

# ON THE GENERALIZED LAGUERRE POLYNOMIALS

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## 1. INTRODUCTION

In this note we shall study two classes of polynomials  $\{g_{n,m}^a(x)\}_{n \in \mathbb{N}}$  and  $\{h_{n,m}^a(x)\}_{n \in \mathbb{N}}$ . These polynomials are generalizations of Panda's polynomials (see [2], [3]). Also, these polynomials are special cases of the polynomials which were considered in [4] and [5]. For  $m = 1$ , the polynomials  $\{g_{n,m}^a(x)\}$  are the well-known Laguerre polynomials  $L_n^a(x)$  (see [6]), i.e.,

$$g_{n,1}^a(x) \equiv L_n^{a-1}(x). \quad (1.0)$$

In this paper the polynomials  $\{g_{n,m}^a(x)\}$  and  $\{h_{n,m}^a(x)\}$  are given by

$$F(x, t) = (1-t^m)^{-a} e^{-\frac{xt}{1-t^m}} = \sum_{n=0}^{\infty} g_{n,m}^a(x) t^n \quad (1.1)$$

and

$$G(x, t) = (1+t^m)^{-a} e^{-\frac{xt}{1+t^m}} = \sum_{n=0}^{\infty} h_{n,m}^a(x) t^n. \quad (1.2)$$

Using (1.1) and (1.2), we shall prove a great number of interesting relations for  $\{g_{n,m}^a(x)\}$  and  $\{h_{n,m}^a(x)\}$ , as well as some mixed relations.

## 2. RECURRENCE RELATIONS AND EXPLICIT REPRESENTATIONS

First we find two recurrence relations of the polynomials  $\{g_{n,m}^a(x)\}$ .

Differentiating (1.1) with respect to  $t$ , we get

$$\begin{aligned} \frac{\partial F(x, t)}{\partial t} &= (1-t^m)^{-a-1} e^{-\frac{xt}{1-t^m}} (amt^{m-1} - amt^{2m-1} - x - x(m-1)t^m) \\ &= (1-t^m) \sum_{n=1}^{\infty} n g_{n,m}^a(x) t^{n-1}. \end{aligned} \quad (2.1)$$

By (2.1) and from (1.1), we obtain the following recurrence relation:

$$\begin{aligned} n g_{n,m}^a(x) - (n-m) g_{n-m,m}^a(x) \\ = am(g_{n-m,m}^{a+1}(x) - g_{n-2m,m}^{a+1}(x)) - x(g_{n-1,m}^{a+1}(x) + (m-1)g_{n-1-m,m}^{a+1}(x)). \end{aligned} \quad (2.2)$$

Again, from (1.1) and (2.1), we get

$$\begin{aligned} n g_{n,m}^a(x) &= -x(g_{n-1,m}^a(x) + (m-1)g_{n-1-m,m}^a(x)) \\ &\quad + (m(a-2) + 2n)g_{n-m,m}^a(x) - (m(a-2) + n)g_{n-2m,m}^a(x), \quad n \geq 2m. \end{aligned} \quad (2.3)$$

**Corollary 2.1:** If  $m = 1$ , then (2.2) and (2.3) yield the corresponding relations for Laguerre polynomials:

$$nL_n^{a-1}(x) - (n-1)L_{n-1}^{a-1}(x) = (a-x)L_{n-1}^a(x) - aL_{n-2}^a(x)$$

and

$$nL_n^a(x) = (2n + a - 2 - x)L_{n-1}^a(x) - (n + a - 2)L_{n-2}^a(x), \quad n \geq 2.$$

In a similar way, from (1.2), we get the following relations:

$$nh_{n,m}^a(x) = (m - 1)xh_{n-1-m,m}^{a+2}(x) - amh_{n-m,m}^{a+1}(x) - xh_{n-1,m}^{a+2}(x), \quad n \geq m,$$

and

$$\begin{aligned} nh_{n,m}^a(x) &= x(m - 1)h_{n-1-m,m}^a(x) - xh_{n-1,m}^a(x) \\ &\quad - (2n + am - 2m)h_{n-m,m}^a(x) - (n + am - 2m)h_{n-2m,m}^a(x), \quad n \geq m. \end{aligned}$$

Starting from (1.1) and (1.2), we get the following explicit representations of the polynomials  $\{g_{n,m}^a(x)\}$  and  $\{h_{n,m}^a(x)\}$ , respectively:

$$g_{n,m}^a(x) = \sum_{i=0}^{[n/m]} \frac{(-1)^{n-mi} (a+n-mi)_i}{i!(n-mi)!} x^{n-mi} \tag{2.4}$$

and

$$h_{n,m}^a(x) = \sum_{i=0}^{[n/m]} \frac{(-1)^{n-(m-1)i} (a+n-mi)_i}{i!(n-mi)!} x^{n-mi} \tag{2.5}$$

