

d -orthogonality of Little q -Laguerre type polynomials

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ABSTRACT

In this paper, we solve a characterization problem in the context of the d -orthogonality. That allows us, on one hand, to provide a q -analog for the d -orthogonal polynomials of Laguerre type introduced by the first author and Douak, and on the other hand, to derive new L_q -classical d -orthogonal polynomials generalizing the Little q -Laguerre polynomials. Various properties of the resulting basic hypergeometric polynomials are singled out. For $d = 1$, we obtain a characterization theorem involving, as far as we know, new L_q -classical orthogonal polynomials, for which we give the recurrence relation and the difference equation.

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1. Introduction

Over the past few years, many works appeared and dealt with the notion of multiple orthogonality [1–11] which is connected with the study of vector Padé approximants, simultaneous Padé approximants, and other problems such as vectorial continued fractions, polynomials solutions of the higher order differential equations (see, for instance, [4,1,12–14]). A convenient framework to discuss examples of multiple orthogonal polynomials consists in considering a subclass of multiple orthogonal polynomials known as d -orthogonal polynomials (see, for instance, [15–35]). To draw up our contribution in this direction, studying further examples, we recall the following notations and definitions.

Let \mathcal{P} be the vector space of polynomials with coefficients in \mathbb{C} and let \mathcal{P}' be its algebraic dual. We denote by $\langle u, f \rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A polynomial sequence $\{P_n\}_{n \geq 0}$ is called a *polynomial set* (PS, for shorter) if and only if $\deg P_n = n$ for all non-negative integers n .

Definition 1.1 (Van Iseghem [30] and Maroni [31]). Let d be a positive integer. A $\text{PS}\{P_n\}_{n \geq 0}$ is called *d -orthogonal* (d -OPS, for short) with respect to the d -dimensional vector of functionals $\Gamma = {}^t(\Gamma_0, \Gamma_1, \dots, \Gamma_{d-1})$ if it satisfies the following orthogonality relations:

$$\begin{cases} \langle \Gamma_k, P_r P_n \rangle = 0, & r > nd + k, n \in \mathbb{N} = \{0, 1, 2, \dots\}, \\ \langle \Gamma_k, P_n P_{nd+k} \rangle \neq 0, & n \in \mathbb{N}, \end{cases}$$

for each integer k belonging to $\mathbb{N}_d = \{0, 1, \dots, d-1\}$.

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In this context, we have the following result.

Theorem 1.2 (Maroni [31]). Let d be a positive integer. A $PS\{P_n\}_{n \geq 0}$ is d -orthogonal if and only if it satisfies a $(d + 1)$ -order recurrence relation of the type:

$$xP_n(x) = \sum_{k=0}^{d+1} \alpha_{k,d}(n)P_{n-d+k}(x),$$

where $\alpha_{d+1,d}(n)\alpha_{0,d}(n) \neq 0, n \geq d$, and by convention, $P_{-n} = 0, n \geq 1$.

This result, for $d = 1$, is reduced to the so-called Favard Theorem [36].

The d -orthogonal polynomials of Laguerre type (d -Laguerre polynomials, for short) were introduced by the first author and Douak [19,20]:

$$\ell_n^{\vec{\alpha}^d}(x) = \ell_n^{(\alpha_1, \dots, \alpha_d)}(x) = {}_1F_d \left(\begin{matrix} -n \\ \alpha_1 + 1, \dots, \alpha_d + 1 \end{matrix}; x \right), \tag{1.1}$$

where ${}_rF_s$ is the hypergeometric series defined by [37]:

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^r (a_j)_k}{\prod_{j=1}^s (b_j)_k} \frac{z^k}{k!},$$

with $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$.

Such polynomials also appear among the solutions of a characterization problem considered by the authors [22,16]. It was shown in [19] that these polynomials are related to Konhauser polynomials [38] Gould–Hopper polynomials [39] and Bateman functions [40]. Recently, the first author and Gaied [25] used the d -Laguerre polynomials to express the components of the Gould–Hopper type polynomials. The first author and Douak [19] stated for d -Laguerre polynomials various properties concerning a differential equation of order $d + 1$, a $(d + 1)$ -order recurrence relation, a generating function defined by means of the hyper-Bessel function, some differentiation formulas and a Koshlyakov formula involving the Meijer G -function.

The 2-Laguerre polynomials have been also studied by the first author and Douak [21] and by Van Assche and Yakubovich [9] in order to solve an open problem formulated by Prudnikov in [10].

Concerning how to obtain the d -dimensional vector of functionals for which the d -orthogonality of the d -Laguerre polynomials holds, the case $d = 2$ was treated by the first author and Douak [21] and the general case was solved by the second and the third authors [34].

The aim of this work is to prove for d -Laguerre polynomials their q -analogous. To this end, we recall first that the Little q -Laguerre polynomials are given in [41, p. 107] by:

$$P_n(x; a|q) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| q; qx \right), \quad a \notin \{0, q^{-1}, \dots\}, \quad 0 < q < 1, \tag{1.2}$$

where ${}_r\phi_s$ is the q -hypergeometric series defined in [41] by:

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} (-1)^{n(1+s-r)} q^{\binom{n}{2}(1+s-r)} \frac{z^n}{(q; q)_n}, \tag{1.3}$$

with $(a_1, \dots, a_r; q)_n := \prod_{j=1}^r (a_j; q)_n, r$ a positive integer or 0 (interpreting an empty product as 1), $(a; q)_0 := 1, (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j), n > 1$, and $(a; q)_\infty := \lim_{n \rightarrow +\infty} (a; q)_n$.

