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ScienceDirect
Applied Mathematics Letters 20 (2007) 982–987
Applied Mathematics Letters
www.elsevier.com/locate/aml

LU factorization of the Vandermonde matrix and its applications

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Received 1 September 2005; accepted 16 October 2006

Abstract

A scaled version of the lower and the upper triangular factors of the inverse of the Vandermonde matrix is given. Two applications of the q-Pascal matrix resulting from the factorization of the Vandermonde matrix at the q-integer nodes are introduced.

Keywords: Vandermonde matrix; Symmetric functions; Triangular and bidiagonal factorization; q-Pascal matrix; q-difference equation

1. Introduction

A Vandermonde matrix is defined in terms of scalars $x_0, x_1, \ldots, x_n$ by

$$V = V(x_0, x_1, \ldots, x_n) = \begin{bmatrix} 1 & x_0 & \ldots & x_0^n \\ 1 & x_1 & \ldots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \ldots & x_n^n \end{bmatrix}.$$

Vandermonde matrices play an important role in approximation problems such as interpolation, least squares and moment problems. The special structure of V makes it possible to investigate not only explicit formulas for LU factors of V and V^{-1} but also fast solutions of a Vandermonde system $V\mathbf{x} = \mathbf{b}$. See [9] and the references therein. Interestingly, complete symmetric functions and elementary symmetric functions appear in the LU factorization of the Vandermonde matrix V and its inverse V^{-1} respectively [8,9]. Taking LU factors into account, [8] deduced one-banded (bidiagonal) factorization of V and hence achieved a well known result that V is totally positive matrix if $0 < x_0 < x_1 < \cdots < x_n$. Note that a matrix is totally positive if the determinant of every square submatrix is positive. The paper [9] investigates the LU factors of V and V^{-1} at $x_0 = 0$, $x_i = 1 + q + \cdots + q^{i-1}$, $i = 1, 2, \ldots, n$, in which q-Pascal and q-Stirling matrices are introduced. Recently, based on [8], the work [12] has scaled the elements of LU of V to give a simpler formulation. There also follows a simpler one-banded factorization of V.

In this work, using [9,12] we simplify the formula [9, Theorem 3.2] for the LU factors of V^{-1} in Section 2, and in turn a shorter proof of one-banded factorization of the upper triangular U is obtained. In Section 3, two applications of the q-Pascal matrix, the subdivision formula for q-Bernstein Bézier curves and the solution of a system of first-order q-difference equations, are presented.

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2006.10.003

2. LU factors of V^{-1}

When $V = LU$ where L is a lower triangular matrix with ones on the main diagonal and U is an upper triangular matrix (Doolittle method), the explicit formulas for the elements of the matrices L and U are given in [8]. However if we let U have ones on the main diagonal (Crout method), namely scaling the elements of upper triangular matrix in the Doolittle method, then we obtain the formulas [12, Theorem 2] and [11, (1.61), (1.62)]. Considering the Crout method on V^{-1}, that is multiplying the matrices $\widehat{D}^{-1}$ and $\widehat{L}^{-1}$ in [9, Theorem 3.2], we obtain the following simplification:

Theorem 2.1. *Let $V^{-1} = U^{-1}L^{-1}$. Then Crout's factorization of V^{-1} satisfies*

$$(U^{-1})_{i,j} = (-1)^{i+j}\sigma_{j-i}(x_0, \dots, x_{j-1}), \quad 0 \leqslant i \leqslant j \leqslant n, \tag{2.1}$$

$$(L^{-1})_{i,j} = \frac{1}{\prod\limits_{\substack{k=0\\k\neq j}}^{i}(x_j - x_k)}, \quad 0 \leqslant j \leqslant i \leqslant n, \tag{2.2}$$

where σ_k denotes the kth elementary symmetric function.

