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**A survey of the Selberg class of  $L$ -functions, part II (\*\*)**

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The first part of this survey, containing Sections 1, 2, 3 and 4, appears in [42]. The second part depends heavily on the notation, definitions and results in part I, to which the reader is referred. Results, equations and sections in part I will simply be quoted by their number, without referring to [42]. Here we report only few definitions. A function  $F(s)$  belongs to the *Selberg class*  $\mathcal{S}$  if

(i) (*ordinary Dirichlet series*)  $F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s}$ , absolutely convergent for  $\sigma > 1$ ;

(ii) (*analytic continuation*) there exists an integer  $m \geq 0$  such that  $(s-1)^m F(s)$  is an entire function of finite order;

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(iii) (functional equation)  $F(s)$  satisfies a functional equation of type  $\Phi(s) = \omega \overline{\Phi}(1-s)$ , where

$$\Phi(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s),$$

say, with  $r \geq 0$ ,  $Q > 0$ ,  $\lambda_j > 0$ ,  $\Re \mu_j \geq 0$  and  $|\omega| = 1$ ;

(iv) (Ramanujan conjecture) for every  $\varepsilon > 0$ ,  $a_F(n) \ll n^\varepsilon$ ;

(v) (Euler product)  $\log F(s) = \sum_{n=1}^{\infty} b_F(n) n^{-s}$ , where  $b_F(n) = 0$  unless  $n = p^m$  with  $m \geq 1$ , and  $b_F(n) \ll n^\vartheta$  for some  $\vartheta < \frac{1}{2}$ .

We recall that the *extended Selberg class*  $S^\sharp$  is the class of the non identically vanishing functions satisfying axioms (i), (ii) and (iii) above. Further, the *degree*  $d_F$  of  $F \in S^\sharp$  is defined by

$$d_F = 2 \sum_{j=1}^k \lambda_j.$$

As in part I, I wish to thank Jurek Kaczorowski and Giuseppe Molteni for carefully reading the manuscript and for their suggestions, and Alessandro Zaccagnini for correcting many misprints. The contents of the entire survey is as follows; part II contains Sections 5, 6 and 7.

1. Classical  $L$ -functions
2. What is an  $L$ -function ?
3. Basic theory of the Selberg class
4. Invariants
5. Linear and non-linear twists
6. Degree  $1 \leq d < 2$
7. Independence
8. Countability and rigidity
9. Polynomial Euler products
10. Analytic complexity
11. Sums of coefficients
12. Miscellanea

## 5 - Linear and non-linear twists

The main tool for the results of Section 6 on the classification of the functions with degree  $1 \leq d < 2$  are the **linear twists**

$$F(s, a) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-na),$$

where  $F \in \mathcal{S}^\sharp$ ,  $\alpha \in \mathbb{R}$  and  $e(x) = e^{2\pi i x}$ . More precisely, the results of Section 6 require certain analytic properties of the linear twists. In order to get a first impression of the relevance of the linear twists, let us consider  $F(s) = L(s, \chi)$  with a primitive Dirichlet character  $\chi \pmod{q}$ , let  $\tau(\chi)$  be its associated Gauss sum and let  $0 \leq \alpha < 1$ . By orthogonality we have

$$(5.1) \quad F(s, \alpha) = \frac{1}{\tau(\bar{\chi})} \sum_{(a, q)=1} \bar{\chi}(a) \zeta\left(s, \frac{a}{q} - \alpha\right),$$

where

$$\zeta(s, \lambda) = \sum_{n=1}^{\infty} e(n\lambda) n^{-s}$$

is the *Lerch zeta function*. It is well known that  $\zeta(s, \lambda)$  has a simple pole at  $s = 1$  if  $\lambda \in \mathbb{Z}$ , otherwise it is an entire function. Therefore,  $F(s, \alpha)$  has a pole at  $s = 1$  and only if  $\alpha = a/q$  with  $(a, q) = 1$ . Thus, for example, information on the modulus  $q$  of the character  $\chi$  can be obtained from the polar structure of the linear twists of  $L(s, \chi)$ .

In order to study the analytic properties of the linear twists, Kaczorowski-Pereelli [26], [28] start with

$$F_N(s, \alpha) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-n\alpha) e^{-n/N},$$

where  $N > 0$ , which is absolutely convergent over  $\mathbb{C}$  and has the integral expression

$$F_N(s, \alpha) = \frac{1}{2\pi i} \int_{(2)} F(s+w) \Gamma(w) z_N^{-w} dw,$$

where  $z_N = \frac{1}{N} + 2\pi i \alpha$  and the integration is on the line from  $2 - i\infty$  to  $2 + i\infty$ . Shifting the line of integration to  $\sigma = -K - \frac{1}{2}$ , where  $K$  is a suitably large positive integer, and using the functional equation of  $F(s)$  we obtain

$$(5.2) \quad F_N(s, \alpha) = R_N(s, \alpha) + \omega Q^{1-2s} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^{1-s}} H_K\left(\frac{n}{Q^2 z_N}, s\right),$$

where  $R_N(s, \alpha)$  is a term arising from the residues, and the functions

$$(5.3) \quad H_K(z, s) = \frac{1}{2\pi i} \int_{(-K-\frac{1}{2})}^r \prod_{j=1}^r \frac{\Gamma(\lambda_j(1-s) + \bar{\mu}_j - \lambda_j w)}{\Gamma(\lambda_j s + \mu_j + \lambda_j w)} \Gamma(w) z^w dw$$