These polynomials are well-known in the theory of orthogonal polynomials as the q -analogs of the Laguerre ones [41]. That suggest us to consider the following characterization problem.

P: Find all d -orthogonal polynomials of type:

$$P_n(x; (b_s), (a_r)|q) = {}_{r+1}\phi_s \left(\begin{matrix} q^{-n}, a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; x \right), \tag{1.4}$$

where $0 < q < 1, r$ and s are positive integers or 0 (interpreting an empty product in (1.3) as 1), $\{a_i; i = 1, \dots, r\}$ and $\{b_j; j = 1, \dots, s\}$ are $r + s$ complex numbers independent of n and x such that $a_i, b_j \neq 1, q^{-1}, q^{-2}, \dots$, and $a_i \neq b_j$.

For $r = s = 1$, these polynomials are reduced to the Little q -Laguerre polynomials given by (1.2).

Such a characterization takes into account the fact that PS which are obtainable from one another by a linear change of variable are assumed equivalent.

Notice by the way that, this problem, for the limiting case $(d, q) = (1, 1)$ was set and treated under different aspects by many authors who took as starting point for their characterizations one of the properties related to the polynomials given by

(1.4) (see [42–44]). Recently, the first author and Douak [19] treated the limiting case $q = 1$ for a general positive integer d and obtained the generalized hypergeometric polynomials of Laguerre type given by (1.1) which are reduced to the classical Laguerre polynomials for $d = 1$.

Our contribution in this direction is to solve the problem P for $0 < q < 1$ and for general positive integer d . That allows us to introduce, as far as we know, new L_q -classical d -OPS of Little q -Laguerre type, which can be viewed as a q -analog of the d -orthogonal polynomials of Laguerre type. Then we focused our analysis on some properties of the obtained polynomials. That turn out to be: Limit relation, generating function, inversion formula, d -dimensional vector of functionals. The case $d = 1$ and $d = 2$ are specially carried out. Incidentally, in the case $d = 1$, we obtain a characterization theorem (Corollary 2.4) involving, as far as we know, new orthogonal polynomials, for which we give a recurrence relation and a difference equation.

The paper is organized as follows: In Section 2, we solve problem P . Section 3 is devoted to study miscellaneous properties of the obtained polynomials. In Section 4, we derive a recurrence relation and a difference equation for the new OPS given by Corollary 2.4.

2. A characterization theorem

As a solution of problem P , we state the following characterization theorem.

Theorem 2.1. *The only d -OPSs of type (1.4) are given by:*

$$P_n(x; (b_s)|q) = {}_{d+1}\Phi_s \left(\begin{matrix} q^{-n}, 0, 0, \dots, 0 \\ b_1, \dots, b_s \end{matrix} \middle| q; x \right), \quad (2.1)$$

where $s = 0, \dots, d$ and $\{b_j; j = 1, \dots, s\}$ are s complex numbers independent of n and x such that $b_j \neq 0, 1, q^{-1}, q^{-2}, \dots$ (for $s = 0$ interpreting an empty product in (1.3) as 1).

In order to prove this theorem, we first state the following lemma.

Lemma 2.2. *The PS $\{P_n\}_{n \geq 0}$ given by the identity (2.1) is a d -OPS.*

Proof. Let $n \geq d$ and $D(x) = \prod_{j=1}^s \left(1 - \frac{b_j}{q}x\right)$. Since $\deg \left[(q^{-n}; q)_d (1-x) \left(\frac{-x}{q}\right)^{d-s} D(x) \right] = d+1$ and the set $\{(q^{-n+d-l}; q)_l (q^{-n-1}x, q)_{d+1-l}\}_{0 \leq l \leq d+1}$ is a basis for $\mathbb{C}_{d+1}[x]$, the vector space of polynomials with coefficient in \mathbb{C} and of degree less or equal $(d+1)$, then there exist $d+2$ complex numbers $\alpha_{l,d}(n); l = 0, \dots, d+1$; such that:

$$(q^{-n}; q)_d (1-x) \left(\frac{-x}{q}\right)^{d-s} D(x) = \sum_{l=0}^{d+1} \alpha_{l,d}(n) (q^{-n+d-l}; q)_l (q^{-n-1}x, q)_{d+1-l}. \quad (2.2)$$

