Siegel modular forms of degree 2: Fourier coefficients, L-functions, and functoriality

(a survey)

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Outline

1. Classical Siegel modular forms of degree 2
   - Definitions
   - The Fourier expansion
   - Hecke operators and L-functions
2. Automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$
   - Local and global representations
   - Adelization of Siegel cusp forms
   - Galois representations
3. Lifts
   - CAP (Saito-Kurokawa lifts)
   - Endoscopic lifts (Yoshida lifts)
   - Lifts from $S_\rho(\Gamma)$ to GL_4 and GL_5
4. Bocherer's conjecture and refinements
   - Bessel periods and Liu's conjecture

Definition of Sp_4

For a commutative ring R, we denote by $\mathrm{Sp}_4(R)$ the set of 4×4 matrices $A \in \mathrm{GL}_4(R)$ satisfying the equation $A^t JA = J$ where $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$.

Definition of $\mathbb{H}_2$

Let $\mathbb{H}_2$ denote the set of complex 2×2 matrices Z such that $Z = Z^t$ and $\mathrm{Im}(Z)$ is positive definite.

$\mathbb{H}_2$ is a homogeneous space for $\mathrm{Sp}_4(\mathbb{R})$ under the action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ + B)(CZ + D)^{-1}$$

Siegel modular forms

A Siegel modular form of degree 2, full level and weight k is a holomorphic function F on $\mathbb{H}_2$ satisfying

$$F(\gamma Z) = \det(CZ + D)^k F(Z),$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z})$,
If in addition, F vanishes at the cusps, then F is called a cusp form.

We define $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ to be the space of cusp forms as above.

Remark. The smallest k for which $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ is non-zero is $k = 10$.

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The space $S_\rho(\Gamma)$

The space $S_\rho(\Gamma)$ consists of holomorphic V-valued function F on $\mathbb{H}_2$ such that

1. $F(\gamma Z) = \rho(CZ + D)F(Z)$, for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$,
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Two remarks:

- $\rho \simeq \det^l \mathrm{sym}^m$ for some integers l, m.
- If $\rho = \det^k$, then $V = \mathbb{C}$, and we get the usual space $S_k(\Gamma)$ of scalar valued weight k cusp forms.

$$\mathcal{P}_2 = \{S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : \qquad a, b, c \in \mathbb{Q}, \ S > 0\},$$

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The Fourier expansion

Let $F(Z) \in S_\rho(\Gamma)$. Then we can write

$$F(Z) = \sum_{S \in \mathcal{P}_2} a(F, S) e^{2\pi i \mathrm{Tr} SZ}, \quad a(F, S) \in V.$$

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Two remarks:

- There exists N such that $a(F,S) = 0$ unless $S \in (1/N)\mathcal{P}_2(\mathbb{Z})$.
- There exists a congruence subgroup $\Gamma' \in \mathrm{SL}_2(\mathbb{Z})$ such that $a(F, A^t SA) = a(F,S)$ for all $A \in \Gamma$, $S \in \mathcal{P}_2$.

(If $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$, then $N = 1$, $\Gamma' = \mathrm{SL}_2(\mathbb{Z})$)

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- There is a Hecke-invariant subspace of $S_k(\mathrm{Sp}_4(\mathbb{Z}))$. (spanned by eigenforms called Saito-Kurokawa lifts.)
- *Most forms are non-lifts.* For example the Saito-Kurokawa space has dimension $\asymp k$ while $\dim(S_k(\mathrm{Sp}_4(\mathbb{Z}))) \asymp k^3$.

Let $\mathbb{A}$ be the adeles of $\mathbb{Q}$ and $\pi = \otimes_v \pi_v$ be a cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$.

- For almost all primes p, π_p is unramified, i.e.,

$$\pi_p \simeq \pi(\chi_0, \chi_1, \chi_2)$$

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- Two possibilities for π_∞ of interest to us are $\pi_\infty \simeq L(k, l)$ (holomorphic discrete series) and $\pi_\infty \simeq L(k, -l)$ (large discrete series) where $k \geq l \geq 0$ are integers. *These are in the same L-packet.*

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Remarks:

- Multiplicity one for π is expected to be true. (Not known at present, but may follow from Arthur)
- Strong multiplicity one for π is FALSE.

Models

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- An alternative to Whittaker models is provided by the Bessel model. These are parametrized by characters Λ of quadratic extensions K.
- If π has a particular Bessel model then so does each π_v. But all π_v having Bessel models DOES NOT imply that π does.
- The Gan-Gross-Prasad conjectures predict that if π is tempered, and each π_v has a Bessel model, then π has a Bessel model if and only if $L(1/2, \pi_K \times \Lambda) \neq 0$.

