligned}$$

where

$$c_{mn} = \frac{(q^{m-N+1};q)_N (q^{n-N+1};q)_N}{(1-q)^{2N} q^{N(m+n-2N)}}.$$

We apply the orthogonality condition for the q-Laguerrre polynomials (2.4) to get

$$\begin{aligned}
&(L_m^{(-N)}(\cdot;q), L_n^{(-N)}(\cdot;q))_q^{(N;A)} \\
= \;& c_{nn} \frac{-\ln q}{(1-q)} \left(\frac{(q;q)_{n-N}}{(1-q)^{n-N}}\right)^2 q^{-2(n-N)(n-N+1/2)} \delta_{mn} \\
= \;& \frac{-\ln q}{(1-q)} \left(\frac{(q;q)_n}{(1-q)^n} q^{(N-n)(n+1/2)}\right)^2 \delta_{mn}.
\end{aligned}$$

□

In order to have a closed form for the norm of the polynomials $L_n^{(-N)}(\cdot;q)$, a natural choice for the diagonal elements of the matrix D would be

$$\kappa_j = \sqrt{\frac{-\ln q}{(1-q)}} \frac{(q;q)_j}{(1-q)^j} q^{(N-j)(j+1/2)}, \qquad 0 \le j \le N-1.$$

This would imply

$$\|L_n^{(-N)}(\cdot;q))\|_q^{(N;A)} = \sqrt{\frac{-\ln q}{(1-q)}} \frac{(q;q)_n}{(1-q)^n} q^{(N-n)(n+1/2)}, \quad n \in \mathbb{N}_0.$$

The discrete orthogonality relation (2.6) allow us to give a discrete bilinear form with respect to which the family $\{L_n^{(-N)}(\cdot;q)\}_{n=0}^\infty$, $(N \in \mathbb{N})$ become orthogonal.

Theorem 4.2. *For each positive integer N and each $c > 0$, there exists a symmetric and positive definite matrix A of order N such that the family of generalized q-Laguerre polynomials $\{L_n^{(-N)}(\cdot; q)\}_{n=0}^{\infty}$ is orthogonal with respect to the inner product $(\cdot,\cdot)_{q,c}^{(N;A)}$ defined by*

(4.2)

$$\begin{aligned}
&(p_1, p_2)_{q,c}^{(N;A)} \\
&= \left(\left(D_q^k p_1(0)\right)_{k=0}^{N-1}\right) A \left(\left(D_q^k p_2(0)\right)_{k=0}^{N-1}\right)^t \\
&\quad + \sum_{k=-\infty}^{\infty} \left(D_q^N p_1(cq^{k-N})\right) \left(D_q^N p_2(cq^{k-N})\right) \frac{q^k}{(-c(1-q)q^k; q)_\infty C}, \quad p_1, p_2 \in \mathbb{P},
\end{aligned}$$

where $C = \sum_{k=-\infty}^{\infty} \frac{q^k}{(-c(1-q)q^k;q)_\infty}$.

The proof of the previous theorem is essentially the same as the one of Theorem 4.1.

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5. Appendix

The set of positive integers is denoted by $\mathbb{N}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers. The set of complex and real numbers is denoted, respectively, by $\mathbb{C}$ and $\mathbb{R}$. All polynomials considered are complex-valued in one complex variable x, and $\mathbb{P}$ stands for the set of all such polynomials. For each $n \in \mathbb{N}_0$, the subset of $\mathbb{P}$ of all polynomials of degree not greater than n is denoted by $\mathbb{P}_n$. By a system of *monic* polynomials we mean a sequence $\{P_n\}_{n=0}^{\infty}$ of polynomials fulfilling $P_n^{(n)}(x) = n!$, $x \in \mathbb{C}$, $n \in \mathbb{N}_0$ (this implies that $\deg P_n = n$ for $n \in \mathbb{N}_0$). Observe that for a sequence of monic polynomials, P_0 is the function defined by $P_0(x) = 1$, $x \in \mathbb{C}$. For notational convenience, we use P_{-1} to denote the null function ($P_{-1}(x) = 0$, $x \in \mathbb{C}$).

For $n \in \mathbb{N}$, a (square) matrix of order n, with complex entries a_{jk}, is denoted by $A = (a_{jk})_{j,k=0}^{n-1}$ (when needed, we use $\left((a_{jk})_{j,k=0}^{n-1}\right)$ instead of $(a_{jk})_{j,k=0}^{n-1}$), and $(a_j)_{j=0}^{n-1} \in \mathbb{C}^n$ stands for the matrix of order $1 \times n$ (equivalently, for the vector) $(a_0, a_1, \ldots, a_{n-1})$. The transpose of a matrix $A = (a_{jk})_{j,k=0}^{n-1}$ is denoted by using the superscript t. As usual, we will identify the only element of a matrix of order 1 with the matrix itself.