Multiplying both sides of the identity (2.2) for $x = q^k$ by

$$\varepsilon(n, k) = \begin{cases} \frac{((-1)^k q^{\binom{k}{2}})^{d-s}}{(q^{-n-1+k}; q)_{d+1-k} (q; q)_k (b_1, \dots, b_s; q)_k} & \text{if } 1 \leq k \leq d+1, \\ \frac{((-1)^k q^{\binom{k}{2}})^{d-s}}{(q^{-n-1+k}; q)_{k-d-1} (q; q)_k (b_1, \dots, b_s; q)_k} & \text{if } d+1 \leq k \leq n+1, \end{cases}$$

we get

$$\frac{(q^{-n}; q)_{k-1} ((-1)^{k-1} q^{\binom{k-1}{2}})^{d-s}}{(q; q)_{k-1} (b_1, \dots, b_s; q)_{k-1}} = \sum_{l=0}^{d+1} \alpha_{l,d}(n) \frac{(q^{-n+d-l}; q)_k ((-1)^k q^{\binom{k}{2}})^{d-s}}{(q; q)_k (b_1, \dots, b_s; q)_k}, \quad 1 \leq k \leq n+1.$$

On the other hand, it is easy to show, from the identity (2.2), that $\sum_{l=0}^{d+1} \alpha_{l,d}(n) = 0$. Consequently

$$\sum_{k=1}^{n+1} \frac{(q^{-n}; q)_{k-1} ((-1)^{k-1} q^{\binom{k-1}{2}})^{d-s}}{(q; q)_{k-1} (b_1, \dots, b_s; q)_{k-1}} x^k = \sum_{k=0}^{n+1} \sum_{l=0}^{d+1} \alpha_{l,d}(n) \frac{(q^{-n+d-l}; q)_k ((-1)^k q^{\binom{k}{2}})^{d-s}}{(q; q)_k (b_1, \dots, b_s; q)_k} x^k.$$

That, by virtue of the identity (2.1), we obtain

$$xP_n(x, (b_s), q) = \sum_{l=0}^{d+1} \alpha_{l,d}(n) P_{n-d+l}(x, (b_s), q), \quad n \geq d. \quad (2.3)$$

By replacing in the identity (2.2) $x = q^{n+1}$, we get:

$$(q^{-n-1}; q)_{d+1} \alpha_{d+1,d}(n) = (q^{-n}; q)_d (1 - q^{n+1}) (-q^n)^{d-s} D(q^{n+1}), \quad n \geq d. \tag{2.4}$$

It follows then $\alpha_{d+1,d}(n) \neq 0, n \geq d$.

Taking into account the fact that, the left-hand side of the identity (2.2) is a polynomial of degree $(d + 1)$, we deduce that $\alpha_{0,d}(n) \neq 0$. We conclude that the PS $\{P_n\}_{n \geq 0}$ satisfies the recurrence relation (2.3) with $\alpha_{d+1,d}(n) \alpha_{0,d}(n) \neq 0$. That, by virtue of Theorem 1.2, leads to the desired result. \square

Proof of Theorem 2.1. This proof is divided in two steps. The first one is devoted to show that $r \geq s$ and $a_i = 0; i = 1, \dots, r$, which will be used in the second step to prove that $b_j \neq 0; j = 1, \dots, s$ and $r = d$.

By using Theorem 1.2, the sequence $\{P_n\}_{n \geq 0}$ satisfies the following recurrence relation:

$$xP_n(x, (b_s), (a_r), q) = \sum_{l=0}^{d+1} \alpha_{l,d}(n) P_{n-d+l}(x, (b_s), (a_r), q), \quad \text{where } n \geq d \text{ and } \alpha_{d+1,d}(n) \alpha_{0,d}(n) \neq 0.$$

Substituting the expression of $P_{n-d+l}(x, (b_s), (a_r), q)$ given by (1.4) in this equality, we obtain:

$$xP_n(x, (b_s), (a_r), q) = \sum_{k=0}^{n+1} \sum_{l=0}^{d+1} \alpha_{l,d}(n) \frac{(q^{-n+d-l}; q)_k (a_1, \dots, a_r; q)_k}{(q; q)_k (b_1, \dots, b_s; q)_k} ((-1)^k q^{\binom{k}{2}})^{s-r} x^k.$$

On the other hand, from the identity (1.4), we have

$$xP_n(x, (b_s), (a_r), q) = \sum_{k=1}^{n+1} \frac{(q^{-n}; q)_{k-1} (a_1, \dots, a_r; q)_{k-1}}{(q; q)_{k-1} (b_1, \dots, b_s; q)_{k-1}} \left((-1)^{k-1} q^{\binom{k-1}{2}} \right)^{s-r} x^k.$$

Then, by identification for $1 \leq k \leq n + 1$, we get:

$$\sum_{l=0}^{d+1} \alpha_{l,d}(n) (q^{-n+d-l}; q)_k = (q^{-n}; q)_{k-1} (1 - q^k) \frac{D(q^k)}{N(q^k)} (-q^{k-1})^{r-s},$$

where $D(x) = \prod_{j=1}^s \left(1 - \frac{b_j}{q} x\right)$ and $N(x) = \prod_{j=1}^r \left(1 - \frac{a_j}{q} x\right)$. Hence, for $d + 1 \leq k \leq n + 1$, we have:

$$Q(q^k) = (-q^{k-1})^{r-s} R(q^k), \tag{2.5}$$

where R and Q are the polynomials defined by

$$Q(x) = \left(\sum_{l=0}^{d+1} \alpha_{l,d}(n) (q^{-n+d-l}; q)_l (q^{-n-1}x; q)_{d+1-l} \right) N(x),$$

$$R(x) = (1 - x)(q^{-n}; q)_d D(x).$$

Observing that: $\deg Q \leq 1 + d + r$ and $\deg R \leq s + 1$. Then $\max(\deg Q, \deg R + r - s) \leq d + 1 + r$ for $s \leq r$ and $\max(\deg Q + s - r, \deg R) \leq d + 1 + s$ for $r \leq s$. Choosing n such that $n > \max(2d + r, 2d + s)$. According to the identity (2.5), we deduce

$$\forall x \in \mathbb{C}; \quad Q(x) = \left(\frac{-x}{q} \right)^{r-s} R(x). \tag{2.6}$$

Taking into account the fact that $R(0) \neq 0$, we obtain $r \geq s$.