Note that a generating function for the elementary symmetric functions is

$$(1 - x_1x)(1 - x_2x)\dots(1 - x_nx) = \sum_{k=0}^{n}(-1)^k\sigma_k(x_1, \dots, x_n)x^k$$

and its recurrence relation is

$$\sigma_k(x_1, \dots, x_n) = \sigma_k(x_1, \dots, x_{n-1}) + x_n\sigma_{k-1}(x_1, \dots, x_{n-1}). \tag{2.3}$$

See [9]. Although the above factorization Theorem 2.1 and the factorization in [12] reduce computational work slightly, they do not reveal a nice structure on the factors L and U at the q-integer nodes, q-Pascal and q-Stirling matrices respectively.

Now let us observe that the sum of the ith row of L^{-1} in (2.2) vanishes for $i = 1, 2, \dots, n$ since $LL^{-1} = I$ and L has leading column consisting of ones. Alternatively, one may show that

$$\sum_{j=0}^{i}\frac{1}{\prod\limits_{\substack{k=0\\k\neq j}}^{i}(x_j - x_k)} = 0, \tag{2.4}$$

using the interpolating polynomial $p_n(x)$ for a function $f(x)$ at distinct points $x_0, x_1, \dots, x_n$ in Newton form:

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_n](x - x_0)\dots(x - x_{n-1}),$$

where the divided difference $f[x_0, x_1, \dots, x_n]$ is expressed as the symmetric sum

$$f[x_0, x_1, \dots, x_n] = \sum_{j=0}^{n}\frac{f(x_j)}{\prod\limits_{\substack{k=0\\k\neq j}}^{n}(x_j - x_k)}. \tag{2.5}$$

Since the interpolating polynomial p_n reproduces a polynomial of degree at most n, see [11], it follows from $f(x) = 1$ that

$$f[x_0, x_1, \dots, x_i] = 0, \quad i = 1, 2, \dots, n.$$

Then Eq. (2.5) reduces to (2.4).

Another important fact is that the entries of V^{-1} can be obtained explicitly from Theorem 2.1 as

$$(V^{-1})_{ij} = (-1)^{n-i} \frac{\sigma_{n-i}(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1})}{\prod_{\substack{k=0 \\ k \neq j}}^{n} (x_j - x_k)}. \tag{2.6}$$

The last formula is well known; see [5,6]. The study [5] finds the formulas for LDU factors of the matrices V and V^{-1} without using properties of elementary or complete symmetric functions. The benefit of the use of symmetric functions is in computing the entries of LU factors of V and V^{-1} recursively; see [9]. The paper [7] analyzes the factorization of the inverse of a Cauchy–Vandermonde matrix as a product of bidiagonal matrices to develop fast algorithms for interpolation.

We end this section by giving a shorter proof expressing U^{-1} as a product of one-banded matrices in [9]. First, for $k = 1, 2, \ldots, n$ define $(n+1) \times (n+1)$ matrices E_k by

$$(E_k)_{ij} = \begin{cases} 1, & i = j \\ -x_{k-1}, & i = j-1 \text{ and } j \geqslant k. \end{cases}$$

It is proved in [9] that $U^{-1} = E_1 E_2 \ldots E_n$. Now using the recurrence relation (2.3) observe that $U^{-1} = E_1 \overline{U}_{n-1}$ where

$$\overline{U}_{n-1} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_{n-1} \end{bmatrix}$$

and $\mathbf{0}$ denotes an appropriate zero matrix, and $n \times n$ matrix U_{n-1} is defined by

$$(U_{n-1})_{ij} = (-1)^{i+j} \sigma_{j-i}(x_1, \ldots, x_{n-1}), \quad 0 \leqslant i \leqslant j \leqslant n-1.$$

Applying the same process once more we have $\overline{U}_{n-1} = E_2 \overline{U}_{n-2}$ where

$$\overline{U}_{n-2} = \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & U_{n-2} \end{bmatrix}$$

and I_2 is the 2×2 identity matrix, and

$$(U_{n-2})_{ij} = (-1)^{i+j} \sigma_{j-i}(x_2, \ldots, x_{n-1}), \quad 0 \leqslant i \leqslant j \leqslant n-2.$$

Thus repeating the above procedure $n-3$ times more, it yields the required bidiagonal product $E_1 E_2 \ldots E_n = U^{-1}$.

