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Zhongying Chen
Yuesheng Li
Zhongshan University
Guangzhou, People's Republic of China

Charles A. Micchelli
IBM T. J. Watson Research Center
Yorktown Heights, New York

Yuesheng Xu
North Dakota State University
Fargo, North Dakota

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Block Cholesky Factorization of Infinite Matrices and Orthonormalization of Vectors of Functions

C.V.M. van der Mee, G. Rodriguez and S. Seatzu

Abstract. The following results on the block Cholesky factorization of bi-infinite and semi-infinite matrices are obtained. A method is proposed for computing the LDM^T- and block Cholesky factors of a bi-infinite banded block Toeplitz matrix. An equivalence relation is introduced to describe when two semi-infinite matrices with entries A_{ij} coincide exponentially as $i, j, i + j \to \infty$. If two equivalent bi-infinite matrices have block Cholesky factorizations, then their block Cholesky factors and their inverses are equivalent. If a bi-infinite block matrix A has a block Cholesky factorization whose lower triangular factor L and its lower triangular inverse decay exponentially away from the diagonal, then the semi-infinite truncation of A has a lower triangular block Cholesky factor whose elements approach those of L exponentially. These results are then applied to studying the asymptotic behavior of vectors of functions obtained by orthonormalizing a large finite set of integer translates of an exponentially decaying vector of functions.

§1. Introduction

In a recent paper [11], several results on the Cholesky factorization of Gram matrices were obtained and applied to the study of the asymptotic behavior of splines obtained by orthonormalizing a large finite set of B-splines, in particular identifying the limiting profile when the knots are equally spaced. Additional properties of the Cholesky factorization of bi-infinite and semi-infinite matrices were obtained in [12] and applied to the study of the limiting profile arising from the orthonormalization of positive integer translates of an exponentially decaying function. In a previous paper [13] some results on the block Cholesky factorization of bi-infinite and semi-infinite Toeplitz matrices were obtained, which, in particular, give a method for computing it in the bi-infinite block tridiagonal case.

The purpose of this paper is to generalize the results on the Cholesky factorization of bi-infinite and semi-infinite matrices obtained in [11, 12] to the block matrix case, and to apply them to the study of the asymptotic behavior of vectors of functions obtained by orthonormalizing positive integer translates of a vector of exponentially decaying functions.

The paper is organized as follows. In Section 2, after a short review of results pertaining to the block LDU-factorization of real bi-infinite block

Toeplitz matrices, we generalize Theorem 5.1 of [13] on the block *LDU*-factorization of bi-infinite block tridiagonal matrices to the banded case. In Section 3, we extend the results of [11, 12] on the Cholesky factorization of bi-infinite and semi-infinite matrices to the block matrix case. In Section 4, we generalize some of the results derived in [12] on the limiting profile of functions obtained by orthonormalizing positive integer translates of an exponentially decaying function to vectors of exponentially decaying functions. In Section 5 we apply this result to a specific example. A crucial ancillary result on the stability of the block Cholesky factors of a positive definite real symmetric matrix perturbed by a matrix small in the Frobenius norm, is proved in the Appendix. This result, which is of independent interest, generalizes a previous result of Sun [16] but has been proved in an entirely different way.

§2. LDM^T Factorization of Banded Block Toeplitz Matrices

Let us first review some results on the block Cholesky factorization of real bi-infinite Toeplitz matrices of the form $(G_{i-j})_{i,j\in\mathbb{Z}}$ where each entry G_{i-j} is a square matrix of order k and $\mathbb{Z}$ is the set of all integers. Such a matrix may be viewed as a bounded linear operator on the Hilbert space $\ell_2(\mathbb{Z})$ of square summable sequences indexed by the integers if and only if its so-called symbol

$$\widehat{G}(z) = \sum_{i=-\infty}^{\infty} z^i G_i, \qquad |z| = 1, \tag{2.1}$$

is essentially bounded, i.e., if all of its entries belong to $L_\infty(\mathbf{T})$ where $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$. In particular, if

$$\sum_{i=-\infty}^{\infty} \|G_i\| < +\infty, \tag{2.2}$$

where any matrix norm can be employed, then $\widehat{G}(z)$ is continuous on $\mathbf{T}$ and the bi-infinite Toeplitz matrix $(G_{i-j})_{i,j\in\mathbb{Z}}$ is bounded on $\ell_2(\mathbb{Z})$. The class of matrix functions $\widehat{G}(z)$ on $\mathbf{T}$ of the form (2.1), where the coefficients G_i satisfy (2.2), is a Banach algebra with respect to the norm $\|\widehat{G}\| := \sum_{i=-\infty}^{\infty} \|G_i\|$, called the Wiener algebra of order k.

Let $G = (G_{i-j})_{i,j\in\mathbb{Z}}$ be a real block Toeplitz matrix satisfying (2.2) where each entry G_{i-j} of G is a square matrix of order k. Then by an *LDU-factorization* of G we mean a representation of G of the form

$$G = LDM^T, \tag{2.3}$$

where the superscript T denotes matrix transposition and $L = (L_{i-j})_{i,j\in\mathbb{Z}}$, $M = (M_{i-j})_{i,j\in\mathbb{Z}}$ and $D = (D_{i-j})_{i,j\in\mathbb{Z}}$ are block Toeplitz matrices having the following properties:

which implies

$$\|L^{-1}E(L^{-1})^T\|_F \leq \|A^{-1}\|\|E\|_F. \tag{A.13}$$

Putting $M = L^{-1}(L+G)$ we now find

$$\begin{cases} L^{-1}(A+E)(L^{-1})^T = I + L^{-1}E(L^{-1})^T \\ L^{-1}(A+E)(L^{-1})^T = L^{-1}(L+G)(L+G)^T(L^{-1})^T = MM^T, \end{cases}$$

where, because of (A.13),

$$\|M - I\|_F \leq \frac{\|E\|_F\|A^{-1}\|(2 - \|E\|_F\|A^{-1}\|)}{(1 - \|E\|_F\|A^{-1}\|)^2}. \tag{A.14}$$

Consequently, as a result of (A.12), (A.14), $G = L(M-I)$ and $\|L\| = \|A\|^{1/2}$ (which follows from the estimate $\|Lx\|^2 = (LL^Tx, x) = \|A^{1/2}x\|^2$ and the identity $\|L\| = \|L^T\|$) we get (A.1), which completes the proof. ■

The proof of Theorem A.1 crucially depends on the boundedness of the projections P and Q onto the strictly upper and strictly lower block triangular parts of a semi-infinite matrix in $\mathcal{F}$. These projections are no longer bounded if $\mathcal{F}$ is replaced by the Banach algebra of all bounded semi-infinite block Toeplitz matrices. Indeed ([4], Example 4.1), the semi-infinite Toeplitz matrix $G = (G_{i-j})_{i,j\in\mathbb{Z}}$ given by $G_0 = 0$ and $G_s = 1/s$ for $s \neq 0$ is bounded with norm $\leq \pi$, but the norms of the strictly upper triangular parts of its antisymmetric $n \times n$ sections have a norm $\geq (4/5)\log n$. Hence the projections P and Q are unbounded on the algebra of bounded bi-infinite Toeplitz matrices. Note that this matrix G does not belong to the Wiener algebra.

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C.V.M. van der Mee, G. Rodriguez and S. Seatzu
Dipartimento di Matematica
Università di Cagliari
Viale Merello 92
09123 Cagliari, Italy
`cornelis@krein.unica.it`
`rodriguez@unica.it`
`seatzu@unica.it`