q-LAGUERRE POLYNOMIALS AND BIG q-BESSEL FUNCTIONS AND THEIR ORTHOGONALITY RELATIONS

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Abstract. The q-Laguerre polynomials correspond to an indetermined moment problem. For explicit discrete non-N-extremal measures corresponding to Ramanujan's ${}_1\psi_1$-summation we complement the orthogonal q-Laguerre polynomials into an explicit orthogonal basis for the corresponding L^2-space. The dual orthogonal system consists of so-called big q-Bessel functions, which can be obtained as a rigorous limit of the orthogonal system of big q-Jacobi polynomials. Interpretations on the $SU(1,1)$ and $E(2)$ quantum groups are discussed.

1. Introduction.

This paper answers two seemingly different questions, which both originated from quantum groups. First, consider the system of Moak's [21] q-Laguerre polynomials with respect to their familiar discrete orthogonality measure, and extend this in an explicit way to a complete orthogonal system of eigenfunctions of a doubly infinite Jacobi matrix originating from analysis on $SU_q(1,1)$, the quantum $SU(1,1)$ group. Second, obtain orthogonality relations and dual orthogonality relations for certain (big) q-Bessel functions originating (see [4]) on $E_q(2)$, the quantum group of plane motions, and give rigorous proofs of these orthogonalities. The two questions are related because the dual orthogonality relations for the big q-Bessel functions turn out to be the orthogonality relations for the completed q-Laguerre polynomials.

It is well known that the moment problem corresponding to Moak's q-Laguerre polynomials is indetermined as a Stieltjes moment problem. Moak [21, Thm. 1, 2] gives several orthogonality measures for the q-Laguerre polynomials: an absolutely continuous measure on $[0,\infty)$ and purely discrete measures supported on the set $\{cq^k \mid k\in\mathbb{Z}\}$ for any constant $c>0$, see §4 for the explicit weights. See also Ismail and Rahman [10] for the explicit calculation of the entire functions in the Nevanlinna parametrisation of the orthogonality measures for the moment problem for the q-Laguerre polynomials. We are interested in the discrete orthogonality measures. From the general theory of orthogonal polynomials it follows that the polynomials are not dense in the corresponding space of quadratically

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integrable functions, since the support is not the set of zeros of an entire function. This is expressed by saying that the measure is not N-extremal, see [1] for more information on moment problems and orthogonal polynomials. In this paper we complement the q-Laguerre polynomials to an orthogonal basis of the L^2-space for the discrete measure by using a certain q-analogue of the Bessel function of order α. The dual basis functions can be recognized as big q-Bessel functions, see below.

q-Analogues of Bessel functions exist in several sorts. Most well-known and probably the oldest ones are Jackson's first and second q-Bessel functions, see Ismail [8]. They occur in many places, including the present paper, but they have the draw-back that they do not form an orthogonal system (possibly they form a biorthogonal system). Other q-analogues of Bessel functions can be obtained as formal limit cases of the three q-analogues of Jacobi polynomials, i.e., of little q-Jacobi polynomials, big q-Jacobi polynomials and Askey-Wilson polynomials. For this reason we propose to speak about little q-Bessel functions, big q-Bessel functions and AW type q-Bessel functions for the corresponding limit cases. Little, big and AW type q-Bessel functions have interpretations on $E_q(2)$, completely analogous to the interpretations of little and big q-Jacobi polynomials and Askey-Wilson polynomials on $SU_q(2)$, the quantum $SU(2)$ group, see Vaksman and Korogodskiĭ [23], Koelink [14], [15], Bonechi et al. [4]. See also Ismail et al. [9] for the Fourier-Bessel transform for the AW type q-Bessel function. The duals of the AW-type q-Bessel functions can also be viewed as the little q-Jacobi functions, which live on $SU_q(1,1)$, see Masuda et al. [20]. These three types of q-Bessel functions satisfy orthogonality relations which can be obtained as formal limits of the corresponding orthogonality relations for q-Jacobi polynomials. In the q-Bessel case it is not sufficient to give orthogonality relations, but one also has to prove completeness, either directly or by giving dual orthogonality relations. For rigorous proofs of orthogonality and completeness we mention three different techniques: (i) spectral theoretic methods, (ii) direct proofs by use of generating functions, (iii) rigorous limit transitions from the orthogonal polynomials case. Usually, only the first method yields both orthogonality and completeness. The second method was used for little q-Bessel functions by Koornwinder and Swarttouw [18], and was sufficient there because of self-duality. The first method was used for little q-Jacobi functions by Kakehi, Masuda and Ueno [11], [12]. In the present paper we give two different proofs, first by the spectral method, and next by method (ii) for the extension of the q-Laguerre orthogonality and by method (iii) for the big q-Bessel orthogonality.

The spectral method, developed in sections 2 and 3, is based on the spectral analysis of a doubly infinite Jacobi matrix, see e.g. Masson and Repka [19]. This operator arises in a natural way from $SU_q(1,1)$. The spectral analysis is very similar to the spectral analysis for second order differential equations (see [6]), such as for the second order differential equation satisfied by the Jacobi functions (see a survey of these functions in [17]). The support of the spectral measure is determined by the zeros of a c-function. We calculate the Green function and we obtain the spectral measure for the doubly infinite Jacobi matrix from the Green function. The spectral measure is discrete and its support falls into two sets; one set corresponding to eigenvectors in terms of the q-Laguerre polynomials and the other set corresponding to eigenvectors in terms of the q-Bessel coefficients. It should be noted that this method corresponds to the one employed by Kakehi, Masuda and Ueno [11],

[12] giving the Plancherel measure for the little q-Jacobi functions from its interpretation as matrix elements of irreducible representations of $SU_q(1,1)$.

The orthogonality relations and squared norms resulting from the spectral analysis are explicitly given in §4. Since we give a basis for this space we also obtain the dual orthogonality relations as an immediate consequence. Using a method of Berg [3] we can easily construct more non-N-extremal orthogonality measures for the q-Laguerre polynomials on the same set $\{cq^k \mid k \in \mathbb{Z}\}$ by perturbing with any of the q-Bessel functions, which is bounded on this set.

In §5 we give a straightforward proof of the orthogonality relations for the q-Laguerre polynomials and q-Bessel functions using two generating functions for the q-Bessel functions. This technique is motivated by [18], [13].

In §6 we obtain the orthogonality relations for the big q-Bessel functions as a rigorous limit of the big q-Jacobi polynomial case. The formal limit is suggested by the limit transition of $SU_q(2)$ group to $E_q(2)$. Big q-Jacobi polynomials have an interpretation as basis elements for the regular representation of $SU_q(2)$ on quantum spheres, see [22]. In the limit transition these basis elements tend to the corresponding basis elements for the regular representation of $E_q(2)$ on quantum hyperboloids, and these latter basis elements can be written as big q-Bessel functions, see [4].

So we see that the same result on q-special functions can be obtained from two different quantum group interpretations: by considering the quantum $SU(1,1)$ group or the quantum group of plane motions. This is also the case for the little q-Jacobi functions studied in [11], [12], which originated as spherical functions on $SU_q(1,1)$, but which can also be viewed as certain q-analogues of Bessel functions and then have an interpretation on the quantum group of plane motions, see [15].

Notation. We follow the notation of Gasper and Rahman [7] for basic (or q-)hypergeometric series. Throughout we assume that $0 < q < 1$.

2. Solutions to a symmetric operator.

Consider the unbounded operator L acting on $\ell^2(\mathbb{Z})$ by

$$
\begin{aligned}
\big(Lu\big)_k &= a_k\, u_{k+1} + b_k\, u_k + a_{k-1}\, u_{k-1}, \qquad u = (u_k)_{k\in\mathbb{Z}}, \\
a_k &= q^{-\frac{1}{2}(k+1)}\sqrt{1+c^{-1}q^{-k}}, \qquad b_k = \sqrt{c^{-1}}(t+t^{-1})q^{-k},
\end{aligned} \tag{2.1}
$$

where $c > 0$ and $t \in \mathbb{R}\backslash\{0\}$ are fixed constants, so that $a_k > 0$ and $b_k \in \mathbb{R}$. The operator is densely defined and symmetric. Split the operator L into two Jacobi matrices J_+ and J_-, see [19]. The coefficients a_k are bounded as $k \to -\infty$, so that the moment problem corresponding to J_- is determined. So the deficiency indices for L are either $(0,0)$ or $(1,1)$. Take $x \in \mathbb{C}\backslash\mathbb{R}$. From the theory of orthogonal polynomials we see that the space of solutions of $Lu = x\,u$ which are ℓ^2 for $k \to -\infty$ is one-dimensional. For the space of solutions of $Lu = x\,u$ which are ℓ^2 for $k \to \infty$ there are two possibilities: (i) the space is one-dimensional if the moment problem for J_+ is determined and in that case L is self-adjoint; (ii) the space is two-dimensional if the moment problem for J_+ is indetermined and in that case L has a one-parameter family of self-adjoint extensions. See Akhiezer [1], Dunford and Schwartz [6, Ch. XII] and Masson and Repka [19] for more information.