3. Applications of the q-Pascal matrix

The Bernstein–Bézier representations are most important tools for computer aided design purposes; see [4]. A parametric Bézier curve P defined by

$$\mathsf{P}(t) = \sum_{i=0}^{n} \mathsf{b}_i \binom{n}{i} t^i (1-t)^{n-i} \quad 0 \leqslant t \leqslant 1 \tag{3.1}$$

where b_i, $i = 0, 1, \ldots, n \in \mathbb{R}^2$ or $\mathbb{R}^3$, are given control points, mimics the shape of the control polygon. In the work [10], the representation (3.1) is generalized by using a one-parameter family of Bernstein–Bézier polynomials, so called q-Bernstein Bézier curves. They were defined as follows:

$$\mathsf{P}(t) = \sum_{i=0}^{n} \mathsf{b}_i \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{j=0}^{n-i-1} (1 - q^j t), \tag{3.2}$$

where an empty product denotes 1, the parameter q is a positive real number and $[r]$ denotes a q-integer, defined by

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1. \end{cases}$$

The q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$ which is the generating function for restricted partitions, see [2], is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n][n-1]\dots[n-r+1]}{[r][r-1]\dots[1]}$$

for $n \geqslant r \geqslant 1$, and has the value 1 when $r = 0$ and the value zero otherwise. Note that this reduces to the usual binomial coefficient when we set $q = 1$ and (3.2) reduces to (3.1). We now generalize the well known subdivision formula, see [4], of the Bernstein Bézier curves which may be used to subdivide the curve P in (3.2).

Theorem 3.1. *Let $B_i^n(t) = \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{j=0}^{n-i-1}(1-q^j t)$ be the q-Bernstein Bézier polynomial and let $c \in (0,1)$ be a fixed real. Then*

$$B_i^n(ct) = \sum_{j=0}^{n} B_i^j(c) B_j^n(t). \tag{3.3}$$

Proof. Let M be an $(n+1)\times(n+1)$ matrix with the elements $M_{ij} = B_j^i(ct)$, that is

$$M_{ij} = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix} c^j t^j \displaystyle\prod_{k=0}^{i-j-1}(1-q^k ct), & 0 \leqslant j < i \leqslant n, \\ c^i t^i, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Since the eigenvalues of the matrix M are distinct it can be written as $M = PDP^{-1}$ where D is a diagonal matrix whose elements $D_{ii} = c^i t^i$ are the eigenvalues of M. It is computed from the product that the elements P_{ij} of P are the entries of the q-Pascal matrix $P_{ij} = \begin{bmatrix} i \\ j \end{bmatrix}$, and the elements of the matrix P^{-1} are $(P^{-1})_{ij} = (-1)^{i-j} q^{(i-j)(i-j-1)/2} \begin{bmatrix} i \\ j \end{bmatrix}$. Now we can write $M = PD_1D_2P^{-1}$, where D_1 and D_2 are diagonal matrices with elements $(D_1)_{ii} = t^i$ and $(D_2)_{ii} = c^i$, $i = 0, 1, \dots, n$. Then it follows from

$$M = PD_1P^{-1}PD_2P^{-1} = RS$$

that the matrices R and S have the entries $R_{ij} = B_j^i(t)$ and $S_{ij} = B_j^i(c)$ respectively. Thus, M has the elements

$$M_{ni} = B_i^n(ct) = \sum_{j=0}^{n} R_{nj} S_{ji} = \sum_{j=0}^{n} B_j^n(t) B_i^j(c), \quad 0 \leqslant i \leqslant n.$$

which completes the proof. □

We note that using the symmetric functions, q-Pascal matrices P and P^{-1} are obtained in the LU factorization of the Vandermonde matrix and in the inverse of the Vandermonde matrix at the q-integer nodes respectively, see [9].

