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Operational formulae for certain classical polynomials - III

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OPERATIONAL FORMULAE FOR CERTAIN CLASSICAL POLYNOMIALS - III

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1. INTRODUCTION

In an earlier paper [1] we found the operational formula

$$(1.1)\qquad \begin{aligned} &\prod_{j=1}^{n} \{x^2D + (2j + a)x + b\} \\ &= \sum_{r=0}^{n} \binom{n}{r} b^{n-r}x^{2r}y_{n-r}(x, a + 2r + 2, b)D^r . \end{aligned}$$

where $y_n(x, a, b)$ is the generalised Bessel polynomials as defined by Krall and Frink [2]. In [1] we also noticed the following consequences of (1.1):

$$(1.2)\qquad b^n y_n(x, a + 2, b) = \prod_{j=1}^{n} \{x^2D + (2j + a)x + b\} \cdot 1$$

$$(1.3)\qquad 2^n y_n(x) = \prod_{j=1}^{n} (x^2D + 2jx + 2) \cdot 1$$

where $y_n(x)$ is the special case of the polynomials $y_n(x, a, b)$,

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obtained by taking $a = b = 2$.

$$(1.4)\qquad \begin{aligned} &b^2\{y_{n+1}(x, a, b) - y_n(x, a, b)\} \\ &= (2n + a)x\{by_n(x, a, b) + nxy_{n-1}(x, a + 2, b)\} \end{aligned}$$

which implies two well-known formulae:

(i) $b\{y_n(x, a + 1, b) - y_n(x, a, b)\} = nx\, y_{n-1}(x, a + 2, b)$

(ii) $b\, y_n'(x, a, b) = n(n + a - 1)y_{n-1}(x, a + 2, b)$;

$$(1.5)\qquad \begin{aligned} &y_{n+m}(x, a, b) \\ &= \sum_{r=0}^{min(m,n)} \binom{m}{r}\binom{n}{r} r!\,(m + 2n + a - 1)_r (x/b)^{2r} y_{n-r}(x, \\ &\qquad a + 2r, b) y_{m-r}(x, a + 2n + 2r, b)\ . \end{aligned}$$

Later in a recent paper [3] we have obtained the operational formula

$$(1.6)\qquad x^{2n}\left[D + \frac{2(nx + 1)}{x^2}\right]^n = \sum_{r=0}^{n}\binom{n}{r} 2^{n-r} x^{2r} y_{n-r}(x, 2 + 2r, 2) D^r\ .$$

which generalises the operational formula derived by Rajagopal [4]:

$$(1.7)\qquad x^{2n}\left[D + \frac{2(nx + 1)}{x^2}\right]^n \cdot 1 = 2^n y_n(x)\ .$$

In [3] we have also derived the following formulae:

$$(1.8)\qquad x^n\left[D - \frac{2x + n + 1}{x}\right]^n = \sum_{r=0}^{n}\binom{n}{r}(-2)^{n-r} x^r \theta_{n-r}(x, 2 + r, 2) D^r\ .$$

where $\theta_n(x, a, b)$ are those polynomials defined by Burchnall [5]:

$$\theta_n(x, a, b) = (-b)^{-n} e^{bx} x^{a+2n-1} D^n (x^{-a-n+1} e^{-bx})$$

$$(1.9)\qquad \frac{x^n}{n!}\left[D + \frac{a + n - x}{x}\right]^n = \sum_{r=0}^{n} \frac{x^r}{r!}\, L_{n-r}^{(a+r)}(x) D^r\ .$$

where $L_n^{(\alpha)}(x)$ is the generalised Laguerre polynomials. In this connection we like to mention that we have been inspired by Carlitz's work [6]. The interesting result of Carlitz is

$$\prod_{j=1}^{n}(xD - x + \alpha + j) = n\,!\sum_{r=0}^{n}\frac{x^r}{r\,!}\,L_{n-r}^{(\alpha+r)}(x)D^r\,. \tag{1.10}$$

wherefrom he obtains $n\,!\,L_n^{(\alpha)}(x) = \prod_{j=1}^{n}(xD - x + \alpha + j)\cdot 1$.

Thus far we have tried to give a systematic development of the operational formulae derived in [1] and [3], for certain classical polynomials. The object of this paper is to discuss in the same line the polynomials $\theta_n(x, a, b)$ as defined by Burchnall.