Note that, for $k \to \infty$,

$$a_k \pm b_k + a_{k-1} = c^{-\frac{1}{2}} q^{-k}\big(q^{\frac{1}{2}} + q^{-\frac{1}{2}} \pm (t+t^{-1})\big) + q^{-\frac{1}{2}}\sqrt{c} + \mathcal{O}(q^{k/2})$$

is bounded from above if $\mp t \geq q^{-\frac{1}{2}}$ or $0 < \mp t \leq q^{\frac{1}{2}}$. Hence we can use the criterion in [2, Ch. VII] to see that J_+ corresponds to a determined moment problem, and hence L is self-adjoint, if $|t| \geq q^{-\frac{1}{2}}$ or $0 < |t| \leq q^{\frac{1}{2}}$.

Remark 2.1. The motivation to consider the operator L comes from an investigation in the Hopf $*$-algebra related to the quantum $SU(1,1)$ group. This is a $*$-algebra with two generators α and γ such that for $0<q<1$ the following relations hold:

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha\alpha^* - q^2\gamma\gamma^* = 1 = \alpha^*\alpha - \gamma^*\gamma.$$

We can represent this $*$-algebra in terms of unbounded operators in $\ell^2(\mathbb{Z})$ by

$$\pi(\gamma)\, e_k = \lambda q^k\, e_k, \qquad \pi(\alpha)\, e_k = \sqrt{1+|\lambda|^2 q^{2k}}\, e_{k-1},$$

for the standard orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ of $\ell^2(\mathbb{Z})$ and where $\lambda \in \mathbb{C}\backslash\{0\}$. Study of the self-adjoint element $\rho = \alpha^*\gamma^* + \gamma\alpha + (t+t^{-1})\gamma\gamma^*$ in these representations leads to the operator in (2.1) after some normalisation. Then we might expect that the spectral resolution of L will give us information on how to construct a possible Haar functional on the subalgebra generated by ρ. For the quantum $SU(2)$ group this is completely rigorous, cf. [16, §5], and we may consider (2.1) as a non-terminating version of the three-term recurrence relation for the orthonormal Al-Salam and Carlitz polynomials.

For $x \in \mathbb{C}\backslash\{0\}$ define

$$V_k^t(x) = \sqrt{(-q^{1-k}/c;q)_\infty}\; q^{\frac{1}{4}k(k+1)} \big(-t\sqrt{c}\big)^k (qt^2;q)_\infty\; {}_1\varphi_1\left(\begin{matrix} -t\sqrt{c}/x \\ qt^2 \end{matrix}; q, xtq^{k+1}\sqrt{c}\right), \tag{2.2}$$

$$U_k(x) = \sqrt{(-q^{1-k}/c;q)_\infty}\; q^{\frac{1}{4}k(k+1)} x^k\; {}_2\varphi_1\left(\begin{matrix} -\sqrt{c}/(tx), -\sqrt{c}t/x \\ 0 \end{matrix}; q, -\frac{q^{1-k}}{c}\right). \tag{2.3}$$

By [7, (4.3.2)] we have for $\pm t \notin \{q^{\frac{1}{2}m} \mid m \in \mathbb{Z}\}$:

$$\begin{gathered} U_k(x) = C_t\, c_t(x)\, V_k^t(x) + C_{t^{-1}}\, c_{t^{-1}}(x)\, V_k^{t^{-1}}(x), \\ c_t(x) = (-\sqrt{c}/xt, qt/x\sqrt{c}, x\sqrt{c}/t;q)_\infty, \qquad C_t^{-1} = (qt^2, t^{-2}, -c, -q/c;q)_\infty. \end{gathered} \tag{2.4}$$

In case $\pm t \in \{q^{\frac{1}{2}m} \mid m \in \mathbb{Z}\}$ we use that for $p \in \mathbb{Z}$ we have

$$(q^{1-p};q)_\infty\; {}_1\varphi_1\left(\begin{matrix} aq^{-p} \\ q^{1-p} \end{matrix}; q, z\right) = \frac{(q/a;q)_\infty}{(q^{p+1}/a;q)_\infty} (azq^{-1})^p (q^{1+p};q)_\infty\; {}_1\varphi_1\left(\begin{matrix} a \\ q^{1+p} \end{matrix}; q, zq^p\right). \tag{2.5}$$

Formula (2.5) is meaningful and can be proved by interpreting its left-hand side for $p \in \mathbb{Z}_{>0}$ as

$$\sum_{k=0}^{\infty} \frac{(aq^{-p};q)_k\, (q^{1-p+k};q)_\infty\, q^{\frac{1}{2}k(k-1)}\, (-z)^k}{(q;q)_k}.$$

From (2.5) we obtain that

$$V_k^{\pm q^{-\frac{1}{2}m}}(x) = (-c)^m \frac{(\mp q^{1-\frac{1}{2}m} x/\sqrt{c};q)_\infty}{(\mp q^{1+\frac{1}{2}m} x/\sqrt{c};q)_\infty} V_k^{\pm q^{\frac{1}{2}m}}(x). \tag{2.6}$$

Lemma 2.2. *Let $x \in \mathbb{C}\backslash\{0\}$, then $V^t(x) = \big(V^t_k(x)\big)_{k\in\mathbb{Z}}$, $V^{t^{-1}}(x) = \big(V^{t^{-1}}_k(x)\big)_{k\in\mathbb{Z}}$ and $U(x) = \big(U_k(x)\big)_{k\in\mathbb{Z}}$ are solutions to $Lu = x\,u$. Furthermore, $V^t(x)$ is ℓ^2 as $k \to \infty$ if and only if $|t| < q^{-\frac12}$ or $t = \pm q^{-\frac12 m}$, $m \in \mathbb{Z}_{\geq 0}$, and $V^{t^{-1}}(x)$ is ℓ^2 as $k \to \infty$ if and only if $|t| > q^{\frac12}$ or $t = \pm q^{\frac12 m}$, $m \in \mathbb{Z}_{\geq 0}$, and $U(x)$ is ℓ^2 as $k \to -\infty$ for all t. If we moreover assume $t \in \mathbb{R}\backslash\{q^{\frac12 m} \mid m \in \mathbb{Z}\}$, then $U(x)$ is ℓ^2 as $k \to \infty$ if and only if $q^{\frac12} < |t| < q^{-\frac12}$.*

Proof. Recall the second order q-difference equation

$$(c - abz)f(qz) + (-(c+q) + (a+b)z)f(z) + (q - z)f(z/q) = 0$$

satisfied by $f(z) = {}_2\varphi_1(a, b; c; q, z)$, see [7, Exercise 1.13]. Take $c = 0$ to see that $U(x)$ satisfies $Lu = x\,u$. It is clear from (2.3) that $U(x)$ is ℓ^2 as $k \to -\infty$. By confluent limit, $f(z) = {}_1\varphi_1(a; c; q, z)$ satisfies

$$(c - az)f(qz) + (-(c+q) + z)f(z) + qf(z/q) = 0.$$

This yields that $V^{t^{\pm1}}(x)$ satisfies $Lu = x\,u$. Now use the theta-product identity

$$(aq^k, q^{1-k}/a; q)_\infty = (-a)^{-k} q^{-\frac12 k(k-1)}(a, q/a; q)_\infty, \qquad \forall\, k \in \mathbb{Z},\ a \in \mathbb{C}\backslash\{0\}, \tag{2.7}$$

to derive the statements on the ℓ^2-behaviour as $k \to \infty$ of $V^{t^{\pm1}}(x)$ together with (2.6).

Use (2.4) to obtain the ℓ^2-behaviour as $k \to \infty$ of $U(x)$. $\square$

For $x = 0$ we can take the limit in $V^t_k(x)$, e.g.

$$V^t_k(0) = \sqrt{(-q^{1-k}/c; q)_\infty}\; q^{\frac14 k(k+1)} \big(-t\sqrt{c}\big)^k (qt^2; q)_\infty\ {}_0\varphi_1(-; qt^2; q, -q^{k+1}ct^2),$$

which is closely related to Jackson's q-Bessel function $J^{(2)}_\alpha(2\sqrt{c}\, q^{\frac12 k}; q)$ for $t^2 = q^\alpha$, see Ismail [8], and also [5], [21]. Explicitly,

$$V^{q^{\frac12\alpha}}_k(0) = \sqrt{(-q^{1-k}/c; q)_\infty}\; q^{\frac14 k(k+1)}(-1)^k c^{\frac12(k-\alpha)}(q; q)_\infty J^{(2)}_\alpha(2\sqrt{c}\, q^{\frac12 k}; q) \tag{2.8}$$

and a similar expression for $V^{q^{-\frac12\alpha}}_k(0)$. We recall the definition of Jackson's q-Bessel function, see [8]:

$$J^{(2)}_\alpha(x; q) := \frac{(q^{\alpha+1}; q)_\infty}{(q; q)_\infty} \left(\frac{x}{2}\right)^\alpha {}_0\varphi_1\left(\genfrac{}{}{0pt}{}{-}{q^{\alpha+1}}; q, -q^{\alpha+1}\frac{x^2}{4}\right). \tag{2.9}$$

Remark. It follows from Lemma 2.2(i) that L has deficiency indices $(1,1)$ for $q^{\frac12} < |t| < q^{-\frac12}$, $t \neq \pm1$, and that the deficiency space $N_{\pm i} = \{u \in \ell^2(\mathbb{Z}) \mid Lu = \pm i\, u\}$ is spanned by $U(\pm i)$.