**Corollary 2.2:** If  $m = 1$ , then (2.6) is the explicit representation of the Laguerre polynomials:

$$L_n^{a-1}(x) = \sum_{i=0}^n \frac{(-1)^{n-i} (a+n-i)_i}{i!(n-i)!} x^{n-i}.$$

Now, differentiating (1.1) with respect to  $x$ , we get

$$Dg_{n,m}^a(x) = -g_{n-1,m}^{a+1}(x), \quad n \geq 1. \tag{2.6}$$

If we differentiate (2.6), with respect to  $x$ ,  $k$  times, we obtain

$$D^k g_{n,m}^a(x) = (-1)^k g_{n-k,m}^{a+k}(x), \quad n \geq k. \tag{2.7}$$

**Corollary 2.3:** Using the idea in [1], from (2.2) and (2.6), we get

$$(n - xD)g_{n,m}^a(x) = (n - m + x(m - 1)D)g_{n-m,m}^a(x) + amD(g_{n+1-2m,m}^a(x) - g_{n+1-m,m}^a(x)).$$

For  $m = 1$  in the last equality and from (1.0), we get

$$(n + (a - x)D)L_n^{a-1}(x) = (n - 1 + aD)L_{n-1}^{a-1}(x).$$

In a similar way, from (1.2), we have

$$Dh_{n,m}^a(x) = -h_{n-1,m}^{a+1}(x)$$

and

$$D^s h_{n,m}^a(x) = (-1)^s h_{n-s,m}^{a+s}(x), \quad n \geq s.$$

### 3. SOME IDENTITIES OF THE CONVOLUTION TYPE

In this section we shall prove some interesting identities related to  $\{g_{n,m}^a(x)\}$  and  $\{h_{n,m}^a(x)\}$ .

First, from (1.1), we find

$$F(x, t) \cdot F(y, t) = (1 - t^m)^{-2a} e^{-\frac{(x+y)t}{1-t^m}} = \sum_{n=0}^{\infty} g_{n,m}^{2a}(x+y)t^n, \tag{3.1}$$

whence we get

$$\sum_{i=0}^n g_{n-i,m}^a(x) g_{i,m}^a(y) = g_{n,m}^{2a}(x+y).$$

**Theorem 3.1:** The following identities hold:

$$g_{n,m}^{2a}(x) = \sum_{j=0}^{[n/m]} \sum_{i=0}^{n-mj} \frac{y^{n-i-mj} (n-i-mj)_j}{j!(n-i-mj)!} g_{i,m}^{2a}(x+y); \tag{3.2}$$

$$\sum_{i=0}^n D^s g_{n-i,m}^a(x) D^s g_{i,m}^a(y) = g_{n-2s,m}^{2a+2s}(x+y), \quad n \geq 2s; \tag{3.3}$$

$$\sum_{i=0}^n D^k g_{n-i,m}^a(x) D^k h_{i,m}^a(x) = g_{n-2k,2m}^{a+k}(2x), \quad n \geq 2k; \tag{3.4}$$

$$\sum_{i=0}^{[(n-k)/m]} \frac{(k)_i}{i!} g_{n-k-mi,2m}^a(2x) = (-1)^k \sum_{i=0}^n g_{n-i-k,m}^{a+k}(x) h_{i,m}^a(x); \tag{3.5}$$

$$\sum_{i=0}^{[(n-k)/m]} (-1)^i \frac{(k)_i}{i!} g_{n-k-mi,2m}^a(2x) = (-1)^k \sum_{i=0}^n h_{n-i-k,m}^{a+k}(x) g_{i,m}^a(x); \tag{3.6}$$

$$\sum_{i=0}^n g_{n-i,m}^a(x) g_{i,m}^b(x) = g_{n,m}^{a+b}(2x). \tag{3.7}$$

**Proof:** From (3.1), we have

$$(1 - t^m)^{-2a} e^{-\frac{xt}{1-t^m}} = e^{-\frac{yt}{1-t^m}} \sum_{n=0}^{\infty} g_{n,m}^{2a}(x+y)t^n,$$

whence

$$\sum_{n=0}^{\infty} g_{n,m}^{2a}(x)t^n = \left( \sum_{n=0}^{\infty} \frac{y^n t^n}{n!} \right) \left( \sum_{k=0}^{\infty} \binom{-n}{k} (-t^m)^k \right) \left( \sum_{n=0}^{\infty} g_{n,m}^{2a}(x+y)t^n \right).$$

Multiplying the series on the right side, then comparing the coefficients to  $t^n$ , by the last equality we get (3.2).