L-functions

The Langlands L-functions

Given a cuspidal automorphic representation π of $G(\mathbb{A})$ and a finite dimensional representation r of the dual group ${}^L G^0$, there exists a global Langlands L-function $L(s, \pi, r)$. It is Eulerian, has degree r at almost all places, and (conjecturally) has a functional equation $s \mapsto 1 - s$, (conjecturally) no poles except in anomalous cases.

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If $G = \mathrm{GSp}_4$, then ${}^L G^0 \simeq \mathrm{GSp}_4(\mathbb{C})$. The two smallest dimensional non-trivial irreducible r we can get are of dimensions 4 and 5.

- $r = \rho_4$. In this case $L(s, \pi, \rho_4)$ is called the spinor L-function.
- $r = \rho_5$. In this case $L(s, \pi, \rho_5)$ is called the standard L-function.

Remark: The analytic properties of $L(s, \pi, \rho_4)$ and $L(s, \pi, \rho_5)$ are essentially known.

- Let $F \in S_\rho(\Gamma)$
- Can lift to a V-valued function $\widetilde{\Phi_F}$ on $\mathrm{Sp}_4(\mathbb{R})$ via

$$\widetilde{\Phi_F}(g) = \rho^{-1}(J(g, iI_2))F(g(iI_2)).$$

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For the last step, we pick local subgroups K'_p of $\mathrm{GSp}_4(\mathbb{Z}_p)$ such that

1. $\mu_2 : K'_p \mapsto \mathbb{Z}_p^\times$ is surjective,
2. $\mathrm{GSp}_4(\mathbb{R}) \prod_p K'_p \bigcap \mathrm{GSp}_4(\mathbb{Q})^+ \subset \Gamma$.

Then $\widetilde{\Phi_F}$ is right invariant under each K'_p.

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- π_F does not depend on the choice of L.
- $\pi_F = \oplus_{i=1}^{t} \pi_F^{(t)}$ with each $\pi_F^{(t)}$ an irreducible, cuspidal automorphic representation.

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- If F is an eigenfunction of the Hecke algebra at every place, then $t = 1$.

Summary: Adelization and deadelization

- Any vector-valued Siegel cusp form F leads to an adelic function Φ_F and then to a (not-necessarily irreducible) cuspidal automorphic representation π_F.
- If F is an eigenfunction of all local Hecke algebras then π_F is irreducible (but π_F may be irreducible even without this).
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- π_F is NOT generic (fails at infinity).
- **De-adelization**: Suppose π is an irreducible cuspidal automorphic represention of $\mathrm{GSp}_4(\mathbb{A})$, with $\pi_\infty \simeq L(k,l)$. Then each vector $\Phi \in \pi$ of suitable type, gives rise to a $F \in S_{\det^l \mathrm{sym}^{k-l}}(\Gamma)$ for some suitable Γ.
- If π_p is generic at all finite places, can pick F uniquely up to multiples so that Γ is paramodular subgroup of correct level.

One can attach Galois representations to Siegel cusp forms.

Theorem (Weissauer)

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ such that $\pi_\infty \simeq L(k, l)$. Then there exists a Galois representation

$$\rho_{\pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(E_\lambda)$$

such that for almost all primes p,

$$\mathrm{Tr}(\rho_{\pi,\lambda}(\mathrm{Fr}_p)) = a(\pi_p).$$

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Corollary (Kowalski-S, 2013)

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform at all primes. Then the set of primes where the Hecke eigenvalue $a_{F,p}$ is 0, has density 0.

Functoriality

Let G, H be connected reductive groups, $u : {}^L H^0 \to {}^L G^0$ a homomorphism. Then, functoriality predicts that given any automorphic representation π on $H(\mathbb{A})$, there exists an automorphic representation π' on $G(\mathbb{A})$ such that for all finite dimensional representations r of ${}^L G^0$,

$$L(s, \pi, u \circ r) = L(s, \pi', r).$$

Theta lifts

Given a symplectic space W and an orthogonal space V, there is a theta correspondence that takes automorphic representations of $\mathrm{Sp}(W)$ or $\widetilde{\mathrm{Sp}}(W)$ to automorphic representations on $\mathrm{SO}(\mathrm{V})$ (and vice-versa).