The Wronskian for two sequences $(u_k)_{k\in\mathbb{Z}}$ and $(v_k)_{k\in\mathbb{Z}}$ is given by

$$[u, v]_k := a_k\big(u_{k+1}v_k - u_k v_{k+1}\big). \tag{2.10}$$

It is independent of k if u and v are two solutions to $Lu = x\,u$, so that we may denote it by $[u, v]$.

Lemma 2.3. *We have the following Wronskians:*

$$[V^t(x), V^{t^{-1}}(x)] = \frac{\sqrt{c}}{t}(t^2, qt^{-2} - 1/c, -cq; q)_\infty,$$
$$[U(x), V^t(x)] = \frac{c_{t^{-1}}(x)}{-t\sqrt{c}}.$$

Proof. The first Wronskian follows by combining (2.10) with Lemma 2.2(ii) and letting $k \to \infty$ using (2.7). The second Wronskian follows from the first and (2.4) for $\pm t \notin \{q^{\frac{1}{2}m} \mid m \in \mathbb{Z}\}$. Next use analytic continuation with respect to t. $\square$

So we find that for $x \in \mathbb{C}\backslash\mathbb{R}$ the solution $U(x)$ and $V^{t^{-1}}(x)$ of $Lu = x\,u$ are linearly independent for $x \in \mathbb{C}\backslash\mathbb{R}$. Moreover, $U(x)$ is ℓ^2 for $k \to -\infty$ and this determines $U(x)$ up to a constant by the considerations in the first paragraph of this section. Furthermore $V^{t^{-1}}(x)$ is ℓ^2 for $k \to \infty$ for $|t| > q^{\frac{1}{2}}$, and for $|t| \geq q^{-\frac{1}{2}}$ this condition determines $V^{t^{-1}}(x)$ up to a constant.

3. Spectral resolution.

In case $|t| > q^{\frac{1}{2}}$ we see $V^{t^{-1}}(x)$ is ℓ^2 for $k \to \infty$ by Lemma 2.2(ii). From now on we assume that $|t| > q^{\frac{1}{2}}$. In case $|t| \geq q^{-\frac{1}{2}}$ we moreover have that L is self-adjoint. The domain of L is given by $\{u \in \ell^2(\mathbb{Z}) \mid Lu \in \ell^2(\mathbb{Z})\}$. In case, $q^{\frac{1}{2}} < |t| < q^{-\frac{1}{2}}$, $t \neq \pm 1$, the deficiency indices of L are $(1,1)$, so that all extensions of L to a self-adjoint operator are parametrised by $U(1) = \mathbb{T}$. In this case, we fix the self-adjoint extension such that $V^{t^{-1}}$ is contained in the domain, which is always possible since $\overline{N_i} = N_{-i}$ because of $a_k, b_k \in \mathbb{R}$. See Dunford and Schwartz [6, Ch. XII] for more information.

Define the Green function for $x \in \mathbb{C}\backslash\mathbb{R}$ by

$$G_x(m,n) := \begin{cases} \dfrac{U_n(x)V_m^{t^{-1}}(x)}{[U(x), V^{t^{-1}}(x)]}, & \text{for } n \leq m, \\[2ex] \dfrac{U_m(x)V_n^{t^{-1}}(x)}{[U(x), V^{t^{-1}}(x)]}, & \text{for } n > m, \end{cases}$$

so that $\big((x-L)^{-1}f\big)_m = \sum_{n=-\infty}^{\infty} G_x(m,n)\, f_n$. For the resolution of the identity E for L, i.e. $L = \int_{\mathbb{R}} t\, dE(t)$, we now have, see [6, Ch. XII],

$$\langle E\big((a,b)\big)v, w\rangle = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} \langle (s - i\varepsilon - L)^{-1}v, w\rangle - \langle (s + i\varepsilon - L)^{-1}v, w\rangle\, ds.$$

Now observe that

$$\langle (s \pm i\varepsilon - L)^{-1}v, w\rangle = \sum_{n\leq m} \frac{U_n(s\pm i\varepsilon)V_m^{t^{-1}}(s\pm i\varepsilon)}{[U(s\pm i\varepsilon), V^{t^{-1}}(s\pm i\varepsilon)]}\,\frac{v_n\bar{w}_m + v_m\bar{w}_n}{2}.$$

Since $V_m^{t^{-1}}(x)$ is an entire function of x, and $U_n(x)$ is analytic in $\mathbb{C}\backslash\{0\}$ we see that the measure $\langle E\big((a,b)\big)v, w\rangle$ has only discrete mass points at the zeros of the Wronskian

$[U(x), V^{t^{-1}}(x)]$. By Lemma 2.3 this corresponds to the zeros of $c_t(x)$ which are the points $\eta_p = -\sqrt{c}q^p/t$, $p \in \mathbb{Z}_{\geq 0}$, and $\xi_p = tq^p/\sqrt{c}$, $p \in \mathbb{Z}$. So the spectrum of L is $\{\eta_p \mid p \in \mathbb{Z}_{\geq 0}\} \cup \{\xi_p \mid p \in \mathbb{Z}\} \cup \{0\}$, and it remains to show that 0 is not in the point spectrum of L. It follows from Moak [21, §7] and (2.8) that $V_k^{t^{-1}}(0)$ is not identically zero for $k < -N$ for some N. The observation now follows from the asymptotic expression for the Jackson q-Bessel function derived by Chen et al. [5, Thm. 4], which shows that the Jackson q-Bessel function in (2.8) increases exponentially with k^2 as $k \to -\infty$.

In order to calculate $E(\{\eta_p\})$ we take the interval (a, b) such that it contains only η_p as a point from the spectrum. Then

$$\begin{aligned}
\langle E\big(\{\eta_p\}\big)v, w\rangle &= \langle E\big((a,b)\big)v, w\rangle \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i}\int_a^b \langle (s - i\varepsilon - L)^{-1}v, w\rangle - \langle (s + i\varepsilon - L)^{-1}v, w\rangle\, ds \\
&= \frac{1}{2\pi i}\oint_{(\eta_p)} \langle (s-L)^{-1}v, w\rangle\, ds \\
&= \sum_{n\leq m} \frac{v_n\bar{w}_m + v_m\bar{w}_n}{2}\frac{1}{2\pi i}\oint_{(\eta_p)} \frac{U_n(s)V_m^{t^{-1}}(s)}{[U(s), V^{t^{-1}}(s)]}\, ds \\
&= \sum_{n\leq m} \frac{v_n\bar{w}_m + v_m\bar{w}_n}{2} U_n(\eta_p)V_m^{t^{-1}}(\eta_p)(-\sqrt{c}t^{-1})\mathrm{Res}_{x=\eta_p}\frac{1}{c_t(x)} \\
&= C_{t^{-1}}c_{t^{-1}}(\eta_p)(-\sqrt{c}t^{-1})\mathrm{Res}_{x=\eta_p}\frac{1}{c_t(x)}\langle v, V^{t^{-1}}(\eta_p)\rangle\langle V^{t^{-1}}(\eta_p), w\rangle
\end{aligned}$$

by (2.4) and $c_t(\eta_p) = 0$. Since ${}_1\varphi_1(q^{-p}; qt^{-2}; q, -cq^{p+k+1}t^{-2}) \sim Cq^{kp}$ for $k \to -\infty$, we obtain $V^{t^{-1}}(\eta_p) \in \ell^2(\mathbb{Z})$ also in a direct manner. Next

$$\mathrm{Res}_{x=\eta_p}\frac{1}{c_t(x)} = \frac{1}{(-q^{1-p}t^2/c, -q^pc/t^2; q)_\infty}\frac{-q^p\sqrt{c}/t}{(q^{-p}; q)_p(q; q)_\infty}$$

and

$$C_{t^{-1}}c_{t^{-1}}(\eta_p) = \frac{(t^2q^{-p}; q)_p}{(qt^{-2}; q)_\infty}c^{-p}q^{-\frac{1}{2}p(p-1)}$$

by (2.7). So that finally,

(3.1)

$$\langle E\big(\{\eta_p\}\big)v, w\rangle = ct^{-2}q^p\frac{(qt^{-2}; q)_p}{(q; q)_p}\frac{1}{(-c/t^2, -qt^2/c, qt^{-2}; q)_\infty}\langle v, V^{t^{-1}}(\eta_p)\rangle\langle V^{t^{-1}}(\eta_p), w\rangle.$$

Take $v = w = V^{t^{-1}}(\eta_p)$ and use that $E\big(\{\eta_p\}\big)V^{t^{-1}}(\eta_p) = V^{t^{-1}}(\eta_p)$ to see that

$$\|V^{t^{-1}}(\eta_p)\|^2 = c^{-1}t^2q^{-p}\frac{(q; q)_p}{(qt^{-2}; q)_p}(q, -c/t^2, -qt^2/c, qt^{-2}; q)_\infty \tag{3.2}$$

Since the projections $E(\{\eta_p\})$ satisfy $E(\{\eta_p\})E(\{\eta_r\}) = \delta_{pr}E(\{\eta_p\})$ we get

$$\begin{aligned}(3.3)\quad \langle V^{t^{-1}}(\eta_p), V^{t^{-1}}(\eta_r)\rangle &= \langle E(\{\eta_p\})V^{t^{-1}}(\eta_p), E(\{\eta_r\})V^{t^{-1}}(\eta_r)\rangle = \\ &\langle V^{t^{-1}}(\eta_p), E(\{\eta_p\})E(\{\eta_r\})V^{t^{-1}}(\eta_r)\rangle = \delta_{pr}\langle V^{t^{-1}}(\eta_p), V^{t^{-1}}(\eta_p)\rangle\end{aligned}$$

and after setting $t^{-2} = q^\alpha$, $\alpha > -1$, we obtain from (3.2) and (3.3) the orthogonality relations for the q-Laguerre polynomials on the set $\{cq^k \mid k \in \mathbb{Z}\}$, $c > 0$, obtained by Moak [21]. It's well known that these polynomials correspond to an indetermined moment problem and that the orthogonality measure supported on $\{cq^k \mid k \in \mathbb{Z}\}$ is not N-extremal, i.e. the polynomials are not dense in the corresponding L^2-space, see [1], [21].