In what follows, we relate the q-Pascal matrix P to an $(n+1)\times(n+1)$ nilpotent matrix H of index $n+1$ defined by

$$H_{ij} = \begin{cases} [i], & \text{if } i = j+1, 0 \leqslant i, j \leqslant n \\ 0, & \text{otherwise.} \end{cases}$$

We first define, see [3, p. 490], the q-analogue of the exponential series

$$E_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!}. \tag{3.4}$$

This series is absolutely convergent only in $|x| < (1-q)^{-1}$ when $|q| < 1$. However, another q-series

$$\mathsf{E}_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]!} \tag{3.5}$$

is convergent for all x and $|q| < 1$.

Theorem 3.2. *The q-Pascal matrix P is given by*

$$P = \sum_{k=0}^{\infty} \frac{H^k}{[k]!}. \tag{3.6}$$

Proof. First we see that the above series (3.6) is indeed finite since $H^k = \mathbf{0}$ for all $k \geqslant n+1$. Then it can be calculated from the definition of H that

$$H^j e_i = [i+j] \dots [i+1] e_{i+j},$$

where $e_i = 0, 1, \dots, n$ denote the unit vectors in $\mathbb{R}^{n+1}$. Now, a generic element on the right of (3.6) is

$$E_q(H)_{ij} = e_i^T E_q(H) e_j = \sum_{k=0}^{n} e_i^T \frac{H^k}{[k]!} e_j.$$

Thus we obtain

$$E_q(H)_{ij} = \sum_{k=0}^{n} e_i^T \frac{[j+k] \dots [j+1]}{[k]!} e_{k+j} = \sum_{k=0}^{n} \frac{[j+k] \dots [j+1]}{[k]!} \delta_{i,j+k},$$

where δ denotes the Kronecker delta function. Shifting the index of the summation gives

$$\frac{[i] \dots [i-j+1]}{[i-j]!} = \begin{bmatrix} i \\ j \end{bmatrix} = P_{ij}$$

and this completes the proof. $\square$

It is well known, see [1], that the initial value problem in $\mathbb{R}^{n+1}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{y}(t) = H_1 \mathbf{y}(t), \qquad \mathbf{y}(0) = \mathbf{y}_0$$

where H_1 denotes the matrix H with $q = 1$, has the solution $\mathbf{y}(t) = e^{H_1 t} \mathbf{y}(0)$. Next we demonstrate that the q-Pascal matrix appears as the solution of the first-order q-difference equation in $\mathbb{R}^{n+1}$. As in [3], we define the q-difference operator $\mathcal{D}_q$ by

$$\mathcal{D}_q f(x) = \frac{f(qx) - f(x)}{qx - x}, \quad x \neq 0.$$

Provided that $f'(x)$ exists,

$$\lim_{q \to 1} \mathcal{D}_q f(x) = f'(x).$$

It can be readily verified that for integers $r \geqslant 1$, $\mathcal{D}_q(x^r) = [r] x^{r-1}$. Then the solution $\mathbf{y}$ of the q-difference equation

$$\mathcal{D}_q \mathbf{y}(t) = H \mathbf{y}(t), \qquad \mathbf{y}(0) = \mathbf{y}_0$$

is $\mathbf{y}(t) = E_q(Ht)\mathbf{y}_0$. It follows from (3.6) that the matrix $E_q(Ht)$ has entries of the q-Pascal matrix

$$E_q(Ht)_{ij} = t^{i-j} \begin{bmatrix} i \\ j \end{bmatrix}, \quad i \geqslant j \geqslant 0.$$

It is worth noting that the polynomial p defined by

$$p(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k,$$

the sum of the nth row of $E_q(Ht)$, is known as the Rogers–Szegö polynomial [2].

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