2. Burchnall defined the polynomials $\Phi_n(x, a, b)$ by

$$\Phi_n(x, a, b) = x^n y_n(x^{-1}, a, b)\,.$$

He showed that $\Phi_n(x, a, b)$ was a solution of

$$\delta(\delta + 1 - a - 2n)z = bx(\delta - n)z\,; \qquad (\delta \equiv xD) \tag{2.1}$$

and that $e^{-bx}\Phi_n(x, a, b)$ was a solution of

$$\delta(\delta + 1 - a - 2n)\omega = -\,bx(\delta - n - a + 2)\omega\,. \tag{2.2}$$

Further he showed that the equation (2.2) had the solution

$$\omega = (\delta - n - a + 1)(\delta - n - a)\ldots(\delta - 2n - a + 2)e^{-bx}, \tag{2.3}$$

wherefrom he deduced that

$$\Phi_n(x, a, b) = (-\,b)^{-n}e^{bx}x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}) \tag{2.4}$$

In particular, when $a = b = 2$, he pointed out that

$$\theta_n(x) \equiv \Phi_n(x, 2, 2) = \left(-\,\frac{1}{2}\right)^n e^{2x}x^{2n+1}D^n(x^{-n-1}e^{-2x}) \tag{2.5}$$

We first mention that we shall write $\theta_n(x, a, b)$ for $\Phi_n(x, a, b)$ throughout this paper. Now we have for any arbitrarily differentiable function y of x:

$$
\begin{aligned}
&x^n D^n(x^{-a-n+1}y) \\
&= \delta(\delta - 1) \ldots (\delta - n + 1)(x^{-a-n+1}y) \\
&= x^{-a-n+1}(\delta - a - n + 1)(\delta - a - n) \ldots (\delta - a - 2n + 2)y \\
&\therefore\ x^{a+2n-1}D^n(x^{-a-n+1}y)
\end{aligned}
$$

$$
(2.6) \qquad = (\delta - a - n + 1)(\delta - a - n) \ldots (\delta - a - 2n + 2)y
$$

Now since the linear operators on the right of (2.6) are commutative, we can write (2.6) as

$$
(2.7) \qquad x^{a+2n-1}D^n(x^{-a-n+1}y) = \prod_{j=1}^{n} (\delta - a - 2n + j + 1)y
$$

Thus we easily have

$$
(2.8) \qquad
\begin{aligned}
&x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}y) \\
&= \prod_{j=1}^{n} (\delta - a - 2n + j + 1)e^{-bx}y
\end{aligned}
$$

But in [3] we have proved that

$$
(2.9) \qquad
\begin{aligned}
&e^{bx}x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}y) \\
&= \sum_{r=0}^{n} \binom{n}{r}(-b)^{n-r}x^r\theta_{n-r}(x, a + r, b)D^r y
\end{aligned}
$$

It therefore follows from (2.8) and (2.9)

$$
(2.10) \qquad
\begin{aligned}
&e^{bx}\prod_{j=1}^{n} (\delta - a - 2n + j + 1)e^{-bx}y \\
&= \sum_{r=0}^{n} \binom{n}{r}(-b)^{n-r}x^r\theta_{n-r}(x, a + r, b)D^r y
\end{aligned}
$$

As a special case of (2.10) we notice that

$$(2.11)\qquad e^{bx}\prod_{j=1}^{n}(\delta - a - 2n + j + 1)e^{-bx} = (-b)^n\theta_n(x, a, b)$$

which may be compared with the remark made by Burchnall in (2.2) and (2.3).

Now we shall find a more interesting operational formula for $\theta_n(x, a, b)$.

From (2.8) we again derive

$$(2.12)\qquad \begin{aligned} &e^{bx}x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}y)\\ &=\prod_{j=1}^{n}(xD - bx - a - 2n + j + 1)y\ . \end{aligned}$$

Now a comparison of (2.9) and (2.12) yields our desired result:

$$(2.13)\qquad \begin{aligned} &\prod_{j=1}^{n}(xD - bx - a - 2n + j + 1)y\\ &=\sum_{r=0}^{n}\binom{n}{r}(-b)^{n-r}x^r\theta_{n-r}(x, a + r, b)D^r y\ . \end{aligned}$$

When $a = b = 2$, we get from (2.13)

$$(2.14)\qquad \begin{aligned} &\prod_{j=1}^{n}(xD - 2x - 2n + j - 1)y\\ &=\sum_{r=0}^{n}\binom{n}{r}(-2)^{n-r}x^r\theta_{n-r}(x, 2 + r, 2)D^r y\ . \end{aligned}$$