On the other hand, from the identity (2.6), it may be seen that the polynomial $\left(\frac{-x}{q}\right)^{r-s} R(x)$ is a multiple of $N(x)$. However, $N(0) \neq 0, N(1) \neq 0$ and, N and D are coprime. Consequently $N \equiv 1$ i.e. $a_i = 0; i = 1, \dots, r$.

Observing that $a_i \neq b_j; i = 1, \dots, r; j = 1, \dots, s$. Then $b_j \neq 0; j = 1, \dots, s$. So that $\deg R = 1 + s$ and $\deg Q = d + 1$. Then according to the identity (2.6), we obtain $r = d$ and $s \leq d$.

To prove the converse we use Lemma 2.2. \square

Remark 2.3. Theorem 2.1 provides the following class of q -polynomials

$$P_n(x; (b_s)|q) = {}_{d+1}\Phi_s \left(\begin{matrix} q^{-n}, 0, \dots, 0 \\ b_1, \dots, b_s \end{matrix} \middle| q; qx \right), \quad s = 0, \dots, d, \tag{2.7}$$

which appears to be a new d -OPS for $(d, s) \neq (1, 1)$. In particular, we have a new orthogonal polynomial set for $(d, s) = (1, 0)$. The PS given by (2.7) will be called Little q -Laguerre type PS, since it is reduced to the Little q -Laguerre PS for $(d, s) = (1, 1)$ and according to the properties obtained in Section 3.

For $d = 1$, **Theorem 2.1** is reduced to the following.

Corollary 2.4. A PS $\{P_n\}_{n \geq 0}$ defined by (1.4) with $d = 1$ is orthogonal if and only if it is the Little q -Laguerre polynomials defined by the identity (1.2) or the polynomials given by:

$$P_n(x|q) = {}_2\Phi_0 \left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix} \middle| q; qx \right). \quad (2.8)$$

3. Miscellaneous properties of the obtained q -polynomials

Our purpose in this section is to state some properties of the obtained d -OPSs, generalizing in a natural way the Little q -Laguerre ones.

3.1. Link between the d -OPS of Laguerre type and the d -OPS of Little q -Laguerre type

We recall that, from the explicit expression of the Little q -Laguerre polynomials $p_n(x; a|q)$ and the Laguerre polynomials $L_n^{(\alpha)}(x)$, we have [41, p. 142]:

$$\lim_{q \rightarrow 1} p_n((1-q)x; q^\alpha|q) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}. \quad (3.1)$$

In this subsection, we use a similar investigation to derive a limit relation between the Little q -Laguerre type polynomials given by (2.7) and the Laguerre type polynomials defined by (1.1), which prove that the obtained Little q -Laguerre type polynomials represent the q -analogs of the d -Laguerre polynomials.

Replacing in the identity (2.7) b_j by q^{α_j+1} ; $j = 1, \dots, s$; and x by $(1-q)^s x$, we obtain:

$$P_n((1-q)^s x; (q^{\alpha_s+1})|q) = \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q^{\alpha_1+1}, \dots, q^{\alpha_s+1}, q; q)_k} (1-q)^{sk} ((-1)^k q^{\binom{k}{2}})^{s-d} (qx)^k.$$

That, by virtue of (1.1), leads to

$$\lim_{q \rightarrow 1} P_n((-1)^d (q-1)^s x; (q^{\alpha_s+1})|q) = \ell_n^{\vec{\alpha}_s}(x), \quad s \leq d,$$

which, for $s = d$, becomes

$$\lim_{q \rightarrow 1} P_n((1-q)^d x; (q^{\alpha_d+1})|q) = \ell_n^{\vec{\alpha}_d}(x).$$

In the particular case $s = d = 1$, this limit relation is reduced to the limit relation given by (3.1).

3.2. Difference formulas

Let q be a real number, Hahn [45] defined a linear operator L_q by

$$L_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad |q| \neq 1,$$

where f is a suitable function for which the second member of this equality exists. This operator tends to the derivative operator D as $q \rightarrow 1$

Definition 3.1. Let $\{P_n\}_{n \geq 0}$ be a d -OPS. Put $Q_n(x) = L_q P_{n+1}(x)$, $n \geq 0$. If the sequence $\{Q_n\}_{n \geq 0}$ is also d -orthogonal, the sequence $\{P_n\}_{n \geq 0}$ is called L_q -classical d -OPS.

For $d = 1$, we meet the notion of the L_q -classical Hahn's property.

Using this definition we have the following.