are rather general cases of **Fox hypergeometric functions**. Since we will eventually let  $N \rightarrow \infty$ , we require some information on  $H_K(-iy, s)$ , especially when

$$y = \frac{n}{2\pi Q^2 a}.$$

An instance of such an approach to the study of the linear twists, in the case  $F(s) = \zeta(s)$ , can be found in Linnik [32]. In fact, starting from (5.1) with  $\alpha = 0$ , Linnik [32] obtained a new proof of the functional equation of the Dirichlet  $L$ -functions using the functional equation of  $\zeta(s)$  in a direct way. In Linnik's special case, the hypergeometric functions (5.3) reduce to simple well known functions, and hence the right hand side of (5.2) becomes rather explicit. This is, unfortunately, not the case in more general situations.

For  $s$  fixed, the general Fox hypergeometric functions have been studied by Braaksma [7]. Roughly speaking, their behaviour depends on the value of the *main parameter*  $\mu$  defined by

$$\mu = 2 \sum_{j=1}^r \lambda_j - 1 = d_F - 1.$$

In the case  $\mu = 0$ , which corresponds to the degree 1 functions, the behaviour is simpler since only the «algebraic part» comes into play, while for  $\mu > 0$  (i.e.  $d_F > 1$ ) the behaviour is more complicated due to the presence of the «exponential part»; we refer to Braaksma [7] for the meaning of the algebraic and exponential parts. We remark that the case  $\mu < 0$ , although not directly related to the linear twists (see Theorem 3.1), has also some interest, and that in this case the situation is simpler thanks to nice convergence properties.

In order to study the linear twists, one has to develop a  $(z, s)$ -variables theory of the hypergeometric functions (5.3). We present here only the result for  $d_F = 1$ , where a clean statement can be given. Let

$$\beta = \prod_{j=1}^r \lambda_j^{2\lambda_j},$$

$\theta_F$  be the *shift* of  $F \in \mathcal{S}^\sharp$  (see Section 4) and, given  $R > 1$ , let  $K = K(R)$  be a suitably large positive integer.

**Theorem 5.1.** ([26], [29]) *Let  $d_F = 1$ ,  $y > 0$  and  $\sigma < R$ . If  $y \neq \beta$  then  $H_K(-iy, s)$  is holomorphic, while  $H_K(-i\beta, s)$  has at most simple poles at the points  $s_k = 1 - k - i\theta_F$  for  $k = 0, 1, \dots$ , with non-vanishing residue at  $s = s_0$ .*

In addition, suitable bounds for  $H_K(-iy, s)$  as  $y \rightarrow \infty$ , required in (5.2), can be obtained; see Kaczorowski-Perelli [26], [29]. Moreover, it is in principle possible to check the vanishing or non-vanishing of the residue at each point  $s_k$  with  $k \geq 1$ , but there are non-trivial complications in details. We will say more on this later on, see Problem 5.3.

We refer to Kaczorowski-Perelli [28] and Kaczorowski [23] for the analytic properties of the hypergeometric functions when  $1 < d_F < 2$ , since in this case the statement is more involved due to the appearance of the above mentioned exponential part. We remark here that such an exponential part is reflected by the exponential factor in the Dirichlet series  $D_F(s, \alpha)$  in Theorem 5.3 below.

The analytic properties of the linear twists of the functions  $F \in \mathcal{S}_1^\sharp$  follow now from (5.2) and Theorem 5.1. Let  $\alpha > 0$  and, in view of (5.2) and Theorem 5.1, define the **critical value**  $n_\alpha$  (of course arising from the equation  $\frac{n}{2\pi Q^2 \alpha} = \beta$ ) by

$$n_\alpha = q_F \alpha,$$

where  $q_F$  is the *conductor* of  $F(s)$  defined in Section 4. Moreover, define  $a_F(n_\alpha) = 0$  if  $n_\alpha \notin \mathbb{N}$ . We have

**Theorem 5.2.** ([26], [29]) *Let  $F \in \mathcal{S}_1^\sharp$  and  $\alpha > 0$ . Then  $F(s, \alpha)$  is entire if  $a_F(n_\alpha) = 0$ , while if  $a_F(n_\alpha) \neq 0$  then  $F(s, \alpha)$  has at most simple poles at the points  $s_k = 1 - k - i\theta_F$  for  $k = 0, 1, \dots$ , with residue at  $s_0$  equal to  $c(F) \frac{a_F(n_\alpha)}{n_\alpha^{i\theta_F}}$  and  $c(F) \neq 0$ .*

Clearly, the vanishing or non-vanishing of the residue at the points  $s_k$  with  $k \geq 1$  is closely related to the analogous problem in Theorem 5.1.