If we differentiate (1.1)  $s$  times, with respect to  $x$ , we find

$$\frac{\partial^s F(x, t)}{\partial x^s} = (-1)^s t^s (1 - t^m)^{-a-s} e^{-\frac{xt}{1-t^m}}. \tag{a}$$

From (a), we get

$$\frac{\partial^s F(x, t)}{\partial x^s} \cdot \frac{\partial^s F(y, t)}{\partial y^s} = \sum_{n=0}^{\infty} g_{n,m}^{2a+2s}(x+y)t^{n+2s}. \tag{i}$$

Since

$$\frac{\partial^s F(x, t)}{\partial x^s} \cdot \frac{\partial^s F(y, t)}{\partial y^s} = \sum_{n=0}^{\infty} \sum_{i=0}^n D^s g_{n-i,m}^a(x) D^s g_{i,m}^a(y)t^n,$$

and, from (i), it follows that

$$\sum_{i=0}^n D^s g_{n-i,m}^a(x) D^s g_{i,m}^a(y) = g_{n-2s,m}^{2a+as}(x+y), \quad n \geq 2s.$$

The last identity is the desired identity (3.3).

Differentiating (1.2)  $k$  times, with respect to  $x$ , we get

$$\frac{\partial^k G(x,t)}{\partial x^k} = (-1)^k t^k (1+t^m)^{-a-k} e^{-\frac{xt}{1+t^m}}. \tag{b}$$

Then, from (a) and (b), we find

$$\frac{\partial^k F(x,t)}{\partial x^k} \cdot \frac{\partial^k G(x,t)}{\partial x^k} = \sum_{n=0}^{\infty} g_{n,2m}^{a+k}(2x) t^{n+2k}. \tag{ii}$$

The left side of (ii) yields

$$\frac{\partial^k F(x,t)}{\partial x^k} \cdot \frac{\partial^k G(x,t)}{\partial x^k} = \sum_{n=0}^{\infty} \sum_{i=0}^n D^k g_{n-i,m}^a(x) D^k h_{i,m}^a(x) t^n. \tag{iii}$$

So, from (ii) and (iii), we get (3.4).

In a similar way, starting from (1.1) and (1.2), we can prove identity (3.5). From (1.1) and (b), we can prove identity (3.6).

In the proof identity (3.7), we start from

$$F^a(x,t) = (1-t^m)^{-a} e^{-\frac{xt}{1-t^m}}, \quad \text{by (1.1),}$$

and

$$F^b(x,t) = (1-t^m)^{-b} e^{-\frac{xt}{1-t^m}}, \quad \text{by (1.1).}$$

So, we obtain

$$F^a(x,t) \cdot F^b(x,t) = \sum_{n=0}^{\infty} g_{n,m}^{a+b}(2x) t^n.$$

On the other side, we have

$$\left( \sum_{n=0}^{\infty} g_{n,m}^a(x) t^n \right) \left( \sum_{n=0}^{\infty} g_{n,m}^b(x) t^n \right) = \sum_{n=0}^{\infty} g_{n,m}^{a+b}(2x) t^n.$$

Identity (3.7) follows by the last equality and the proof of Theorem 3.1 is completed.

**Corollary 3.1:** If  $m = 1$  in (3.2), (3.3), and (3.7), then we get

$$L_n^{2a-1}(x) = \sum_{j=0}^n \sum_{i=0}^{n-j} \frac{y^{n-i-j} (n-i-j)_i}{j!(n-i-j)!} L_i^{2a-1}(x+y),$$

$$\sum_{i=0}^n D^s L_{n-i}^{a-1}(x) D^s L_i^{a-1}(y) = L_{n-2s}^{2a+2s-1}(x+y),$$

and

$$\sum_{i=0}^n L_{n-i}^{a-1}(x) L_i^{b-1}(x) = L_n^{a+b-1}(2x),$$

respectively.

Furthermore, we shall prove some more general results.

**Theorem 3.2:**

$$\sum_{i_1+\dots+i_k=n} g_{i_1,m}^{a_1}(x_1) \cdots g_{i_k,m}^{a_k}(x_k) = g_{n,m}^{a_1+\dots+a_k}(x_1+\dots+x_k); \tag{3.8}$$

$$\sum_{i_1+\dots+i_k=n} h_{i_1,m}^{a_1}(x_1) \cdots h_{i_k,m}^{a_k}(x_k) = h_{n,m}^{a_1+\dots+a_k}(x_1+\dots+x_k); \tag{3.9}$$

$$\begin{aligned} & \sum_{s=0}^n \sum_{i_1+\dots+i_k=n-s} g_{i_1,m}^a(x_1) \cdots g_{i_k,m}^a(x_k) \cdot \sum_{j_1+\dots+j_k=s} h_{j_1,m}^a(x_1) \cdots h_{j_k,m}^a(x_k) \\ &= \sum_{i_1+\dots+i_k=n} g_{i_1,2m}^a(2x_1) \cdots g_{i_k,2m}^a(2x_k). \end{aligned} \tag{3.10}$$