Similarly,

$$\langle E(\{\xi_p\})v, w\rangle = C_{t^{-1}}c_{t^{-1}}(\xi_p)(-\sqrt{c}t^{-1})\mathrm{Res}_{x=\xi_p}\frac{1}{c_t(x)}\langle v, V^{t^{-1}}(\xi_p)\rangle\langle V^{t^{-1}}(\xi_p), w\rangle$$

with

$$\begin{aligned}\mathrm{Res}_{x=\xi_p}\frac{1}{c_t(x)} &= \frac{tq^p}{\sqrt{c}}\frac{(-1)^{p-1}q^{\frac12 p(p-1)}}{(-cq^{-p}t^{-2}, q, q; q)_\infty}, \\ C_{t^{-1}}c_{t^{-1}}(\xi_p) &= \left(-\frac{c}{t^2}\right)^p q^{-p^2}\frac{1}{(-q^{1+p}/c; q)_\infty}.\end{aligned}$$

Hence,

$$\langle E(\{\xi_p\})v, w\rangle = q^p\frac{(-q^{p+1}t^2/c; q)_\infty}{(-q^{p+1}/c, q, q, -ct^{-2}, -qt^2/c; q)_\infty}\langle v, V^{t^{-1}}(\xi_p)\rangle\langle V^{t^{-1}}(\xi_p), w\rangle.$$

Then, similarly as before, we obtain

$$(3.4)\qquad \langle V^{t^{-1}}(\xi_p), V^{t^{-1}}(\xi_r)\rangle = \delta_{pr}q^{-p}\frac{(-q^{p+1}/c, q, q, -ct^{-2}, -qt^2/c; q)_\infty}{(-q^{p+1}t^2/c; q)_\infty}.$$

Remark. The vector $V^{t^{-1}}(\xi_p)$ belongs to $\ell^2(\mathbb{Z})$ for $p \in \mathbb{Z}$. This can also be seen as follows. First apply to (2.3) the transformation formula

$$(3.5)\qquad {}_1\varphi_1\left(\begin{matrix}a\\c\end{matrix}; q, z\right) = \frac{(z; q)_\infty}{(c; q)_\infty}\, {}_1\varphi_1\left(\begin{matrix}az/c\\z\end{matrix}; q, c\right)$$

(a limit case of Heine's transformation formula [7, (1.4.5)]). Combination of (2.3), (3.5) and (2.5) yields that

$$\begin{aligned}V_k^{t^{-1}}(\xi_p) = \frac{(-q^{p+1}/c; q)_\infty\,(-t^2c^{-1}q^p)^p}{\sqrt{(-q^{1-k}/c; q)_\infty}(qt^{-2}; q)_\infty}\, q^{\frac14 k(k+1)}\,(tc^{-\frac12}q^p)^k(qt^{-2}; q)_\infty \\ \times\ (q^{1-p-k}; q)_\infty\, {}_1\varphi_1\left(\begin{matrix}-cq^{-p}\\q^{1-p-k}\end{matrix}; q, q^{1-p-k}t^{-2}\right),\end{aligned}$$

so that we obtain the ℓ^2-behaviour as $k \to -\infty$ of $V^{t^{-1}}(x)$.

4. Basis for L^2-space.

By $L^2(\mu^{(\alpha;c)})$ we denote the space of square integrable functions on the set $\{cq^k \mid k \in \mathbb{Z}\}$, $c>0$, with positive weight $q^{k(\alpha+1)}/(-cq^k;q)_\infty$ at cq^k, $k\in\mathbb{Z}$, i.e. $f\in L^2(\mu^{(\alpha;c)})$ if

$$\mathcal{L}(|f|^2) := \sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}}{(-cq^k;q)_\infty}|f(cq^k)|^2 < \infty.$$

Here we take $\alpha > -1$.

Recall the q-Laguerre polynomials introduced by Moak [21], see also [7],

$$L_n^{(\alpha)}(x;q) := \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1}\end{matrix};q,-xq^{n+\alpha+1}\right). \tag{4.1}$$

We also define the functions

$$\begin{aligned} M_p^{(\alpha;c)}(x;q) :=& \frac{(q^{\alpha+1};q)_\infty}{(q,-cq^{\alpha+1};q)_\infty}\,{}_1\varphi_1\left(\begin{matrix} -cq^{\alpha-p} \\ q^{\alpha+1}\end{matrix};q,\frac{xq^{p+1}}{c}\right) \\ =& \frac{(xq^{p+1}/c;q)_\infty}{(q,-cq^{\alpha+1};q)_\infty}\,{}_1\varphi_1\left(\begin{matrix} -x \\ xq^{p+1}/c\end{matrix};q,q^{\alpha+1}\right)\end{aligned} \tag{4.2}$$

for $p\in\mathbb{Z}$. (The second equality is by (3.5).)

Theorem 4.1. *The functions $M_p^{(\alpha;c)}(\cdot;q)$, $p\in\mathbb{Z}$, together with the q-Laguerre polynomials $L_n^{(\alpha)}(\cdot;q)$, $n\in\mathbb{Z}_{\geq 0}$, form an orthogonal basis for $L^2(\mu^{(\alpha;c)})$, $c>0$, $\alpha>-1$. Explicitly,*

$$\begin{aligned} \mathcal{L}\big(L_n^{(\alpha)}(\cdot;q)L_p^{(\alpha)}(\cdot;q)\big) &= \delta_{n,p}q^{-p}\frac{(q^{\alpha+1};q)_p}{(q;q)_p}\frac{(q,-cq^{\alpha+1},-q^{-\alpha}/c;q)_\infty}{(q^{\alpha+1},-c,-q/c;q)_\infty}, \\ \mathcal{L}\big(M_p^{(\alpha;c)}(\cdot;q)M_r^{(\alpha;c)}(\cdot;q)\big) &= \delta_{p,r}cq^\alpha q^{-p}\frac{(-q^{p+1}/c,-q^{-\alpha}/c;q)_\infty}{(-q^{p+1-\alpha}/c,-cq^{\alpha+1};q)_\infty}\frac{1}{(-c,-q/c;q)_\infty}, \\ \mathcal{L}\big(M_p^{(\alpha;c)}(\cdot;q)L_n^{(\alpha)}(\cdot;q)\big) &= 0.\end{aligned}$$

Note that $M_p^{(\alpha;c)}(x;q)$ depends on c, unlike $L_n^{(\alpha)}(x;q)$. It should also be observed that $M_p^{(\alpha;c)}(cq^k;q)$ is bounded as $k\to\infty$ and that $M_p^{(\alpha;c)}(cq^k;q)\to 0$ for $|q^{p-\alpha}/c|<1$ for $k\to-\infty$ as follows from (2.5). Using the method of Berg [3] we can construct more orthogonality measures for the q-Laguerre polynomials supported on the set $\{cq^k\mid k\in\mathbb{Z}\}$. Let then $|M_p^{(\alpha;c)}(cq^k;q)|\leq K$ for some K for any $p\in\mathbb{Z}$ satisfying $|q^{p-\alpha}/c|<1$. Then the q-Laguerre polynomials are also orthogonal with respect to the positive discrete measure with masses $\big(1+sK^{-1}M_p^{(\alpha;c)}(cq^k;q)\big)q^{k(\alpha+1)}/(-cq^k;q)_\infty$ at cq^k, $k\in\mathbb{Z}$, for any $s\in[-1,1]$, cf. [3, Prop. 4.1]. We may also take suitable linear combinations of the functions $M_p^{(\alpha;c)}(x;q)$.