In order to state the properties of the linear twists of functions  $F \in \mathcal{S}_d^\sharp$  with  $1 < d < 2$  we need a few more definitions. Let

$$\kappa = \frac{1}{d_F - 1}, \quad A = (d_F - 1) q_F^{-\kappa}, \quad s^* = \kappa \left( s + \frac{d_F}{2} - 1 + i\theta_F \right)$$

and

$$D_F(s, \alpha) = \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^s} e \left( A \left( \frac{n}{\alpha} \right)^\kappa \right).$$

Moreover, let  $\sigma^*$  denote the real part of  $s^*$  and  $\sigma_a(F)$  be the abscissa of absolute convergence of  $F(s)$ . We have

**Theorem 5.3.** ([28]) *Let  $1 < d < 2$ ,  $F \in S_d^\sharp$ ,  $\alpha > 0$  and  $J \geq 1$  be an integer. Then there exist a constant  $c_0 \neq 0$  and polynomials  $P_j(s)$ , with  $0 \leq j \leq J - 1$  and  $P_0(s) = c_0$  identically, such that for  $\sigma^* > \sigma_a(F)$*

$$(5.4) \quad F(s, \alpha) = q_F^{\kappa s} \sum_{j=0}^{J-1} \alpha^{\kappa(d_F s - d_F/2 + i\theta_F + j)} P_j(s) D_F(s^* + j\kappa, \alpha) + G_J(s, \alpha)$$

where  $G_J(s, \alpha)$  is holomorphic for  $s$  in the half-plane  $\sigma^* > \sigma_a(F) - \kappa J$  and continuous for  $\alpha > 0$ .

Note that  $\sigma^* > \sigma$  for  $\sigma > \frac{1}{2}$  and  $1 < d_F < 2$ , hence (5.4) shows a kind of *overconvergence phenomenon* for  $F(s, \alpha)$ , which will be exploited in Section 6. Observe also that, contrary to Theorem 5.2 where a clean description of the analytic properties of the linear twists of degree 1 functions is given, in this case the properties of the linear twists are related to those of certain *non-linear* twists. However, due to the above overconvergence phenomenon, Theorem 5.3 provides a non-trivial continuation of  $F(s, \alpha)$  to a strip to the left of  $\sigma = 1$ . This will be important in Section 6, where Theorems 5.2 and 5.3 will be applied to obtain a complete classification of the functions  $F \in S_d^\sharp$  with  $1 \leq d < 5/3$ .

The relation in (5.4) between the linear twists and suitable non-linear twists is just a special case of a more general theory, based on the properties of the Fox hypergeometric functions as in Kaczorowski-Perelli [28], where a *general non-linear twist* of  $F \in S^\sharp$  is related to its *conjugate non-linear twist*. We do not enter such a general theory in this survey.

We remark here that by axiom (i) we have  $\sigma_a(F) \leq 1$  for every  $F \in S^\sharp$ , but the exact value of  $\sigma_a(F)$  is not known in general. Assuming the Selberg orthonormality conjecture, in Section 3 we saw that  $\sigma_a(F) = 1$  for every  $F \in S \setminus \{1\}$ . In the general case we raise the following

**Problem 5.1.** *Is it true that  $\sigma_a(F) = 1$  for every  $F \in S_d^\sharp$  with  $d > 0$ ?*

Now we turn to a discussion of a special type of non-linear twist. For a given  $F \in S_d^\sharp$  and  $\alpha \in \mathbb{R}$  we consider the (*canonical*) **non-linear twist**

$$(5.5) \quad F_d(s, \alpha) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-n^{1/d} \alpha),$$

Clearly, if  $d = 1$  the non-linear twist coincides with the linear twist. Moreover, it

turns out that Theorem 5.2 is a special case of a more general result describing the analytic properties of the non-linear twists of functions of *any* degree  $d > 0$ . In fact, following the argument leading to (5.2) with  $F_d(s, \alpha)$  in place of  $F(s, \alpha)$  and performing a suitable change of variable, the involved hypergeometric functions still have  $\mu = 0$ . Therefore, defining for every  $d > 0$  and  $\alpha > 0$  the  $d$ -**critical value** by

$$n_\alpha = q_F d^{-d} \alpha^d$$

and writing again  $a_F(n_\alpha) = 0$  if  $n_\alpha \notin \mathbb{N}$ , an argument similar to Theorem 5.2 in this case yields

**Theorem 5.4.** ([29]) *Let  $d > 0$ ,  $F \in S_d^\sharp$  and  $\alpha > 0$ . Then  $F_d(s, \alpha)$  is entire if  $a_F(n_\alpha) = 0$ , while if  $a_F(n_\alpha) \neq 0$  then  $F_d(s, \alpha)$  has at most simple poles at the points*

$$s_k = \frac{d+1}{2d} - \frac{k}{d} - i \frac{\theta_F}{d} \quad k = 0, 1, \dots,$$

with residue  $\overline{a_F(n_\alpha)} c_k(F, \alpha)$  and  $c_0(F, \alpha) \neq 0$ .

Clearly, Theorem 5.4 reduces to Theorem 5.2 for  $d = 1$ , and again the vanishing or non-vanishing of the residue at the points  $s_k$  with  $k \geq 1$  is closely related to the analogous problem in Theorem 5.1.

Uniform versions of Theorem 5.4 can also be obtained, in the sense that uniform bounds on vertical strips can be proved for functions in suitable families  $\mathcal{F} \subset S^\sharp$ . Roughly speaking, the functions in such families  $\mathcal{F}$  must have bounded degree and bounded  $\mu$ -coefficients, and the estimates are obtained starting from analogous estimates for families of hypergeometric functions. We refer to Theorem 2 of Kaczorowski-Perelli [29] for such results.