As particular cases of (2.13) and (2.14) we note that

$$(2.15)\qquad (-b)^n\theta_n(x, a, b) = \prod_{j=1}^{n}(xD - bx - a - 2n + j + 1)\cdot 1$$

$$(2.16)\qquad (-2)^n\theta_n(x) = \prod_{j=1}^{n}(xD - 2x - 2n + j - 1)\cdot 1$$

In this connection it is interesting to note that a comparison of (1.10) with the result [7]:

$$(-2)^n\theta_n\left(\frac{x}{2}\right) = n\,!\,L_n^{(-2n-1)}(x)\ ;$$

implies that

$$(2.17)\qquad (-2)^n\theta_n\left(\frac{x}{2}\right) = \prod_{j=1}^{n}(xD - x - 2n + j - 1)\cdot 1\ .$$

Lastly we like to mention a consequence of the formula (2.16). To this end, we observe from (2.16)

$$(2.18)\qquad \begin{aligned} -2(xD - 2x - n)\theta_n(x) &= (xD - 2x - 2n)(xD - \\ &- 2x - 2n + 1)\theta_{n-1}(x) \end{aligned}$$

which implies

$$(2.19)\qquad \begin{aligned} &2\left\{x\theta_n' - (2x + n)\theta_n\right\} + x^2\theta_{n-1}'' - 2(2x^2 + 2nx - x)\theta_{n-1}' \\ &+ 2\left\{2x^2 + 2(2n - 1)x + 2n^2 - n\right\}\theta_{n-1} = 0\ . \end{aligned}$$

To verify the truth of (2.19) we observe [5, formulae (15), (16)]

$$(2.20)\qquad \theta_n' - \theta_n = -x\theta_{n-1}$$

$$(2.21)\qquad \theta_{n+1} - x^2\theta_{n-1} = (2n + 1)\theta_n$$

At first we shall prove the formula

$$(2.22)\qquad x\theta_{n-1}' + \theta_n = (x + 2n - 1)\theta_{n-1}$$

which is not mentioned in Burchnall's paper. For this, we easily notice from (2.20) and (2.21)

$$\theta_{n+1} + x(\theta_n' - \theta_n) = (2n + 1)\theta_n$$

$$\text{or,}\qquad x\theta_n' + \theta_{n+1} = (x + 2n + 1)\theta_n\ .$$

Now we have

$$
\begin{aligned}
& 2\{x\theta_n' - (2x + n)\theta_n\} \\
(2.23) \qquad & = 2x(\theta_n - x\theta_{n-1}) - 2(2x + n)\theta_n \\
& = -2x^2\theta_{n-1} - 2(n + x)\theta_n
\end{aligned}
$$

Thus eliminating θ_n between (2.19) and (2.22) with the help of (2.23), we obtain

$$(2.24) \qquad x\theta''_{n-1} - 2(x + n - 1)\theta'_{n-1} + 2(n - 1)\theta_{n-1} = 0$$

which is the differential equation for $\theta_{n-1}(x)$ and which may be compared with Burchnall's form:

$$(2.25) \qquad \delta(\delta - 2n - 1)\theta_n = 2x(\delta - n)\theta_n \, .$$

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REFERENCES

[1] CHATTERJEA S. K.: Operational formulae for certain classical polynomials - I, communicated to the Quart. Jour. Math. (Oxford).

[2] KRALL H. L. and FRINK O.: A new class of orthogonal polynomials: The Bessel polynomials, Trans. Amer. Math. Soc., Vol. 65, pp. 100-115, (1949).

[3] CHATTERJEA S. K.: Operational formulae for certain classical polynomials - II, communicated to the Rend. Sem. Mat. Univ. Padova.

[4] RAJAGOPAL A. K.: On some of the classical orthogonal polynomials, Amer. Math. Monthly, Vol. 67, pp. 166-169, (1960).

[5] BURCHNALL J. L.: The Bessel polynomials, Canad. Jour. Math., Vol. 3, pp. 62-68, (1951).

[6] CARLITZ L.: A note on the Laguerre polynomials, Michigan Math. Jour., Vol. 7, pp. 219-223, (1960).

[7] ——————: A note on the Bessel polynomials, Duke Math. Jour., Vol. 24, pp. 151-162, (1957).