Proof. Apply the results of the previous section with $t^{-2}=q^\alpha$. The first statement corresponds to $\langle V^{t^{-1}}(\eta_p),V^{t^{-1}}(\eta_r)\rangle = \delta_{p,r}\|V^{t^{-1}}(\eta_p)\|^2$, the second statement corresponds to $\langle V^{t^{-1}}(\xi_p),V^{t^{-1}}(\xi_r)\rangle = \delta_{p,r}\|V^{t^{-1}}(\xi_p)\|^2$ and the last one to $\langle V^{t^{-1}}(\xi_p),V^{t^{-1}}(\eta_r)\rangle = 0$. □

Remark 4.2. The first statement of Theorem 4.1 corresponds to Moak's discrete orthogonality relations [21, Thm. 2] for the q-Laguerre polynomials. The second relation can be rewritten as

(4.3)
$$\sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}(q^{\alpha+1};q)_\infty}{(-cq^k;q)_\infty(q;q)_\infty} {}_1\varphi_1\left(\begin{matrix}-cq^{\alpha-p}\\ q^{\alpha+1}\end{matrix};q,q^{p+k+1}\right) \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty} {}_1\varphi_1\left(\begin{matrix}-cq^{\alpha-r}\\ q^{\alpha+1}\end{matrix};q,q^{r+k+1}\right)$$
$$= \delta_{p,r} cq^\alpha q^{-p} \frac{(-q^{p+1}/c;q)_\infty}{(-q^{p+1-\alpha}/c;q)_\infty} \frac{(-cq^{\alpha+1},-q^{-\alpha}/c;q)_\infty}{(-c,-q/c;q)_\infty} =$$
$$\sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}(q^{k+p+1};q)_\infty}{(-cq^k;q)_\infty(q;q)_\infty} {}_1\varphi_1\left(\begin{matrix}-cq^{k}\\ q^{k+p+1}\end{matrix};q,q^{\alpha+1}\right) \frac{(q^{k+r+1};q)_\infty}{(q;q)_\infty} {}_1\varphi_1\left(\begin{matrix}-cq^{k}\\ q^{k+r+1}\end{matrix};q,q^{\alpha+1}\right),$$

for $p,r\in\mathbb{Z}$, which can be viewed as a q-analogue of the Hankel transform for the first equality or as a q-analogue of the Hansen-Lommel orthogonality relations for the second equality, cf. [18], and the limit case $c\to 0$ corresponds to [18, Prop. 2.6]. The last statement shows that these q-Bessel functions are orthogonal to the q-Laguerre polynomials;

$$\sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}}{(-cq^k;q)_\infty} L_n^{(\alpha)}(cq^k;q) \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty} {}_1\varphi_1\left(\begin{matrix}-cq^{\alpha-r}\\ q^{\alpha+1}\end{matrix};q,q^{r+k+1}\right) = 0, \quad r\in\mathbb{Z},\ n\in\mathbb{Z}_{\geq 0}.$$

Since the statement is that $\{V^{t^{-1}}(\eta_p)\}_{p\in\mathbb{Z}_{\geq 0}}$ and $\{V^{t^{-1}}(\xi_p)\}_{p\in\mathbb{Z}}$ form an orthogonal basis for the Hilbert space $\ell^2(\mathbb{Z})$, we also find the dual orthogonality relations

$$\sum_{p=0}^{\infty} \frac{V_k^{t^{-1}}(\eta_p)V_l^{t^{-1}}(\eta_p)}{\|V^{t^{-1}}(\eta_p)\|^2} + \sum_{p=-\infty}^{\infty} \frac{V_k^{t^{-1}}(\xi_p)V_l^{t^{-1}}(\xi_p)}{\|V^{t^{-1}}(\xi_p)\|^2} = \delta_{k,l}. \tag{4.4}$$

The first sum is the Poisson kernel for the q-Laguerre polynomials evaluated at one; it can also be derived from the Christoffel-Darboux formula and the limit transition of the q-Laguerre polynomials to Jackson's q-Bessel function, see [21]. Explicitly, from [21, (4.11), Thm. 5] we get

$$\begin{aligned}(4.5)\quad & \sum_{p=0}^{N} \frac{q^p(q;q)_p}{(q^{\alpha+1};q)_p} L_p^{(\alpha)}(x;q)L_p^{(\alpha)}(y;q) = \\ & \frac{(q;q)_N}{(q^{\alpha+1};q)_N}\frac{1}{x-y}\big(xL_N^{(\alpha+1)}(x;q)L_N^{(\alpha)}(y;q) - yL_N^{(\alpha+1)}(y;q)L_N^{(\alpha)}(x;q)\big) \xrightarrow{N\to\infty} \\ & \frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty}\frac{(xy)^{-\frac{1}{2}\alpha}}{x-y}\big(\sqrt{x}J_{\alpha+1}^{(2)}(2\sqrt{x};q)J_\alpha^{(2)}(2\sqrt{y};q) - \sqrt{y}J_{\alpha+1}^{(2)}(2\sqrt{y};q)J_\alpha^{(2)}(2\sqrt{x};q)\big),\end{aligned}$$

where the right hand side is well-defined for $x=y$ using l'Hôpital's formula. Using $t^{-2}=q^\alpha$

we get

$$\sum_{p=0}^{\infty}\frac{V_k^{t^{-1}}(\eta_p)V_l^{t^{-1}}(\eta_p)}{\|V^{t^{-1}}(\eta_p)\|^2}=\frac{q^{\frac12(\alpha+1)(k+l)}(-1)^{k+l}}{(-cq^k,-cq^l;q)_\infty^{\frac12}}\frac{(-c,-q/c,q^{\alpha+1};q)_\infty}{(-cq^{\alpha+1},-q^{-\alpha}/c,q;q)_\infty}$$
$$\times\sum_{p=0}^{\infty}\frac{q^p(q;q)_p}{(q^{\alpha+1};q)_p}L_p^{(\alpha)}(cq^k;q)L_p^{(\alpha)}(cq^l;q)=\frac{q^{\frac12(k+l)}(-1)^{k+l}c^{-\alpha-\frac12}(-c,-q/c;q)_\infty}{(-cq^k,-cq^l;q)_\infty^{\frac12}(-cq^{\alpha+1},-q^{-\alpha}/c;q)_\infty}$$
$$\times\frac{1}{q^k-q^l}\Big(q^{\frac12 k}J_{\alpha+1}^{(2)}(2\sqrt{c}q^{\frac12 k};q)J_\alpha^{(2)}(2\sqrt{c}q^{\frac12 l};q)-q^{\frac12 l}J_{\alpha+1}^{(2)}(2\sqrt{c}q^{\frac12 l};q)J_\alpha^{(2)}(2\sqrt{c}q^{\frac12 k};q)\Big)$$

by (4.5). Calculating the second sum in (4.4) is straightforward, so that we now have obtained the following corollary.

Corollary 4.3. *The following orthogonality relations hold;*

$$\delta_{k,l}cq^{-k}(-cq^k;q)_\infty\frac{(-cq^{\alpha+1},-q^{-\alpha}/c;q)_\infty}{(-c,-q/c;q)_\infty}=$$
$$\frac{c^{\frac12-\alpha}}{q^k-q^l}\Big(q^{\frac12 k}J_{\alpha+1}^{(2)}(2\sqrt{c}q^{\frac12 k};q)J_\alpha^{(2)}(2\sqrt{c}q^{\frac12 l};q)-q^{\frac12 l}J_{\alpha+1}^{(2)}(2\sqrt{c}q^{\frac12 l};q)J_\alpha^{(2)}(2\sqrt{c}q^{\frac12 k};q)\Big)+$$
$$q^{\alpha(\frac12(k+l)-1)}\sum_{p=-\infty}^{\infty}q^p\frac{(-q^{p+1-\alpha}/c;q)_\infty}{(-q^{p+1}/c;q)_\infty}\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}{}_1\varphi_1\left(\begin{matrix}-cq^{\alpha-p}\\ q^{\alpha+1}\end{matrix};q,q^{k+p+1}\right)$$
$$\times\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}{}_1\varphi_1\left(\begin{matrix}-cq^{\alpha-p}\\ q^{\alpha+1}\end{matrix};q,q^{l+p+1}\right),$$

In this form Corollary 4.3 is reminiscent of the Hankel transform, whereas if we use the transformation for the ${}_1\varphi_1$-series of (4.2), the orthogonality relations remind us of the Hansen-Lommel orthogonality relations, cf. [18]. Of course, we may also replace the expression for the Jackson q-Bessel function by the Poisson kernel for the q-Laguerre polynomials evaluated at one, cf. (4.5), to obtain the dual orthogonality relations involving a sum over $\mathbb{Z}$ and over $\mathbb{Z}_{\geq 0}$.

5. Direct proof of the orthogonality relations of Theorem 4.1.

In this section we present a direct analytic proof using summation and transformation formulas of the orthogonality relations of Theorem 4.1, but not of the completeness. The first statement of the orthogonality relations is well known, see Moak [21], so we concentrate on the last two. The method is based on manipulating generating functions, cf. [18], [13]. We start with the following lemma giving generating functions for the q-Bessel functions under consideration.

Lemma 5.1. *We have the following generating functions:*
(i) *For $0<|z|<|b|^{-1}$ we have*

$$\sum_{p=-\infty}^{\infty}z^pb^p\frac{(q^{p+1};q)_\infty}{(aq^p/b;q)_\infty}{}_1\varphi_1\left(\begin{matrix}aq^p/b\\ q^{p+1}\end{matrix};q,bx\right)=\frac{(q,az,x/z;q)_\infty}{(a/b,bz;q)_\infty},$$

(ii) *For* $|d|<|w|<1$ *we have*

$$\sum_{r=-\infty}^{\infty} w^r (q^{r+1};q)_\infty\, {}_1\varphi_1\left(\begin{matrix} dq^{r+1}/y \\ q^{r+1}\end{matrix};q,y\right) = \frac{(d,q,y/w;q)_\infty}{(w,d/w;q)_\infty}.$$

Proof. Case (i) is a limit case of [13, Prop. 2.1]. Use the q-binomial theorem [7, (1.3.2)] to expand $(az;q)_\infty/(bz;q)_\infty$ in a power series of z for $|z|<|b|^{-1}$, and [7, (1.3.16)] to expand $(x/z;q)_\infty$ in a power series in z^{-1} for $|z|>0$. Multiply the resulting series to find the result.