The polar structure in Theorem 5.4 deserves a brief discussion. First of all we raise the following

**Problem 5.2.** *Find a heuristic explanation for the simple pole of  $F_d(s, \alpha)$  at  $s_0 = \frac{d+1}{2d} - i \frac{\theta_F}{d}$  (for suitable values of  $\alpha$ ).*

Concerning the possible poles at the points  $s_k$  with  $k \geq 1$ , there are cases (for example when  $F(s)$  equals  $\zeta(s)$  or  $L(s, \chi)$ ) where all residues vanish but at  $s_0$ . In fact, it follows from Theorem 6.2 below that this is the case for *all*  $F \in S_1^\sharp$ . However, there are cases in degree 2 where infinitely many such residues do not

vanish, see the explicit formulae for  $H_K(s, -i/2)$  in the proof of Lemma 1 of Kaczorowski [22]. Moreover, some evidence for the non-vanishing of the residue at  $s_1$  will be presented in Section 11, again for degree 2 functions. In general, we raise the following

**Problem 5.3.** *Investigate the vanishing or non-vanishing of the residue of  $F_d(s, \alpha)$  at the points  $s_k$  with  $k \geq 1$  (for suitable values of  $\alpha$ ).*

We conclude illustrating a typical application of the hypergeometric functions with main parameter  $\mu < 0$ . Given a function  $F \in S^\sharp$ , let  $F_d(s, \alpha)$  be defined by (5.5) also when  $d > d_F$ ; a classical instance is

$$\zeta_d(s, \alpha) = \sum_{n=1}^{\infty} e(-n^{1/d} \alpha) n^{-s}$$

with  $d > 1$  and  $\alpha > 0$ . We have

**Theorem 5.5.** ([29]) *Let  $F \in S^\sharp$ ,  $d > d_F$  and  $\alpha > 0$ . Then  $F_d(s, \alpha)$  is an entire function.*

Note that Theorem 5.5 has been proved in Kaczorowski-Perelli [29] only in the case  $F(s) = \zeta(s)$ , see Lemma 4.1 of [29]. However, it is remarked there that the argument is general, and in fact it proves the stated result. It is interesting to note that the case  $F(s) = \zeta(s)$  of Theorem 5.5 is due to Hardy [18] (see also n. 3 of the Miscellaneous Examples at the end of Chapter IX of Titchmarsh [53]), although Hardy's proof is quite different from that in [29]. The proof in [29] is based on the theory of the hypergeometric functions with  $\mu < 0$ , and represents a much simplified version of the basic argument in Theorem 5.4.

## 6 - Degree $1 \leq d < 2$

At the beginning of Section 2 we raised two questions. We gave an answer to the first question but not yet to the second one, asking if *all*  $L$ -functions are already known. An answer to such question is given by the following impressive conjecture.

**Conjecture 6.1.** (main conjecture) *The Selberg class  $S$  coincides with the class of the  $GL(n)$  automorphic  $L$ -functions.*

Conjecture 6.1, if true, lies very deep. In fact, on the one hand it *morally* implies the truth of the Langlands program, since the  $L$ -functions of arithmetic, al-

gebraic and geometric nature (morally in  $\mathcal{S}$ ) would become special cases of automorphic  $L$ -functions. On the other hand, if one accepts that  $\mathcal{S}$  is the class of all  $L$ -functions, then Conjecture 6.1 implies that *all*  $L$ -functions are already known. Moreover, Conjecture 6.1 immediately implies *almost all* the other conjectures in this survey, since most such conjectures are known in the case of the automorphic  $L$ -functions (but *not all*, for example GRH).

Since the automorphic  $L$ -functions have integer degree, we can split Conjecture 6.1 into two parts as follows.

Conjecture 6.2. (general converse theorem) For  $d \in \mathbb{N}$

$$\mathcal{S}_d = \{\text{automorphic } L\text{-functions of degree } d\}.$$

Conjecture 6.3. (degree conjecture) For  $d \notin \mathbb{N}$

$$\mathcal{S}_d = \emptyset.$$

Since the standard  $\gamma$ -factors of the automorphic  $L$ -functions have all the  $\lambda$ -coefficients equal to  $\frac{1}{2}$ , it is clear that Conjectures 6.2 and 6.3 imply and motivate Conjectures 4.1 and 4.2 (restricted to  $\mathcal{S}$ ).

Although we are mainly concerned with the Selberg class  $\mathcal{S}$ , it is interesting to raise similar problems for  $\mathcal{S}^\sharp$  as well.

Problem 6.1. What does  $\mathcal{S}_d^\sharp$  contain for  $d \in \mathbb{N}$ ?

We remark here that  $\mathcal{S}^\sharp$  is a model, introduced in Kaczorowski-Perelli [26], for the class of the « $L$ -functions without Euler product». However, there are well known classes of  $L$ -functions which in general do not fall into  $\mathcal{S}^\sharp$ , see for instance the vector spaces of  $L$ -functions associated with holomorphic modular forms (where the functional equation has no conjugation on the right hand side). Possibly, a better definition of  $\mathcal{S}^\sharp$  is obtained by allowing a slightly more general type of functional equation, for example relating  $F(s)$  to  $G(1-s)$  instead of  $\overline{F}(1-s)$ , where  $G(s)$  satisfies the same properties of  $F(s)$ . At any rate, we do not expect substantial differences between the properties of such a class and  $\mathcal{S}^\sharp$ .