Proposition 3.2. The Little q -Laguerre type polynomials defined by (2.7) are L_q -classical d -orthogonal polynomials. Moreover they satisfy:

$$L_q^k P_n(x; (b_s)|q) = \xi_{n,k}(q) P_{n-k}(q^{k(s-d)} x; (q^k b_s)|q), \quad k \geq 1, \quad (3.2)$$

where $\xi_{n,k}(q) = \frac{q^{k((-1)^k q^{\binom{k}{2}})^{s-d} (q^{-n}; q)_k}{\prod_{j=1}^s (b_j; q)_k (1-q)^k}$.

Proof. According to (2.7), we have

$$L_q^k P_n(x; (b_s)|q) = \frac{q(-1)^{s-d} (1-q^{-n})}{\prod_{j=1}^s (1-b_j)} \frac{(1-q^{-n})}{(1-q)} \sum_{k=0}^n \frac{(q^{-(n-1)}; q)_{k-1}}{(qb_1, \dots, qb_s; q)_{k-1} (q; q)_{k-1}} \left((-1)^{k-1} q^{\binom{k-1}{2}} \right)^{s-d} (q^{s-d+1} x)^{k-1}.$$

Hence

$$L_q P_n(x; (b_s)|q) = \frac{q(-1)^{s-d} (1 - q^{-n})}{\prod_{j=1}^s (1 - b_j) (1 - q)} P_{n-1}(q^{s-d}x; (qb_s)|q). \tag{3.3}$$

So, according to Definition 3.1, $\{P_n(x; (b_s)|q)\}_{n \geq 0}$ is L_q -classical.

Now, the iteration of (3.3) leads to (3.2). \square

Remark 3.3. • For $d = s = k = 1$, the identity (3.2) is reduced to the well-known difference formula associated to the Little q -Laguerre polynomials [41, p. 108]:

$$L_q P_n(x; a|q) = -\frac{q^{-n+1}(1 - q^n)}{(1 - q)(1 - aq)} P_{n-1}(x; aq|q).$$

- If we replace x by $(-1)^d(q - 1)^s$ and b_j by q^{α_j+1} ; $j = 1, \dots, s$; in (3.2), and we take the limit $q \rightarrow 1$, we obtain the following differential relations established by the first author and Douak [19]:

$$D^k \ell_n^{\vec{\alpha}_s}(x) = \frac{(-1)^k n!}{(n - k)! \prod_{j=1}^s (\alpha_j + 1)_k} \ell_{n-k}^{\vec{\alpha}_s+k}(x), \quad k \geq 1.$$

3.3. Generating functions

Proposition 3.4. The Little q -Laguerre type polynomials defined by (2.7) are generated by:

$$(t; q)_{\infty d+1} \phi_s \left(\begin{matrix} 0, \dots, 0 \\ b_1, \dots, b_s \end{matrix} \middle| q; xt \right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} P_n(x; (b_s)|q) t^n. \tag{3.4}$$

Proof. From the identity (2.7), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} P_n(x; (b_s)|q) t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k} q^{\binom{n+k}{2}}}{(q; q)_{n+k}} \frac{(q^{-(n+k)}; q)_k ((-1)^k q^{\binom{k}{2}})^{s-d}}{(b_1, \dots, b_s, q; q)_k} (xqt)^k t^n \\ &= \sum_{k=0}^{\infty} \frac{((-1)^k q^{\binom{k}{2}})^{s-d} (xt)^k}{(b_1, \dots, b_s, q; q)_k} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (-t)^n}{(q; q)_n}, \end{aligned}$$

as we wanted to show. \square

Remark 3.5. • For $s = d = 1$ the identity (3.4) is reduced to the well-known generating function associated to the Little q -Laguerre polynomials [41, p. 108]:

$$\frac{(t; q)_{\infty}}{(xt; q)_{\infty 0}} \phi_1 \left(\begin{matrix} - \\ b_1 \end{matrix} \middle| q; b_1 xt \right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} P_n(x; b_1|q) t^n.$$

- For $s = d$, replacing in (3.4) x by $(1 - q)^d x$, t by $(1 - q)t$ and b_j by q^{α_j+1} ; $j = 1, \dots, d$; and letting $q \rightarrow 1$, we get

$$e^t {}_0F_d \left(\begin{matrix} - \\ \alpha_1 + 1, \dots, \alpha_d + 1 \end{matrix} ; -xt \right) = \sum_{n=0}^{\infty} \ell_n^{\vec{\alpha}_d}(x) \frac{t^n}{n!}.$$

This generating function was given by the first author and Douak [20, p. 353].

3.4. Inversion formulas

Proposition 3.6. The Little q -Laguerre type polynomials defined by (2.7) verify the following inversion formula:

$$x^n = ((-1)^n q^{\binom{n}{2}})^{s-d} (b_1, \dots, b_s; q)_n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} P_k(x; (b_s)|q), \tag{3.5}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$.