**Proof:** From (1.1), we get

$$F^{a_1}(x_1, t) \cdots F^{a_k}(x_k, t) = \sum_{n=0}^{\infty} g_{n,m}^{a_1+\dots+a_k}(x_1+\dots+x_k) t^n.$$

Further, we have the following identity:

$$\sum_{n=0}^{\infty} \sum_{i_1+\dots+i_k=n} g_{i_1,m}^{a_1}(x_1) \cdots g_{i_k,m}^{a_k}(x_k) t^n = \sum_{n=0}^{\infty} g_{n,m}^{a_1+\dots+a_k}(x_1+\dots+x_k) t^n.$$

Identity (3.8) follows immediately from the last equality. In a similar way, from (1.2), we can prove (3.9).

Now we shall prove (3.10). From (1.1) and (1.2), we have

$$F(x_1, t) \cdots F(x_k, t) \cdot G(x_1, t) \cdots G(x_k, t) = (1-t^{2m})^{-ka} e^{\frac{2(x_1+\dots+x_k)t}{1-t^2m}}.$$

So we get

$$\begin{aligned} & \left( \sum_{n=0}^{\infty} \sum_{i_1+\dots+i_k=n} g_{i_1,m}^a(x_1) \cdots g_{i_k,m}^a(x_k) t^n \right) \cdot \left( \sum_{n=0}^{\infty} \sum_{j_1+\dots+j_k=n} h_{j_1,m}^a(x_1) \cdots h_{j_k,m}^a(x_k) t^n \right) \\ &= \sum_{n=0}^{\infty} g_{n,2m}^{ka}(2x_1+\dots+2x_k) t^n. \end{aligned}$$

Comparing the coefficients to  $t^n$  in the last equality, we get (3.10) and the proof of Theorem 3.2 is completed.

**Corollary 3.2:** If  $m = 1$ , using (1.0), then (3.8) becomes

$$\sum_{i_1+\dots+i_k=n} L_{i_1}^{a_1-1}(x) \cdots L_{i_k}^{a_k-1}(x) = L_n^{a_1+\dots+a_k-1}(x_1+\dots+x_k).$$

**Corollary 3.3:** If  $x_1 = x_2 = \dots = x_k = x$  and  $a_1 = a_2 = \dots = a_k = a$ , then (3.8) becomes

$$\sum_{i_1+\dots+i_k=n} g_{i_1,m}^a(x) \cdots g_{i_k,m}^a(x) = g_{n,m}^{ka}(kx). \tag{3.11}$$

**Corollary 3.4:** If  $m = 1$ , then (3.11) yields

$$\sum_{i_1+\dots+i_k=n} L_{i_1}^{a-1}(x) \cdots L_{i_k}^{a-1}(x) = L_n^{ka-1}(kx).$$

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**Errata for "Generalizations of Some Identities Involving the Fibonacci Numbers"**

by **Fengzhen Zhao & Tianming Wang**

*The Fibonacci Quarterly* **39.2** (2001):165-167

On page 166, (10) should be

$$\sum_{a+b+c=n} U_{ak}U_{bk}U_{ck} = \frac{U_k^2}{2(V_k^2 - 4q^k)^2} ((n-1)(n-2)V_k^2 U_{nk} - q^k V_k (4n^2 - 6n - 4)U_{(n-1)k} + q^{2k} (4n^2 - 4)U_{(n-2)k}), \quad n \geq 2.$$

Hence, on page 167, (13) should be

$$\sum_{a+b+c=n} F_{ak}F_{bk}F_{ck} = \frac{F_k^2}{2(L_k^2 - 4(-1)^k)^2} ((n-1)(n-2)L_k^2 F_{nk} - (-1)^k L_k (4n^2 - 6n - 4)F_{(n-1)k} + (4n^2 - 4)F_{(n-2)k}), \quad n \geq 2.$$

In the meantime, line 14 and line 16 of page should be, respectively,

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50} (9(n-1)(n-2)F_{2n} - 3(4n^2 - 6n - 4)F_{2n-2} + (4n^2 - 4)F_{2n-4}).$$

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50} ((15n^2 - 63n + 66)F_{2n-3} + (10n^2 - 36n + 44)F_{2n-4}).$$

Line 19 of page 167 should be:  $+ (4n^2 - 4)P_{(n-2)k}), \quad n \geq 2.$