Coming back to the description of  $\mathcal{S}^\sharp$ , we expect that the degree conjecture holds for  $\mathcal{S}^\sharp$  as well.

Conjecture 6.4. (strong degree conjecture)  $\mathcal{S}_d^\sharp = \emptyset$  for  $d \notin \mathbb{N}$ .

Let  $V^\sharp(Q, \lambda, \mu, \omega)$  denote the real vector space of the functions of  $\mathcal{S}^\sharp \cup \{0\}$

satisfying a given functional equation. A much weaker version of Conjecture 6.4 is

**Problem 6.2.** *Prove that  $\dim V^{\sharp}(Q, \lambda, \mu, \omega) < \infty$  if  $d \notin \mathbb{N}$ .*

Probably,  $\dim V^{\sharp}(Q, \lambda, \mu, \omega) = \infty$  for certain degree 2 functional equations (see Chapter II of Hecke [19]), thus the condition  $d \notin \mathbb{N}$  in Problem 6.2 appears to be crucial.

Somehow, the degree conjecture reflects the *arithmetical nature* of the Selberg class. Although  $S^{\sharp}$  is obtained dropping the two arithmetical axioms of  $S$ , Conjecture 6.4 suggests that  $S^{\sharp}$  still has some arithmetical content. Indeed, by axiom (i) every  $F \in S^{\sharp}$  is an *ordinary* Dirichlet series, *i.e.* the «frequencies» are *integers*. We may therefore ask if the analog of the degree conjecture fails once axiom (i) is weakened to allow *general* Dirichlet series (see Section 2 for the definition). It turns out that this is essentially the case, as the following simple result shows. For *any* choice of  $(Q, \lambda, \mu, \omega)$  as in axiom (iii), let  $\mathcal{D}(Q, \lambda, \mu, \omega)$  denote the real vector space of the *somewhere* absolutely convergent general Dirichlet series satisfying axioms (ii) and (iii). We have

**Theorem 6.1.** ([30])  $\mathcal{D}(Q, \lambda, \mu, \omega)$  *has an uncountable basis.*

Therefore, the degree conjecture definitely fails in this case. The proof is based on Hecke's theory of modular forms associated with the groups  $G(\lambda)$ , see Chapter II of Hecke [19], which provides examples of suitable  $L$ -functions with arbitrary non-negative  $\mu$ -coefficient. Theorem 6.1 is slightly unsatisfactory due to the «somewhere absolutely convergent» general Dirichlet series in the definition of  $\mathcal{D}(Q, \lambda, \mu, \omega)$ . Therefore we raise

**Problem 6.3.** *Is the analog of Theorem 6.1 true with «somewhere absolutely convergent» replaced by «absolutely convergent for  $\sigma > 1$ » in the definition of  $\mathcal{D}(Q, \lambda, \mu, \omega)$ ?*

In Section 3 we presented the results for degree  $0 \leq d < 1$  (Theorems 3.1, 3.2 and 3.3), which confirm Conjectures 6.2 and 6.3 in that range. We remark here that a very simple proof of Theorem 3.1 can be obtained as a corollary of Theorem 5.4, see Kaczorowski-Perelli [29]. In fact, let  $0 < d < 1$ ,  $F \in S^{\sharp}_d$ ,  $m$  with  $a_F(m) \neq 0$  and let  $\alpha$  be such that  $n_{\alpha} = m$ . Hence the non-linear twist  $F_d(s, \alpha)$  has a pole on the line  $\sigma = \frac{d+1}{2d} > 1$ , a contradiction.

Now we turn to the next case, *i.e.* the classification of the  $L$ -functions of  $S_1$ . By Conjecture 6.2 one expects that these are the Dirichlet  $L$ -functions with primitive

characters and their shifts (see Section 3), and this is in fact the case as Theorem 6.3 below shows. We start with a complete description of the functions  $F \in S_1^\sharp$ . For a positive integer  $q$  and complex numbers  $\xi = \eta + i\theta$  and  $\omega^*$  with  $|\omega^*| = 1$ , we denote by  $S_1^\sharp(q, \xi, \omega^*)$  the set of  $F \in S_1^\sharp$  such that (see Section 4)

$$q_F = q, \quad \xi_F = \xi, \quad \omega_F^* = \omega^*.$$

Since  $q_F, \xi_F = \eta_F + i\theta_F$  and  $\omega_F^*$  are invariants,  $S_1^\sharp$  is a disjoint union of these classes. Moreover, we write

$$V_1^\sharp(q, \xi, \omega^*) = S_1^\sharp(q, \xi, \omega^*) \cup \{0\}.$$

If  $\chi$  is a Dirichlet character we denote by  $f_\chi$  its conductor and by  $\chi^*$  the primitive character inducing  $\chi$ . We also denote by  $\omega_{\chi^*}$  the  $\omega$ -factor in the standard functional equation of  $L(s, \chi^*)$  and, for  $\eta \in \{-1, 0\}$ , we write

$$\mathfrak{X}(q, \xi) = \begin{cases} \{\chi \pmod{q} \text{ with } \chi(-1) = 1\} & \text{if } \eta = -1 \\ \{\chi \pmod{q} \text{ with } \chi(-1) = -1\} & \text{if } \eta = 0. \end{cases}$$

Further,  $\chi_0$  denotes the principal character  $\pmod{q}$ .