For the proof of (ii) we expand $1/(w;q)_\infty$ in a power series in w for $|w|<1$ and $(y/w;q)_\infty/(d/w;q)_\infty$ in a power series in w^{-1} for $|d|<|w|$ using [7, (1.3.2)] twice. Combine this to write the right hand side as a Laurent series in w with coefficients given by a ${}_2\varphi_1$-series. Then use the limit case $b\to 0$ of Heine's transformation [7, (1.4.6)] to get the result. $\square$

To prove the last part of Theorem 4.1 we use Lemma 5.1(i) after replacing p by $k+r$, a by $-bcq^{-r}$ and x by $q^{\alpha+1}/b$ to get

$$\sum_{k=-\infty}^{\infty} \frac{(zb)^{k+r}}{(-cq^k;q)_\infty}(q^{k+r+1};q)_\infty\, {}_1\varphi_1\left(\begin{matrix} -cq^k \\ q^{k+r+1}\end{matrix};q,q^{\alpha+1}\right) = \frac{(q,-bzcq^{-r},q^{\alpha+1}/(bz);q)_\infty}{(-cq^{-r},bz;q)_\infty}$$

and specialising $zb=q^{\alpha+1+m}$, $\alpha>-1$, shows that the right hand side is zero for $m\in\mathbb{Z}_{\geq 0}$ whereas the left hand side is a non-zero multiple of $\mathcal{L}$ applied to $M_r^{(\alpha;c)}(x;q)x^m$. So $M_r^{(\alpha;c)}(x;q)$ is orthogonal to all monomials, hence to all polynomials implying the last statement of Theorem 4.1.

To prove the orthogonality relations for the functions $M^{(\alpha;c)}(x;q)$ of Theorem 4.1 we first deduce the following result.

Proposition 5.2. *For* $|y|<1$, $l\in\mathbb{Z}$, *we have*

$$\sum_{k=-\infty}^{\infty} \frac{y^k}{(aq^k/b;q)_\infty}(q^{k+1};q)_\infty {}_1\varphi_1\left(\begin{matrix} aq^k/b \\ q^{k+1}\end{matrix};q,y\right)(q^{k-l+1};q)_\infty {}_1\varphi_1\left(\begin{matrix} dq^{k-l+1}/y \\ q^{k-l+1}\end{matrix};q,y\right)$$
$$= \begin{cases} 0, & \text{for } l<0, \\ d^l\,\dfrac{(ay/(bd);q)_l(d,q,q;q)_\infty}{(q;q)_l(a/b;q)_\infty}, & \text{for } l\geq 0.\end{cases}$$

Proof. Choose $w=y/(bz)$ in Lemma 5.1(ii) and multiply with the generating function of Lemma 5.1(i) with x replaced by y/b. This then gives the following identity

$$\frac{(az;q)_\infty(d,q,q;q)_\infty}{(bzd/y;q)_\infty(a/b;q)_\infty} = \sum_{l=-\infty}^{\infty} z^l$$
$$\times \sum_{k=-\infty}^{\infty}\frac{y^{k-l}b^l}{(aq^k/b;q)_\infty}(q^{k+1};q)_\infty {}_1\varphi_1\left(\begin{matrix} aq^k/b \\ q^{k+1}\end{matrix};q,y\right)(q^{k-l+1};q)_\infty {}_1\varphi_1\left(\begin{matrix} dq^{k-l+1}/y \\ q^{k-l+1}\end{matrix};q,y\right)$$

under the condition $|y/b| < |z| < \min(|b|^{-1}, |y/(db)|)$. So we need $|y| < 1$, $|d| < 1$ to have a non-empty region of analyticity of the right hand side as function of z. The left hand side is an analytic function of z in $|z| < |y/(db)|$ and it can be expanded in a power series in z using the q-binomial theorem [7, (1.3.2)]. Comparing coefficients at both sides gives the result for $|d| < 1$.

Next we use analytic continuation with respect to d. For fixed $l \in \mathbb{Z}$ the right hand side is analytic in d. To see that the left hand side is analytic in d we note that $f_k(d) = (q^{k-l+1};q)_\infty {}_1\varphi_1(dq^{k-l+1}/y; q^{k-l+1}; q, y)$ is analytic in d. Moreover, for $k \geq l$ we easily estimate $|f_k(d)| \leq (-|d/y|, -|y|; q)_\infty$, so that the convergence of the sum for $k \to \infty$ is uniform in d on compact subsets since $|y| < 1$.

In order to obtain uniform convergence as $k \to -\infty$ we use (2.5) so that for $k \leq -l$ we can estimate $|f_k(d)| \leq |d^{l-k}(y/d;q)_{l-k}|(-|d/y|, -|y|; q)_\infty$. On the other hand we see that by (2.5) and (2.7) the coefficient for the function $f_k(d)$ as $k \to \infty$ behaves like $C(-1)^k q^{\frac{1}{2}k(k+1)}$. Hence the convergence for $k \to -\infty$ is also uniform in d on compact subsets. Hence both sides of the equality of this proposition are analytic in d and coincide on $|d| < 1$, so it holds for all d. $\square$

There are many ways of linking w and z such that the products of the generating functions of Lemma 5.1 simplify. Taking $w = x/z$ and $bx = y$ leads to the same result. The case $az = d/w$ also gives an interesting result, which may be obtained as a limit case of [13, Prop. 2.2].

In Proposition 5.2 we replace k by $k+p$, l by $-r+p$ for $r, p \in \mathbb{Z}$, next we take $y = q^{\alpha+1}$, $\alpha > -1$, and $aq^m = -cb$ to get

$$\sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}(q^{k+p+1};q)_\infty}{(-cq^k;q)_\infty} {}_1\varphi_1\left(\begin{matrix} -cq^k \\ q^{k+p+1} \end{matrix}; q, q^{\alpha+1}\right) (q^{k+r+1};q)_\infty {}_1\varphi_1\left(\begin{matrix} dq^{k+r-\alpha} \\ q^{k+r+1} \end{matrix}; q, q^{\alpha+1}\right)$$
$$= \begin{cases} 0, & \text{for } p < r, \\ q^{-p(\alpha+1)} d^{p-r} \dfrac{(-cq^{\alpha-p+1}/d;q)_{p-r}(d, q^{p-r+1}, q; q)_\infty}{(-cq^{-p};q)_\infty}, & \text{for } p \geq r. \end{cases}$$

Using (2.7) we see that this reduces to the second equation of (4.3) after specialising $d = -cq^{\alpha-r}$, since $(q^{1+r-p};q)_{p-r} = \delta_{r,p}$ for $p \geq r$. So we have proved (4.3).

6. Proof of the dual orthogonality relations.

In this section we prove the dual orthogonality relations of Corollary 4.3 by using a rigorous limit transition from the big q-Jacobi polynomials to the q-Bessel functions under consideration, which we will call now *big q-Bessel functions*. The orthogonality relations for the big q-Jacobi polynomials tend to the dual orthogonality relations given in Corollary 4.3. This then gives, together with the analytic proofs of the previous section a complete alternative analytic proof of Theorem 4.1.

For $\alpha > -1$, $c > 0$, $k \in \mathbb{Z}$, $x \in \mathbb{R}$ put

$$\mathcal{J}^c_{\alpha,k}(x;q) := {}_1\varphi_1\left(\begin{matrix} x^{-1} \\ q^{\alpha+1} \end{matrix}; q, -c^{-1}xq^{k+\alpha+2}\right) \tag{6.1}$$

$$= (-c^{-1}q^{k+1};q)_\infty \, {}_2\varphi_1\left(\begin{matrix} q^{\alpha+1}x, 0 \\ q^{\alpha+1} \end{matrix}; q, -c^{-1}q^{k+1}\right). \tag{6.2}$$

Here we used for the second identity the tranformation

$$
{}_1\varphi_1\left(\begin{matrix}a\\c\end{matrix};q,x\right)=(ax/c;q)_\infty\;{}_2\varphi_1\left(\begin{matrix}c/a,0\\c\end{matrix};q,ax/c\right),
$$

which is a limit case of Heine's transformation formula [7, (1.4.6)]. Then formulas (4.1) and (4.2) can be rewritten as

$$
L_n^{(\alpha)}(c^{-1}q^{k+1};q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,\mathcal{J}^c_{\alpha,k}(q^n;q)\quad(n\in\mathbb{Z}_{\geq 0}),\tag{6.3}
$$

respectively

$$
M_p^{(\alpha;c^{-1})}(c^{-1}q^{k+1};q)=\frac{(q^{\alpha+1};q)_\infty}{(q,-c^{-1}q^{\alpha+1};q)_\infty}\,\mathcal{J}^c_{\alpha,k}(-cq^{p-\alpha};q)\quad(p\in\mathbb{Z}).\tag{6.4}
$$

The (dual) orthogonality relations in Corollary 4.3 can be rewritten by substitution of (4.5), (6.3) and (6.1). We obtain for $k,l\in\mathbb{Z}$:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(\mathcal{J}^c_{\alpha,k}\mathcal{J}^c_{\alpha,l})(q^n;q)\,\frac{q^n\,(q^{n+1};q)_\infty}{(q^{n+\alpha+1};q)_\infty}\\
&\qquad+\sum_{p=-\infty}^{\infty}(\mathcal{J}^c_{\alpha,k}\mathcal{J}^c_{\alpha,l})(-cq^{p-\alpha-1};q)\,\frac{cq^{p-\alpha-1}\,(-cq^{p-\alpha};q)_\infty}{(-cq^p;q)_\infty}\\
&\qquad\qquad=\delta_{k,l}q^{-k(\alpha+1)}\,\frac{(q;q)_\infty^2}{(q^{\alpha+1};q)_\infty^2}\,\frac{(-cq^{-\alpha-1},-q^{\alpha+2}c^{-1},-q^{k+1}c^{-1};q)_\infty}{(-c,-qc^{-1};q)_\infty}\,.
\end{aligned}\tag{6.5}
$$

We will show that the orthogonality relations (6.5) can be rigorously obtained as a limit case of the orthogonality relations for big q-Jacobi polynomials.