Proof. According to the identity (3.4), we have

$$\sum_{k=0}^{\infty} \frac{((-1)^k q^{\binom{k}{2}})^{s-d} (xt)^k}{(b_1, \dots, b_s, q; q)_k} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} P_k(x; (b_s)|q) \frac{t^n}{(q; q)_{n-k}}.$$

Equating the coefficients of t^n , we obtain (3.5). \square

Remark 3.7. • In the particular case $d = s = 1$, the identity (3.5) is reduced to the following inversion formula associated to the Little q -Laguerre polynomials:

$$x^n = (a; q)_n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} P_k(x; a|q).$$

- For $s = d$, replacing in (3.5) x by $(1 - q)^d x$, t by $(1 - q)t$ and b_j by $q^{\alpha_j + 1}$; $j = 1, \dots, d$; and letting $q \rightarrow 1$, we obtain the known inversion formula [20, p. 353]:

$$x^n = \prod_{j=1}^d (\alpha_j + 1)_n \sum_{k=0}^n (-1)^n \binom{n}{k} \ell_k^{\vec{\alpha}_d}(x).$$

3.5. d -dimensional vector of functionals

Our interest here is to determinate the d -dimensional vector of functionals for which we have the d -orthogonality of the Little q -Laguerre type polynomials.

As proved in [32], a $\text{PS}\{P_n\}_{n \geq 0}$ is d -orthogonal with respect to a d -dimensional vector of functionals $\Gamma = {}^t(\Gamma_0, \Gamma_1, \dots, \Gamma_{d-1})$ if and only if it is also d -orthogonal with respect to the vector $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$, where the functionals u_0, u_1, \dots, u_{d-1} are the d first elements of the dual sequence $\{u_n\}_{n \geq 0}$ associated to the $\text{PS}\{P_n\}_{n \geq 0}$ and defined by $\langle u_r, P_n \rangle = \delta_{n,r}$; $r \geq 0$, $n \geq 0$. Consequently, for the considered polynomials in this subsection, we determine the d first elements of the corresponding dual sequence to derive the d -dimensional vector of functionals ensuring the d -orthogonality of these polynomials. That leads to the following.

Proposition 3.8. The Little q -Laguerre type polynomials defined by (2.7) are d -orthogonal with respect to the d -dimensional vector of functionals $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$ given by

$$\langle u_r, x^n \rangle = ((-1)^n q^{\binom{n}{2}})^{s-d} (b_1, \dots, b_s; q)_n \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\binom{r}{2}}, \quad n \geq r. \quad (3.6)$$

Proof. For $n \geq r$, according to the inversion formula given by (3.5), we have

$$\langle u_r, x^n \rangle = ((-1)^n q^{\binom{n}{2}})^{s-d} (b_1, \dots, b_s; q)_n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} \langle u_r, P_k(x; (b_s)|q) \rangle.$$

That, by virtue of the definition of a dual sequence, leads to (3.6). \square

Next, we consider two particular cases.

3.5.1. Little q -Laguerre PS ($d = s = 1$)

By letting $d = s = 1$ and $b_1 = aq$ in (3.6), we obtain

$$\langle u_0, x^n \rangle = (aq; q)_n = \frac{(aq; q)_\infty}{(aq^{n+1}; q)_\infty} = (aq; q)_\infty \sum_{k=0}^{\infty} \frac{(aq)^k}{(q; q)_k} \delta_{q^k}[x^n].$$

Thus, we rediscover the well-known linear functional ensuring the orthogonality of the Little q -Laguerre PS given by [41, p. 107]:

$$\sum_{k=0}^{\infty} \frac{(aq)^k}{(q; q)_k} P_n(q^k; aq|q) P_m(q^k; aq|q) = \frac{(aq)^n}{(aq; q)_\infty} \frac{(q; q)_n}{(aq; q)_n} \delta_{nm}.$$

3.5.2. Little q -Laguerre 2-OPS ($d = s = 2$)

For $d = s = 2$, Proposition 3.8 is reduced to the following.

Corollary 3.9. The Little q -Laguerre type polynomials defined by (2.7) with $d = s = 2$ are 2-orthogonal with respect to the 2-dimensional vector of functionals $\mathcal{U} = {}^t(u_0, u_1)$ given by

$$\langle u_r, x^n \rangle = \sum_{k=0}^{\infty} \varphi_{r,2}(k; b_1, b_2) \delta_{q^k}[x^n], \quad r = 0, 1, \tag{3.7}$$

where

$$\begin{aligned} \varphi_{0,2}(k; b_1, b_2) &= (b_1, b_2; q)_{\infty} \frac{b_2^k}{(q; q)_k} V_k^{(\frac{b_2}{b_1})}(0; q), \\ \varphi_{1,2}(k; b_1, b_2) &= \frac{(b_1, b_2; q)_{\infty}}{(q-1)} \frac{b_2^k}{(q; q)_k} \left[V_k^{(\frac{b_2}{b_1})}(0; q) - \frac{1-q^k}{b_2} V_{k-1}^{(\frac{b_2}{b_1})}(0; q) \right], \end{aligned}$$

with $V_k^a(x; q)$ the Al-Salam–Carlitz II polynomials defined by [41, p. 114]:

$$V_k^a(x; q) = {}_2\phi_0 \left(\begin{matrix} q^{-k}, x \\ - \end{matrix} \middle| q; \frac{q^k}{a} \right). \tag{3.8}$$