**Theorem 6.2.** ([26]) *i) If  $F \in S_1^\sharp$ , then  $q_F \in \mathbb{N}$ , the sequence  $a_F(n)n^{i\theta_F}$  is periodic of period  $q_F$  and  $\eta_F \in \{-1, 0\}$ .*

*ii) Every  $F \in S_1^\sharp(q, \xi, \omega^*)$ , with  $q \in \mathbb{N}$ ,  $\eta \in \{-1, 0\}$  and  $|\omega^*| = 1$ , can be uniquely written as*

$$F(s) = \sum_{\chi \in \mathfrak{X}(q, \xi)} P_\chi(s + i\theta) L(s + i\theta, \chi^*)$$

where  $P_\chi \in S_0^\sharp\left(\frac{q}{f_{\chi^*}}, \omega^* \overline{\omega_{\chi^*}}\right)$ . Moreover,  $P_{\chi_0}(1) = 0$  if  $\theta \neq 0$ .

*iii) For  $q, \xi$  and  $\omega^*$  as above,  $V_1^\sharp(q, \xi, \omega^*)$  is a real vector space with*

$$\dim_{\mathbb{R}} V_1^\sharp(q, \xi, \omega^*) = \begin{cases} \left\lceil \frac{q}{2} \right\rceil + 1 & \text{if } \xi = -1 \\ \left\lceil \frac{q-1-\eta}{2} \right\rceil & \text{otherwise.} \end{cases}$$

Note that by Theorem 6.2 the functions in  $S_1^\sharp$  satisfy the Ramanujan conjecture. Adding the Euler product axiom, from Theorem 6.2 we obtain

**Theorem 6.3.** ([26]) *Let  $F \in \mathcal{S}_1$ . If  $q_F = 1$ , then  $F(s) = \zeta(s)$ . If  $q_F \geq 2$ , then there exists a primitive Dirichlet character  $\chi \pmod{q_F}$  with  $\chi(-1) = -(2\eta_F + 1)$  such that  $F(s) = L(s + i\theta_F, \chi)$ .*

We give a sketch of the first steps of the proof of Theorem 6.2, thus showing the relevance of the polar structure of  $F(s, \alpha)$  in Theorem 5.2. Choose  $m$  with  $a_F(m) \neq 0$  and let  $\alpha = \frac{m}{q_F}$ . Then the linear twist  $F(s, \alpha)$  has a simple pole at  $s = 1 - i\theta_F$ , and hence the same holds for  $F(s, \alpha + 1)$  by the  $\alpha$ -periodicity of linear twists. Therefore  $n_{\alpha+1} = q_F \left( \frac{m}{q_F} + 1 \right) \in \mathbb{N}$ , thus  $q_F \in \mathbb{N}$ . Similarly, to show the periodicity of the coefficients we choose  $\alpha = \frac{n}{q_F}$ . Then  $F(s, \alpha)$  has residue equal to  $c(F) \frac{a_F(n)}{n^{i\theta_F}}$  at  $s = 1 - i\theta_F$ , and hence  $F(s, \alpha + 1)$  has residue  $c(F) \frac{a_F(n + q_F)}{(n + q_F)^{i\theta_F}}$  at the same point, and the periodicity follows. Once the periodicity of the coefficients is established, Dirichlet characters enter the game, and  $F(s)$  is expressed as a linear combination of Dirichlet  $L$ -functions over the Dirichlet polynomials of  $\mathcal{S}_1^\sharp$ . The full description of the functions in  $\mathcal{S}_1^\sharp$  follows then from a careful analysis of the functional equations satisfied by  $F(s)$  and by the involved Dirichlet  $L$ -functions.

We refer to Soundararajan [51] for a different proof of Theorem 6.3. For previous partial results towards Theorems 6.2 and 6.3, and for related results, we refer to Bochner [1], Vignéras [55], Gérardin-Li [17], Conrey-Ghosh [10] and Funakura [16], and to the literature quoted there.

We remark that Theorem 6.2 confirms (ii) of Theorem 4.5 in the case of degree 1 functions: the triplet  $(q_F, \omega_F^*, \xi_F)$  determines the functional equation of  $F \in \mathcal{S}_1^\sharp$ . Moreover, Theorem 6.3 clarifies the name and the meaning of the invariant  $\eta_F$ .

A simple consequence of Theorem 6.3 is

**Corollary 6.1.** ([37]) *The normalized  $L$ -functions  $L_f(s)$  associated with holomorphic newforms  $f(z)$  on congruence subgroups of  $SL(2, \mathbb{Z})$  are primitive.*

This is proved by contradiction, assuming that  $L_f(s) = L(s + i\theta_1, \chi_1) L(s + i\theta_2, \chi_2)$  with  $\chi_j$  primitive Dirichlet characters and  $\theta_j \in \mathbb{R}$ . Taking the Rankin-Selberg convolution of both sides (or twisting by a suitable character), the order of pole at  $s = 1$  leads to a contradiction.