Using $\mathrm{PD}^\times \simeq SO(3)$ and $\mathrm{PGSp}_4 \simeq SO(5)$, we have

$$\begin{array}{ccccccc} \pi & \mathrm{PD}^\times(\mathbb{A}) & \xleftarrow{\text{Wald}} & \widetilde{\mathrm{SL}}(\mathbb{A}) & \xrightarrow{\theta} & \mathrm{PGSp}_4(\mathbb{A}) & \Pi \\ \uparrow & & & & & & \downarrow \\ f & & & & & & F \end{array} \quad (1)$$

This allows us to take a classical cusp form f of weight $2k-2$ for $\Gamma_0(N)$, and produce a Siegel cusp form $F \in S_k(\Gamma)$ for some Γ.

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This allows us to take a classical cusp form f of weight $2k-2$ for $\Gamma_0(N)$, and produce a Siegel cusp form $F \in S_k(\Gamma)$ for some Γ.
For this to work, we need

$$(-1)^{|S|} = \varepsilon(1/2, \pi_f),$$

where S is a set of places including ∞ where π_f is discrete series.
If f has full level, we must have k even, and we recover the classical Saito-Kurokawa lift $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$.

The Saito-Kurokawa lift and functoriality

How does the correspondence $\pi_f \mapsto \Pi$ from automorphic representations of PGL_2 to PGSp_4 fit with functoriality?

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We have a map of L-groups

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Theorem (Schmidt (2005))

Π is the functorial lift of $\pi_f \otimes \pi_S$ under the above embedding, where π_S is the unique subquotient of $\mathrm{Ind}(||^{1/2}, ||^{-1/2})$ that is unramified exactly at the places outside S.

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Theorem (Andrianov, Piatetski-Shapiro, Weissauer, Pitale-Schmidt)

Suppose that $k > 2$ and $F \in S_k(\Gamma)$ generates an irreducible cuspidal automorphic representation Π of $\mathrm{PGSp}_4(\mathbb{A})$. Then the following are equivalent.

1. *F is a Saito-Kurokawa lift, or a quadratic twist of it.*
2. *Π is CAP with respect to the Siegel parabolic.*
3. *Π is CAP.*
4. *Π is non-tempered at some unramified prime (i.e., F does not satisfy the Ramanujan bound)*
5. *$L(s, \Pi \times \chi, \rho_4)$ has a pole for some quadratic character χ.*

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Under suitable conditions, the resulting lift is non-zero, cuspidal, and of the form $L(k, l)$ at infinity. Forms F obtained from such Π are called endoscopic lifts, or Yoshida lifts.

Various lifts to $S_\rho(\Gamma)$

- Saito-Kurokawa (CAP) lifts F, with strange properties. For these, there exists a classical newform f and a quadratic character χ (possibly trivial) such that

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- Yoshida (Endoscopic) lifts. For these there exist two classical newforms f_1, f_2 such that

$$L(s, \pi_F) = L(s, \pi_{f_1})L(s, \pi_{f_2})$$

Various lifts to $S_\rho(\Gamma)$

- Saito-Kurokawa (CAP) lifts F, with strange properties. For these, there exists a classical newform f and a quadratic character χ (possibly trivial) such that

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- Lifts from $\mathrm{GL}_2(K)$, where K is a quadratic field. These may be viewed as the non-split version of Yoshida lifts. See papers of Roberts–Johnson-Leung (real quadratic) and Berger–Dembele–Pacetti–Sengun (imaginary quadratic).

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This should follow (follows?) from the work of Arthur using methods of the trace formula. However, using the converse theorem, Pitale, Schmidt and I proved:

Theorem (Pitale–Schmidt–S, 2012)

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be an eigenform for all Hecke operators, and not of Saito-Kurokawa type. Let π_F be the associated cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. Then π_F admits a strong lifting to an automorphic representation Π_4 of $\mathrm{GL}_4(\mathbb{A})$, and a strong lifting to an automorphic representation Π_5 of $\mathrm{GL}_5(\mathbb{A})$. Both Π_4 and Π_5 are cuspidal.

Method

We prove the following theorem and then apply the converse theorem

Theorem

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform that is not a Saito-Kurokawa lift and π be any cuspidal automorphic representation of GL_2. Then $L(s, F \times \pi)$ is absolutely convergent for $\mathrm{Re}(s) > 1$, has meromorphic continuation to the entire complex plane, and the completed L-function satisfies the usual functional equation, is entire, and bounded in vertical strips.

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The starting point for this is an integral representation due to Furusawa. We generalize Furusawa's formula, use this generalization to prove meromorphic continuation, functional equation and boundedness, and then prove a pullback formula and a seesaw argument to prove entireness.