Proof. According to (3.6) with $d = s = 2$, we deduce

$$\langle u_r, x^n \rangle = (-1)^r q^{\binom{r}{2}} \frac{(q^{r+1}; q)_{\infty} (b_1, b_2; q)_{\infty}}{(q; q)_{\infty}} \frac{(q^{n-r+1}; q)_{\infty}}{(q^{n+1}; q)_{\infty} (b_1 q^n, b_2 q^n; q)_{\infty}}, \quad r = 0, 1. \tag{3.9}$$

On the other hand, the polynomials defined by (3.8) are generated by [41, p. 115]:

$$\frac{(xt; q)_{\infty}}{(t, at; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{a^n}{(q; q)_n} V_n^{(a)}(x; q) t^n.$$

Replacing in this identity $x = 0, t = b_1 q^n$ and $a = \frac{b_2}{b_1}$, we obtain

$$\frac{1}{(b_1 q^n, b_2 q^n; q)_{\infty}} = \sum_{k=0}^{\infty} \frac{b_2^k}{(q; q)_k} V_k^{(\frac{b_2}{b_1})}(0; q) q^{nk}. \tag{3.10}$$

We consider the following two cases.

Case 1. $r = 0$: For this case (3.9) is reduced to

$$\langle u_0, x^n \rangle = (b_1, b_2; q)_{\infty} \frac{1}{(b_1 q^n, b_2 q^n; q)_{\infty}}. \tag{3.11}$$

Substituting (3.10) in (3.11), we get the desired result.

Case 2. $r = 1$: For this case (3.9) becomes

$$\langle u_1, x^n \rangle = -\frac{1-q^n}{1-q} \frac{(b_1, b_2; q)_{\infty}}{(b_1 q^n, b_2 q^n; q)_{\infty}}.$$

That, by virtue of (3.10), leads to (3.7). \square

4. An orthogonal polynomial set

From Corollary 2.4, the resulting orthogonal polynomials are the Little q -Laguerre polynomials defined by (1.2) and the polynomials given by (2.8), which seems to be new. Throughout the literature several properties associated with the Little q -Laguerre polynomials are scattered (see, for instance, [41]). Next, we establish a recurrence relation and a difference equation for the polynomials given by (2.8).

4.1. A recurrence relation

Proposition 4.1. The orthogonal polynomials defined by (2.8) verify the following recurrence relation:

$$xP_n(x|q) = q^{2n+1}P_{n+1}(x|q) + q^n(1 - q^n(1 + q))P_n(x|q) + q^n(q^n - 1)P_{n-1}(x|q). \tag{4.1}$$

Proof. According to (2.3) with $d = 1$, we have

$$xP_n(x|q) = \alpha_{2,1}(n)P_{n+1}(x|q) + \alpha_{1,1}(n)P_n(x|q) + \alpha_{0,1}(n)P_{n-1}(x|q).$$

From (2.4) with $(d, s) = (1, 0)$, we get $\alpha_{2,1}(n) = q^{2n+1}$.

Equalizing the coefficients of x^{n+1} in (2.2) with $(d, s) = (1, 0)$, we obtain $\alpha_{0,1}(n) = q^n(q^n - 1)$.

That, by virtue of (2.2) with $(d, s) = (1, 0)$ and $x = 1$, leads to $\alpha_{1,1}(n) = q^n(1 - q^n(1 + q))$. \square

Remark 4.2. From the recurrence relation given by (4.1) and Favard Theorem [36], it is easy to prove that the obtained orthogonal polynomials defined by (2.8) are quasi-definite orthogonal polynomials.

4.2. Difference equation

Proposition 4.3. The orthogonal polynomials defined by (2.8) are L_q -classical. Moreover they satisfy the following difference equation:

$$\alpha_n(x)L_q^2P_n(x|q) - \beta_n(x)L_qP_n(x|q) - q^{-4n}(1 - q^{-n})P_n(x|q) = 0, \quad (4.2)$$

where

$$\begin{aligned} \alpha_n(x) &= (1 - q^{-n})^2 \left[\frac{q^{-n+1}(2 - q^{-n-1} - q^{-n})}{1 - q^{-n}}x + (1 - q^{-n-1}) \left(\frac{q^{-1}}{q^{-n} - 1} - 1 \right) - q^{-1} \right] \\ &\quad + q^{-1}(2 - q^{-n-1} - q^{-n})(2 - q^{-n} - q^{-n+1}), \\ \beta_n(x) &= q^{-2n}(1 - q^{-n}) \left[\frac{-q^{-n+1}(2 - q^{-n-1} - q^{-n})}{1 - q^n}x + (1 - q^{-n})(1 + q) - q^{-n-1} \right. \\ &\quad \left. - \frac{(1 - q^{-n-1})(1 - q^{-n+1} - q)}{q^{-n} - 1} \right]. \end{aligned} \quad (4.3)$$

Proof. According to Proposition 4.1, we have

$$P_n(x|q) = [q^{-n+1}x + q^{n-1}(1 + q) - 1]q^{-n}P_{n-1}(x|q) - q^{-1}(1 - q^{-n+1})P_{n-2}(x|q). \quad (4.4)$$