We remark that the classical converse theorem of Weil [56] characterizes the  $GL(2)$   $L$ -functions by means of their twists by Dirichlet characters. Roughly

speaking, a similar philosophy applies in general to the  $GL(n)$  converse theorems, see Cogdell and Piatetski-Shapiro [8]. In fact, the  $GL(n)$   $L$ -functions are characterized in terms of suitable Rankin-Selberg convolutions, and the twists are of course special instances of such convolutions. Since Theorems 6.2 and 6.3 can be regarded as general converse theorems for degree 1 functions, it is clear that the use of the linear twists in their proofs fits well into the above philosophy. Note that the axioms of  $\mathcal{S}$  do not explicitly include any property of the Rankin-Selberg convolutions; this increases the difficulties in proving general converse theorems in  $\mathcal{S}$ . We refer to Conrey-Farmer [9] for an interesting alternative to Weil's converse theorem, where the twists are replaced by the Euler product, although at present this approach produces much less general results.

Another instance showing the relevance of the twists in the problems of this section is the following. Given  $F \in \mathcal{S}^\sharp$  and a Dirichlet character  $\chi$ , define the **twist** of  $F(s)$  as

$$F^\chi(s) = \sum_{n=1}^{\infty} \frac{a_F(n) \chi(n)}{n^s}.$$

The following two natural conjectures about twists are given in Kaczorowski-Petrelli [27]; see also Selberg [50] for other conjectures on twists.

**Conjecture 6.5.** (twist conjecture) *Let  $F \in \mathcal{S}$  with  $q_F \in \mathbb{N}$ ,  $m \in \mathbb{N}$  with  $(m, q_F) = 1$  and let  $\chi \pmod{m}$  be a primitive Dirichlet character. Then  $F^\chi \in \mathcal{S}$ .*

In addition, in [27] it is also conjectured that  $F^\chi(s)$  is primitive if and only if  $F(s)$  is primitive.

**Conjecture 6.6.** (twisted conductor conjecture) *Assume the twist conjecture. Then*

$$q_{F^\chi} = q_F m^{d_F}.$$

It is easy to see that *the conductor conjecture* (see Section 4) and *Conjectures 6.5 and 6.6 imply the degree conjecture*, since  $q_F m^{d_F} \in \mathbb{N}$  for all  $(m, q_F) = 1$  implies that  $d_F \in \mathbb{N}$ . Although the twists (by Dirichlet characters) and the linear twists are closely related, a problem of some interest is to give a proof of Theorem 6.3 closer to the spirit of Weil's converse theorem.

**Problem 6.4.** *Give a proof of Theorem 6.3 using the twists (by Dirichlet characters) instead of the linear twists.*

Turning to the range  $1 < d < 2$  we have

**Theorem 6.4.** ([28])  $S_d^\sharp = \emptyset$  for  $1 < d < 5/3$ . Moreover, for  $1 < d < 2$  there exist no  $F \in S_d^\sharp$  with a pole at  $s = 1$ .

The second part of Theorem 6.4 follows immediately from Theorem 5.3 with  $J = 1$ . Indeed, assuming that  $F(s)$  has a pole at  $s = 1$ , choosing  $\alpha = 1$  and exploiting the  $\alpha$ -periodicity of the linear twists, thanks to the overconvergence phenomenon the right hand side of (5.4) is holomorphic at  $s = 1$ , a contradiction. The holomorphic case is definitely more involved. By a Fourier transform argument, equation (5.4) is transformed into an identity of type

$$(6.1) \quad \Sigma_1(x) = x^{-\kappa/2} \Sigma_2(x) + O(x^{\sigma_a(F) - \kappa + \varepsilon}),$$

where  $\Sigma_1(x)$  and  $\Sigma_2(x)$  are certain exponential sums. Moreover,  $\Sigma_1(x)$  is concentrated at the integers, while the same phenomenon is not visible in  $\Sigma_2(x)$ . This fact is exploited by computing the  $L^2$ -norm, weighted by the function  $e(x)$ , of both sides of (6.1). The weight is clearly irrelevant on the left hand side, but produces some cancellation on the right hand side when  $1 < d < 5/3$ , thus getting a contradiction in that range.

We remark that in order to avoid the use of the Ramanujan conjecture (axiom (iv)), in the proof of Theorem 6.4, the following lemma of some independent interest concerning a form of *Rankin-Selberg convolution* in  $S^\sharp$  is used. For  $F \in S^\sharp$  and  $\sigma > 2\sigma_a(F)$  define

$$(6.2) \quad F \times \bar{F}(s) = \sum_{n=1}^{\infty} |a_F(n)|^2 n^{-s}.$$

**Lemma 6.1.** ([28]) *Let  $1 < d < 2$  and  $F \in S_d^\sharp$ . Then  $F \times \bar{F}(s)$  is holomorphic for  $\sigma > \sigma_a(F) - \kappa$  apart from a simple pole at  $s = 1$ .*

Note that the simple pole at  $s = 1$  of  $F \times \bar{F}(s)$  is in agreement with the fact that all  $F \in S_d$  with  $1 < d < 2$  (if any !) are primitive, and the Rankin-Selberg convolution (6.2) of a primitive function is expected to have simple pole at  $s = 1$  (in agreement with the Selberg orthonormality conjecture).

J. Kaczorowski observed that the same range  $1 < d < 5/3$  in Theorem 6.4 would follow from a direct application of the well known conjecture that  $(\varepsilon, \frac{1}{2} + \varepsilon)$  is an exponent pair (see Chapter 3 of Montgomery [35]) to a certain exponential sum closely linked to the exponential sums in (6.1). Therefore, Theorem 6.4 appears to be the limit of the method in that respect. However, the whole range  $1 < d < 2$  would follow from a natural multidimensional analog of such a conjecture.

re. In fact, using a more refined Fourier transform argument giving an error  $O(x^{\sigma_a(F) - \frac{3}{2}k + \varepsilon})$  in (6.1), by a  $k$ -fold iteration of (6.1) one can get an expression for the coefficients  $a_F(n)$  involving a certain  $k$ -dimensional exponential sum. The full range  $1 < d < 2$  would then follow from a square-root cancellation bound for such a sum. We refer to Kaczorowski-Perelli [28] for further remarks on this subject.