The Kronecker delta is denoted by δ_{jk}, and $(\cdot)_n$ denotes the so-called shifted factorial (also, Pochhammer symbol), defined by

$$(x)_0 = 1, \qquad (x)_{n+1} = x(x+1)\cdots(x+n), \quad n \in \mathbb{N}_0, \quad x \in \mathbb{C}.$$

The binomial coefficient for complex numbers α, β is

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}, \quad -\alpha, -\beta, -(\alpha-\beta) \notin \mathbb{N},$$

and the hypergeometric series ${}_mF_n$ is

$${}_mF_n\left(\begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_m)_k}{(b_1)_k \cdots (b_n)_k} \frac{x^k}{k!}, \quad m, n \in \mathbb{N}_0,$$

being the parameters b_j such that $(b_1)_k, \ldots, (b_n)_k \neq 0$ for all $k \in \mathbb{N}_0$. When $m = 0$ ($n = 0$) the numerator (denominator) of $(a_1)_k \cdots (a_m)_k/(b_1)_k \cdots (b_n)_k$ becomes 1. Clearly, if one of the numerator parameters fulfills $-a_j \in \mathbb{N}_0$, then the hypergeometric series is a polynomial of degree $\min_{1\leq j\leq m}\{-a_j : -a_j \in \mathbb{N}_0\}$.

The symbol $(\cdot; q)_n$ denotes the so-called q-shifted factorial, defined by

$$(x;q)_0 = 1, \qquad (x;q)_{n+1} = \prod_{k=0}^{n}(1 - xq^k), \quad n \in \mathbb{N}_0, \quad x \in \mathbb{C}.$$

We also define

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1 - xq^k), \quad x \in \mathbb{C}.$$

A q-analogue of the complex number x is $[x]_q = (1-q^x)/(1-q)$, since $\lim_{q\uparrow 1}[x]_q = x$. By a similar reason we consider $(q^x;q)_n/(1-q)^n$ as a q-analogue of $(x)_n$. The q-gamma function is defined by

$$\Gamma_q(x) = ((q;q)_\infty/(q^x;q)_\infty)(1-q)^{1-x},$$

and it is a q-analogue of the gamma function since $\lim_{q\uparrow 1}\Gamma_q(x) = \Gamma(x)$.

The q-binomial coefficient, for complex numbers α and β, is

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)}, \qquad -\alpha, -\beta, -(\alpha-\beta) \notin \mathbb{N},$$

and of course $\lim_{q\uparrow 1}\left[{\alpha \atop \beta} \right]_q = \binom{\alpha}{\beta}$ for $-\alpha, -\beta, -(\alpha-\beta) \notin \mathbb{N}$.

For $m, n \in \mathbb{N}_0$, the q-hypergeometric (also, basic hypergeometric) series ${}_m\phi_n$ is defined by

$$\begin{aligned} &{}_m\phi_n\left(\begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{array} \middle| q; x \right) \\ (5.1) \qquad &= \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_m;q)_k}{(b_1;q)_k \cdots (b_n;q)_k} \frac{(-1)^{(n-m+1)k}}{q^{(m-n-1)k(k-1)/2}} \frac{x^k}{(q;q)_k}, \end{aligned}$$

being the parameters b_j such that $(b_1;q)_k, \ldots, (b_n;q)_k \neq 0$ for all $k \in \mathbb{N}_0$. When $m = 0$ ($n = 0$) the numerator (denominator) of the coefficient

$$(a_1;q)_k \cdots (a_m;q)_k/(b_1;q)_k \cdots (b_n;q)_k$$

becomes 1. In case that one of the numerator parameters a_j equals q^{-n} for a nonnegative integer n, the q-hypergeometric series is a polynomial. We have

$$\lim_{q\uparrow 1} {}_m\phi_n\left(\begin{array}{c} q^{a_1},\dots,q^{a_m} \\ q^{b_1},\dots,q^{b_n}\end{array}\middle|\, q;(q-1)^{n-m+1}x\right) = {}_mF_n\left(\begin{array}{c} a_1,\dots,a_m \\ b_1,\dots,b_n\end{array}\middle|\, x\right),$$

and this relation states the q-analogy between the two hypergeometric series.

Basic hypergeometric series are used to define q-extensions of elementary and special functions. For example, one of the q-analogues of the exponential function reads

$$e_q(x) = {}_1\phi_0\left(\begin{array}{c} 0 \\ -\end{array}\middle|\, q;x\right) = \sum_{k=0}^{\infty}\frac{x^k}{(q;q)_k}.$$

As a consequence of the q-binomial theorem

$${}_1\phi_0\left(\begin{array}{c} a \\ -\end{array}\middle|\, q;x\right) = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \qquad |x|<1,$$

it follows that $e_q(x) = 1/(x;q)_\infty$ for $|x|<1$.

In closing, let us recall that the q-derivative operator $D_q : \mathbb{P}\to\mathbb{P}$ is defined by

$$D_q p(x) = \begin{cases} \dfrac{p(x)-p(qx)}{(1-q)x}, & x\neq 0, \\ p'(0), & x=0. \end{cases}$$

Clearly, for each $p\in\mathbb{P}$ we have $\lim_{q\uparrow 1} D_q p(x) = p'(x)$. Further, the nth iteration of the q-derivative operator is recursively defined by means of $D_q^0 = I$ (I is the identity operator) and $D_q^{n+1} = D_q\circ D_q^n$ for $n\in\mathbb{N}_0$.

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