Substituting (3.2) with $(d, s, k) = (1, 0, 1)$ in this identity, we obtain

$$L_qP_n(x|q) = [q^{-n+1}x + q^{n-2}(1 + q) - 1] \frac{1 - q^{-n}}{1 - q^{n-1}}L_qP_{n-1}(x|q) + q^{-1}(1 - q^{-n})L_qP_{n-2}(x|q). \quad (4.5)$$

Letting L_q operate on both sides of (4.4) gives

$$L_qP_n(x|q) = [q^{-n+2}x + q^{n-1}(1 + q) - 1]q^{-n}L_qP_{n-1}(x|q) + q^{-2n+1}P_{n-1}(x|q) - q^{-1}(1 - q^{-n+1})L_qP_{n-2}(x|q). \quad (4.6)$$

Multiply (4.5) and (4.6) by, respectively, $(1 - q^{-n+1})$ and $(1 - q^{-n})$ and add

$$(2 - q^{-n} + q^{-n+1})L_qP_n(x|q) = (1 - q^{-n})^2L_qP_{n-1}(x|q) + q^{-2n+1}(1 - q^{-n})P_{n-1}(x|q). \quad (4.7)$$

On the other hand, a combination of (4.5) and (4.6) leads to

$$\begin{aligned} 0 &= \left[q^{-n+1} \left(\frac{2 - q^{-n} - q^{-n+1}}{1 - q^{n-1}} \right) x + \left((1 - q^{-n}) \left(\frac{q^{-1}}{q^{-n+1} - 1} - 1 \right) - q^{-1} \right) \right] L_qP_{n-1}(x|q) \\ &\quad - q^{-2n+1}P_{n-1}(x|q) + q^{-1}(2 - q^{-n} - q^{-n+1})L_qP_{n-2}(x|q). \end{aligned} \quad (4.8)$$

Shifting $n \rightarrow n + 1$ in (4.8), we get

$$\begin{aligned} q^{-2n-1}P_n(x|q) &= \left[q^{-n} \left(\frac{2 - q^{-n-1} - q^{-n}}{1 - q^n} \right) x + (1 - q^{-n-1}) \left(\frac{q^{-1}}{q^{-n} - 1} - 1 \right) - q^{-1} \right] L_qP_n(x|q) \\ &\quad + q^{-1}(2 - q^{-n-1} - q^{-n})L_qP_{n-1}(x|q). \end{aligned} \quad (4.9)$$

Multiply (4.7) and (4.9) by, respectively, $q^{-1}(2 - q^{-n-1} - q^{-n})$ and $-(1 - q^{-n})^2$ and add

$$-q^{-2n-1}(1 - q^{-n})^2P_n(x|q) = q^{-2n}(1 - q^{-n})(2 - q^{-n-1} - q^{-n})P_{n-1}(x|q) - \alpha_n \left(\frac{x}{q} \right) L_qP_n(x|q), \quad (4.10)$$

where α_n is given by (4.3).

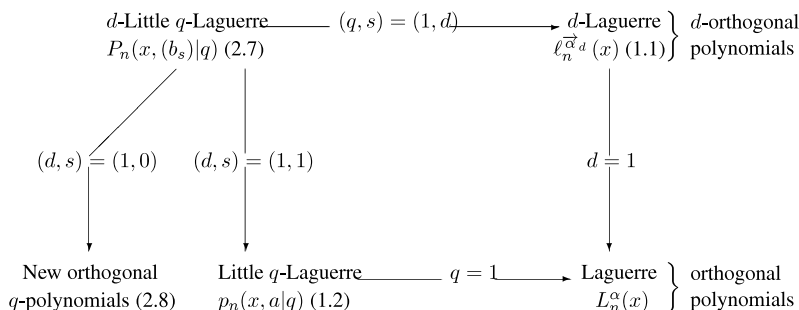
Now, letting L_q operate on both sides of (4.10), we obtain

$$\begin{aligned} 0 &= q^{-2n}(1 - q^{-n})(2 - q^{-n-1} - q^{-n})L_qP_{n-1}(x|q) - \alpha_n(x)L_q^2P_n(x|q) + q^{-2n}(1 - q^{-n}) \\ &\quad \times [2 + q - q^{-n}(q^{-1} + 1 + q)]L_qP_n(x|q). \end{aligned} \quad (4.11)$$

Multiply (4.11) and (4.9) by, respectively, q^{-1} and $q^{-2n}(q^{-n} - 1)$ and add, we obtain (4.2). \square

5. Concluding remarks

Many papers dealing with the d -orthogonality notion, generalized some known characterization theorems for orthogonal polynomials to the d -orthogonality (see, [15,18,19,23,24,33,32,31]), especially, for continuous and discrete cases. But, the basic case was considered only in [35], who give the d -orthogonal polynomials analogous to q -Al-Salam–Carlitz’s ones, using for this purpose a suitable generating function. In this paper, we use another technique based on the basic hypergeometric representation of the considered polynomials to state a characterization theorem dealing with q -polynomials analogous to the Little q -Laguerre ones. The obtained d -orthogonal polynomials can be viewed as a q -extension of the d -orthogonal polynomials of Laguerre type [19], since we rediscovery some properties of these polynomials for the limiting case $q = 1$. We summarize this point of view in the following scheme.



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