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Corollary

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be an eigenform for all Hecke operators, and not of Saito-Kurokawa type. Let π_F be the associated cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. Let π'_F be the representation obtained by switching the Archimedean $L(k,k)$ to $L(k,-k)$. Then π'_F is also cuspidal automorphic (and now also generic!).

Fourier coefficients are mysterious objects

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be an eigenform. Assume k even. Recall the Fourier expansion

$$F(Z) = \sum_{S \in \mathcal{P}_2(\mathbb{Z})} a(F,S) e^{2\pi i \mathrm{Tr} SZ}.$$

We have $a(F, A^t SA) = a(F,S)$ for all $A \in \mathrm{SL}_2(\mathbb{Z})$, $S \in \mathcal{P}_2(\mathbb{Z})$.

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Let us focus on the Fourier coefficients $a(F,S)$ with $\mathrm{disc}(S)$ a (negative) fundamental discriminant.
If $\mathrm{disc}(S_1) = d_1 \neq d_2 = \mathrm{disc}(S_2)$ are two such fundamental discriminants, then we cannot understand one from the other via Hecke operators.

Put $K = \mathbb{Q}(\sqrt{d})$ and let Cl_K denote the ideal class group of K. Then $\mathrm{SL}_2(\mathbb{Z})-$equivalence classes of binary quadratic forms of discriminant d are in natural bijective correspondence with the elements of Cl_K. Define

$$R(f, K) = \sum_{c \in \mathrm{Cl}_K} a(f, c). \qquad (2)$$

Böcherer's conjecture

There exists a constant c_f depending only on f such that for any imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with $d < 0$ a fundamental discriminant, we have

$$|R(f, K)|^2 = c_f \cdot |d|^{k-1} \cdot w(K)^2 \cdot L(1/2, \pi_f \times \chi_d).$$

Thus, Bocherer's conjecture predicts that the sum of Fourier coefficients of discriminant d is essentially a L-value.

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1. What is *exactly* the constant c_f?
2. Instead of a plain sum, what if we weigh them by a character Λ?
3. What is the proper generalization to the case of $S_\rho(\Gamma)$, or to general automorphic representations π of $\mathrm{GSp}_4(\mathbb{A})$?

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All of these have now been addressed by very general and exact conjectures (Furusawa-Martin-Shalika, Prasad–Takloo-Bighash, Gan-Gross-Prasad, Liu,..).

Let π be an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. For any automorphic form ϕ in the space of π, we can define a global Bessel period $B(\phi, \Lambda)$ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$B(\phi, \Lambda) = \int_{\mathbb{A}^\times T_S(F)\backslash T_S(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} \phi(tu)\Lambda^{-1}(t)\theta_S^{-1}(n)dndt. \tag{3}$$

where S is a symmetric matrix, K a quadratic extension attached to S, Λ a Hecke character of K, $T_S \simeq K^\times$ the non-split torus of GL_2.

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Two key points:

- π has a Bessel model of type (K, Λ) if and only if $B(\phi, \Lambda) \neq 0$ for some ϕ.
- If ϕ is the adelization of some $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$, and Λ corresponds to a character of Cl_K, then

$$B(\phi, \Lambda) = e^{-2\pi Tr(S)} \sum_{c \in \mathrm{Cl}_K} \Lambda^{-1}(c)a(F, c).$$

A conjecture of Yifeng Liu

Let π, Λ be as above. Suppose that for almost all places v of F, the local representation π_v is generic. Let ϕ be any automorphic form in the space of π. Then

$$\frac{|B(\phi,\Lambda)|^2}{\langle \phi,\phi\rangle} = \frac{C_T}{4}\frac{\zeta(2)\zeta(4)L(1/2,\pi\otimes\mathcal{AI}(\Lambda))}{L(1,\mathrm{sym}^2\pi)L(1,\chi_d)}\prod_v I_v(\phi_v).$$

where $I_v(\phi_v)$ is an explicit local integral, equal to 1 almost everywhere.

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Remarks:

- This formulation excludes the Saito-Kurokawa (CAP) lift.
- Liu proved this conjecture for all endoscopic lifts.
- Note that the conjecture implies that

$$B(\phi,\Lambda)\neq 0 \Rightarrow L(1/2,\pi\otimes\mathcal{AI}(\Lambda))\neq 0$$

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Compute $I_v(\phi_v)$ in some specific ramified cases, and thus formulate the precise refinement of Bocherer's conjecture for various Siegel cusp forms with level.

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Thank you!