Let $0<a<q^{-1}$, $b>-q^{-1}$, $c>0$. *Big q-Jacobi polynomials* are defined by

$$
P_k(x;a,b,-c;q):={}_3\varphi_2\left(\begin{matrix}q^{-k},abq^{k+1},x\\aq,-cq\end{matrix};q,q\right)\quad(x\in\mathbb{R},\ k\in\mathbb{Z}_{\geq 0}).\tag{6.6}
$$

Formally we have

$$
\lim_{r\to\infty}P_{r-k}(q^{\alpha+1}x;q^\alpha,b,-cq^{-r-1};q)=\frac{1}{(-c^{-1}q^{k+1};q)_\infty}\,\mathcal{J}_k(x;\alpha,c;q).\tag{6.7}
$$

(Substitute (6.6) and (6.2) and take termwise limits in the ${}_3\varphi_2$.) The orthogonality relations for big q-Jacobi polynomials are given by

$$
\begin{aligned}
&\int_{-cq}^{aq}(P_kP_l)(x;a,b,-c;q)\,\frac{(x/a,-x/c;q)_\infty}{(x,-bx/c;q)_\infty}\,d_qx\\
&\qquad\qquad=\delta_{k,l}\,M\,\frac{1-abq}{1-abq^{2k+1}}\,\frac{(q,bq,-abq/c;q)_k}{(abq,aq,-cq;q)_k}\,(acq^2)^k\,q^{k(k-1)/2},
\end{aligned}\tag{6.8}
$$

where

$$M := \frac{(1-q)\,aq\,(q,-c/a,-aq/c,abq^2;q)_\infty}{(aq,bq,-cq,-abq/c;q)_\infty}\,,$$

see [7, (7.3.12)–(7.3.14)]. (Note the error on the right-hand side of [7, (7.3.13)]: the factor $(-ac)^n$ must be replaced by $(-acq^2)^{-n}$.)

Formally the orthogonality relations (6.5) can be obtained as a limit case of the orthogonality relations (6.8) for big q-Jacobi polynomials. In (6.8) just replace k by $r-k$, l by $r-l$, c by cq^{r-1}, and a by q^α, and let $r \to \infty$. Because of these limit results we call the functions $x \mapsto \mathcal{J}_k(x;\alpha,c;q)$ *big q-Besel functions.*

For $b=0$ we can transform the right-hand side of (6.6) by the transformation formula [7, Exercise 1.15(i)]:

$$\text{(6.9)} \quad \widetilde{P}_k(x;a,0,-c;q) := (-c^{-1}q^{-k};q)_k\, P_k(x;a,0,-c;q) = {}_2\varphi_1\left(\begin{matrix} q^{-k}, aq/x \\ aq \end{matrix}; q, -x/c\right).$$

Formula (6.7) for $b=0$ can also be obtained as a formal termwise limit by substituting (6.9) and (6.1) and by taking termwise limits in the ${}_2\varphi_1$.

Fix $\alpha > -1$, $c > 0$, and let $r \in \mathbb{Z}_{\geq 0}$. From (6.8) we get orthogonality relations for the functions $x \mapsto \widetilde{P}_{r-k}(q^{\alpha+1}x;q^\alpha,0,-cq^{-r-1};q) \quad (k=r,r-1,\dots)$:

$$\text{(6.10)} \quad \begin{aligned} &\sum_{n=0}^{\infty} (\widetilde{P}_{r-k}\widetilde{P}_{r-l})(q^{n+\alpha+1};q^\alpha,0,-cq^{-r-1};q)\; \frac{q^n\,(q^{n+1},-c^{-1}q^{\alpha+n+r+2};q)_\infty}{(q^{\alpha+n+1};q)_\infty} \\ &+ \sum_{p=-r}^{\infty} (\widetilde{P}_{r-k}\widetilde{P}_{r-l})(-cq^p;q^\alpha,0,-cq^{-r-1};q)\; \frac{cq^{p-\alpha-1}\,(-cq^{p-\alpha},q^{p+r+1};q)_\infty}{(-cq^p;q)_\infty} \\ &\qquad = \delta_{k,l}\, \frac{(q,-cq^{-r-\alpha-1},-q^{\alpha+r+2}/c;q)_\infty}{(q^{\alpha+1},-cq^{-r};q)_\infty}\, \frac{(q,-c^{-1}q^{k+1};q)_{r-k}}{(q^{\alpha+1};q)_{r-k}}\, q^{(\alpha+1)(r-k)}. \end{aligned}$$

Here $k,l = r,r-1,\dots$. Note that the orthogonality relations (6.10) and (6.5) have the same structure. We want to show that we can take a rigorous limit for $r \to \infty$ of (6.10) which yields (6.5) preserving this structure.

Proposition 6.1. *Fix $\alpha > -1$, $c > 0$, $k \in \mathbb{Z}$. Then, for each $x \in \mathbb{R}$ we have the pointwise limit*

$$\text{(6.11)} \qquad \lim_{r\to\infty} \widetilde{P}_{r-k}(q^{\alpha+1}x;q^\alpha,0,-cq^{-r-1};q) = \mathcal{J}^c_{\alpha,k}(x;q).$$

Furthermore, if $M > 0$ then

$$\text{(6.12)} \quad \left.\begin{aligned} \left|\widetilde{P}_{r-k}(q^{\alpha+1}x;q^\alpha,0,-cq^{-r-1};q)\right| \\ \left|\mathcal{J}^c_{\alpha,k}(x;q)\right| \end{aligned}\right\} \leq {}_1\phi_1\left(\begin{matrix} -M^{-1} \\ q^{\alpha+1} \end{matrix}; q, \frac{-q^{\alpha+k+2}M}{c}\right)$$

$$\textit{for } |x| \leq M \textit{ and } r = k, k+1, \dots .$$

Proof. Write

$$\widetilde{P}_{r-k}(q^{\alpha+1}x;q^\alpha,0,-cq^{-r-1};q) = {}_2\phi_1\left(\begin{matrix} q^{-r+k}, x^{-1} \\ q^{\alpha+1} \end{matrix}; q, -\frac{q^{\alpha+r+2}x}{c}\right) = \sum_{j=0}^\infty t_r(j,x)$$

with

$$t_r(j,x) := \frac{(q^{-r+k},x^{-1};q)_j}{(q^{\alpha+1},q;q)_j}\,(-c^{-1}q^{\alpha+r+2}x)^j \quad \text{(vanishing if } j > r-k\text{)}.$$

Also write

$$\mathcal{J}^c_{\alpha,k}(x;q) = {}_1\phi_1\left(\begin{matrix} x^{-1} \\ q^{\alpha+1} \end{matrix}; q, -c^{-1}xq^{k+\alpha+2}\right) = \sum_{j=0}^\infty t(j,x)$$

with

$$t(j,x) := \frac{(x^{-1};q)_j}{(q^{\alpha+1},q;q)_j}\,q^{\frac{1}{2}j(j-1)}\,(c^{-1}xq^{k+\alpha+2})^j.$$

Then

$$\lim_{r\to\infty} t_r(j,x) = t(j,x) \tag{6.13}$$

and

$$|t_r(j,x)| \le T(j,M) \quad \text{for } r \ge k \text{ and } |x| \le M, \tag{6.14}$$

where

$$T(j,M) := \frac{(-M^{-1};q)_j}{(q^{\alpha+1},q;q)_j}\,q^{\frac{1}{2}j(j-1)}\,(c^{-1}Mq^{k+\alpha+2})^j$$

and

$$\sum_{j=0}^\infty T(j,M) = {}_1\phi_1\left(\begin{matrix} -M^{-1} \\ q^{\alpha+1} \end{matrix}; q, -c^{-1}Mq^{k+\alpha+2}\right) < \infty.$$

Then the limit (6.11) follows from (6.13) and (6.14) by dominated convergence.

For the proof of (6.14) we have used that, for $i = 0,1,\ldots,r-k-1$ and $|x| \le M$:

$$\begin{aligned} \big|(1-q^{-r+k+i})(1-x^{-1}q^i)q^{r-k}x\big| &= \big|(q^i-q^{r-k})(x-q^i)\big| \\ &\le q^i(M+q^i) = (1+M^{-1}q^i)q^iM. \qquad \square \end{aligned}$$

For $\widetilde{P}_{r-k}(-cq^p;q^\alpha,0,-cq^{-r-1};q)$ as $-r \le p \le -k$, $r\to\infty$ we need a more refined estimate in order to be able to take limits in (6.10).

Proposition 6.2. *Fix $\alpha > -1$, $c > 0$, $k \in \mathbb{Z}$, Let $r \in \{k, k+1, \dots\}$. Then, for $p \in \{-k, -k-1, \dots\}$ we have*

$$\widetilde{P}_{r-k}(-cq^p; q^\alpha, 0, -cq^{-r-1}; q) = (q;q)_{r-k} \frac{(-cq^p;q)_{-k-p}}{(q^{\alpha+1}, q; q)_{-k-p}} (-c^{-1}q^{\alpha-p+1})^{-k-p} \times {}_2\phi_2\left(\begin{matrix} q^{-r-p}, -cq^{-k} \\ q^{\alpha+1-k-p}, q^{-k-p+1} \end{matrix}; q, -c^{-1}q^{\alpha-k-p+r+2}\right) \tag{6.15}$$

and

$$\begin{aligned}\mathcal{J}^c_{\alpha,k}(-cq^{p-\alpha-1}; \alpha, c; q) &= (q;q)_\infty \frac{(-cq^p;q)_{-p-k}}{(q^{\alpha+1};q;q)_{-p-k}} (-c^{-1}q^{-p+\alpha+1})^{-p-k} \\ &\quad \times {}_1\phi_2\left(\begin{matrix} -cq^{-k} \\ q^{-p-k+1}, q^{\alpha+1-p-k} \end{matrix}; q, -c^{-1}q^{-2p-k+\alpha+2}\right) \\ &= \lim_{r\to\infty} \widetilde{P}_{r-k}(-cq^p; q^\alpha, 0, -cq^{-r-1}; q).\end{aligned} \tag{6.16}$$

Furthermore,

$$\left.\begin{matrix} \left|\widetilde{P}_{r-k}(-cq^p; q^\alpha, 0, -cq^{-r-1}; q)\right| \\ \left|\mathcal{J}^c_{\alpha,k}(-cq^{p-\alpha-1}; \alpha, c; q)\right| \end{matrix}\right\} \le \frac{(-cq^p;q)_{-k-p}}{(q^{\alpha+1}, q; q)_{-k-p}} (c^{-1}q^{\alpha-p+1})^{-k-p} \times {}_1\phi_2\left(\begin{matrix} -cq^{-k} \\ q^{\alpha+1}, q \end{matrix}; q, c^{-1}q^{k+\alpha+2}\right) \tag{6.17}$$

for $r \in \{k, k+1, \dots\}$ and $p \in \{-k, -k-1, \dots\}$.

Proof. By Jackson's tranformation formula [7, (1.5.4)] we have

$$\begin{aligned}{}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) &= \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2\left(\begin{matrix} a, c/b \\ c, az \end{matrix}; q, bz\right) \\ &= \frac{1}{(z;q)_\infty} \sum_{j=0}^\infty \frac{(a, c/b; q)_j\, (azq^j; q)_\infty}{(c, q; q)_j} q^{\frac12 j(j-1)} (-bz)^j.\end{aligned}$$

Put $z := q^{-s+1}a^{-1}$, where $s \in \mathbb{Z}_{\ge 0}$. Then $(azq^j;q)_\infty = (q^{-s+j+1};q)_\infty$, so in the last sum the summation will start at $j = s$. Replace the summation index j by $j+s$ and write the resulting sum again as a ${}_2\phi_2$. Then we obtain for $s \in \mathbb{Z}_{\ge 0}$:

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, a^{-1}q^{-s+1}\right) = \frac{(q;q)_\infty\, (c/b;q)_s\, b^s}{(qa^{-1};q)_\infty\, (c;q)_s}\, {}_2\phi_2\left(\begin{matrix} aq^s, cq^s/b \\ cq^s, q^{s+1} \end{matrix}; q, qa^{-1}b\right).$$

This yields (6.15) (here $s = -k-p$).

The first identity in (6.16) can be derived in a similar way by starting with

$${}_1\phi_1\left(\begin{matrix} a \\ c \end{matrix}; q, z\right) = \frac{1}{(c;q)_\infty} \sum_{j=0}^\infty \frac{(az/c;q)_j\, (q^j z;q)_\infty}{(q;q)_j} q^{\frac12 j(j-1)} (-c)^j$$

(this is (3.5) with the right-hand side expanded) and then putting $z := q^{-s+1}$ ($s \in \mathbb{Z}_{\geq 0}$).

Expand the right-hand side of (6.15). So

$$\widetilde{P}_{r-k}(-cq^p; q^\alpha, 0, -cq^{-r-1}; q) = \sum_{j=0}^{\infty} t_r(j; p)$$

with

$$\begin{aligned} t_r(j,p) := {} & (q;q)_{r-k} \frac{(-cq^p;q)_{-k-p}}{(q^{\alpha+1},q;q)_{-k-p}} (-c^{-1}q^{\alpha-p+1})^{-k-p} \\ & \times \frac{(q^{-r-p}, -cq^{-k};q)_j}{(q^{\alpha+1-k-p}, q^{-k-p+1}, q;q)_j} q^{\frac{1}{2}j(j-1)} (c^{-1}q^{\alpha-k-p+r+2})^j \quad \text{(vanishing if } j > r+p\text{)}. \end{aligned}$$

Then

$$|t_r(j,p)| \leq T(j,p) \quad \text{for } r \in \{k, k+1, \dots\} \text{ and } p \in \{-k, -k-1, \dots\}, \tag{6.18}$$

where

$$T(j,p) := \frac{(-cq^p;q)_{-k-p}}{(q^{\alpha+1},q;q)_{-k-p}} (c^{-1}q^{\alpha-p+1})^{-k-p} \frac{(-cq^{-k};q)_j}{(q^{\alpha+1},q,q;q)_j} q^{j(j-1)} (c^{-1}q^{k+\alpha+2})^j$$

and $\sum_{j=0}^{\infty} T(j,p)$ equals the right-hand side of (6.17).

For the proof of (6.18) we have used that, for $i = 0, 1, \dots, r+p-1$,

$$\left|(1-q^{-r-p+i})q^{r-p-k}\right| = \left|(q^i - q^{r+p})q^{-2p-k}\right| \leq q^i q^k.$$

For fixed p, the limit formula in (6.16) follows from (6.11), but it follows also by taking a termwise limit for $r \to \infty$ on the right-hand side of (6.15) and by using dominated convergence in view of (6.18). $\square$

Proof of (6.5). Suppose $k \geq l$ (without loss of generality). As $r \to \infty$, the right-hand side of (6.10) tends to the right-hand side of (6.5), and each term on the left-hand side of (6.10) tends to the corresponding term on the left-hand side of (6.5) (because of (6.11)). We will show that the left-hand side of (6.10) tends rigorously to the left-hand side of (6.5) by splitting up the left-hand side of (6.10) as

$$\sum_{n=0}^{\infty} + \sum_{p=-l+1}^{\infty} + \sum_{p=-k+1}^{l} + \sum_{p=-\infty}^{-k},$$

and by using dominated convergence for each of the three infinite sums.

As for the first sum, by (6.12) the n^{th} term is bounded in absolute value by Cq^n, where $C > 0$ and independent of n. Similarly, the p^{th} term in the second sum is bounded in absolute value by Cq^p.

By (6.17), the p^{th} term in the fourth sum is bounded in absolute value by

$$C\frac{(-cq^p;q)_{-k-p}}{(q^{\alpha+1},q;q)_{-k-p}}\,\frac{(-cq^p;q)_{-l-p}}{(q^{\alpha+1},q;q)_{-l-p}}\,(c^{-1}q^{\alpha-p+1})^{-k-l-2p}q^p\,\frac{(-cq^{p-\alpha};q)_\infty}{(-cq^p;q)_\infty}\le A\,q^{p^2}\,B^p,$$

where $A,B,C>0$ and independent of p. $\square$

Remark. As we said in the Introduction the limit transition (6.7) from big q-Jacobi polynomials to big q-Bessel functions is inspired by what happens at a quantum group level.

The regular representation of the quantum group $SU_q(2)$ on Podleś' quantum spheres naturally decomposes into irreducibles, generated by suitable spherical functions which turn out to depend on a single variable. Such functions can thus be identified with ordinary big q-Jacobi polynomials, see [22].

In the limit transition from $SU_q(2)$ to the Euclidean quantum group $E_q(2)$, quantum spheres are replaced by quantum hyperboloids, i.e., algebras in two generators z and $\bar{z}$ such that $z\bar{z}=q^2\bar{z}z+1-q^2$. The corresponding regular representation decomposes into irreducibles, each of which has an ∞-dimensional basis consisting of certain formal power series in $1-\bar{z}z$, see [4]. This makes it possible to identify such power series with ordinary functions, and more precisely with q-Bessel functions of the form $\mathcal{J}^c_{\alpha,k}(x;q)$.

More precisely, the paper [4] gives in (4.9) the basis elements of the irreducible representation space as explicit formal power series $J_r^{(q)}$. After the substitution $\bar{z}^jz^j=(1-\bar{z}z;q^{-2})_j$ (see their Remark (4.10) (ii)) one can identify the series $J_r^{(q)}$ for $\mathcal{E}=-q^{2k}$ with $\mathcal{J}_{r,k}^{q^2/(q^2-1)^2}(1-\bar{z}z;q^2)$ as defined by (6.1).

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