From Theorem 6.4 we easily obtain still another proof of Theorem 3.1, see [28]. Indeed, assuming that there exists a function  $F \in S_d^\sharp$  with  $0 < d < 1$ , we may clearly assume (shifting if necessary) that  $F(1) \neq 0$ . Therefore  $\zeta(s)F(s)$  is a polar function in  $S_d^\sharp$  with  $1 < d < 2$ , a contradiction. The same argument shows that *the degree conjecture restricted to the polar  $L$ -functions implies the degree conjecture* (and similarly for the strong degree conjecture). We conclude by stressing the importance of the  $\alpha$ -periodicity of the linear twists, which plays a fundamental role in the proof of the structure theorems for  $1 \leq d < 2$ .

Now we come back to the measure theoretic approach to invariants outlined in Section 4. We recall that a numerical invariant  $I$  is *additive* if  $I(FG) = I(F) + I(G)$ , and that the  $H$ -invariants are additive; hence in particular so is the degree. We refer to Section 4 for the definition of continuous invariants, and recall that in this case  $I(S)$  and  $I(S^\sharp)$  are Lebesgue measurable by Theorem 4.6. We also recall that, given  $\mathcal{C} \subset \mathbb{R}$ ,  $\mathcal{C} - \mathcal{C}$  denotes the set of all real numbers of the form  $a - a'$  with  $a, a' \in \mathcal{C}$ .

The next result is a **0–1 law for additive invariants** which, in view of the degree conjectures, is particularly interesting in the case of the degree.

**Theorem 6.5.** ([31]) *Let  $I$  be a continuous, additive and real-valued invariant. Then either the set  $I(S)$  has Lebesgue measure 0, or  $I(S) - I(S) = \mathbb{R}$ . The same holds for  $I(S^\sharp)$  as well.*

A similar result holds in the case of the root number, in the sense that the sets of values  $\omega^*(S)$  and  $\omega^*(S^\sharp)$  taken by the root number  $\omega_F^*$  over  $S$  and  $S^\sharp$  are either of measure zero or coincide with the unit circle  $T^1$ ; see Kaczorowski-Perelli [31]. Further, similar results hold for certain subclasses of  $S^\sharp$ , for example  $S_d$  and  $S_d^\sharp$  with a fixed  $d$ . Note that we already met related examples, involving the invariants  $\omega_F^*$  and  $\xi_F$  ( $\omega_F^*$  is not additive), where both alternatives happen. Indeed, from Theorems 6.2 and 6.3 we see that

$$\begin{aligned} \omega^*(S_1^\sharp) &= T^1 & \text{and} & & \omega^*(S_1) & \text{is countable} \\ \eta(S_1^\sharp) &= \eta(S_1) = \{-1, 0\} & \text{and} & & \theta(S_1^\sharp) &= \theta(S_1) = \mathbb{R}. \end{aligned}$$

In the case of the Selberg class  $\mathcal{S}$ , a stronger form of Theorem 6.5 is suggested by the following conjecture in Kaczorowski-Perelli [31].

**Conjecture 6.7.** (0–1 law conjecture) *Let  $I$  be a continuous, additive and real-valued invariant. Then either the set  $I(\mathcal{S})$  is countable or it contains a half-line.*

We conclude the discussion of the measure theoretic results with the following simple conditional result related to the degree conjectures. Let

$$E = \{d > 0 : d \notin \mathbb{N} \text{ and } \mathcal{S}^\sharp \neq \emptyset\}$$

be the «exceptional set» for the degree conjecture. Clearly,  $E$  is measurable by Theorem 4.6, and let  $\mu(E)$  denote its Lebesgue measure.

**Theorem 6.6.** ([14]) *Assume that every  $F \in \mathcal{S}_d^\sharp$  with  $d \in \mathbb{N}$  has a  $\gamma$ -factor with all  $\lambda_j = \frac{1}{2}$ . Then  $E \cap \mathbb{Q} = \emptyset$  and  $\mu(E) = 0$ .*

The same result holds for  $\mathcal{S}$  as well, under the same assumption restricted to the integer degrees of  $\mathcal{S}$ . Since Theorem 6.6 is an unpublished result, we give a detailed proof.

**Proof of Theorem 6.6.** Suppose first that there exists a function  $F(s)$  of degree  $d_F = \frac{d}{n} \in \mathbb{Q} \setminus \mathbb{N}$ , with  $(d, n) = 1$  and  $n > 1$ . Then  $F^n \in \mathcal{S}_d^\sharp$  with  $d \in \mathbb{N}$ , hence by the assumption and Theorem 4.1, the  $n$ -th power of the  $\gamma$ -factor of  $F(s)$  satisfies an identity of type

$$(6.3) \quad \gamma(s)^n = c_0 Q^s \prod_{j=1}^l \Gamma\left(\frac{s}{2} + \mu_j\right)^{n_j},$$

where the  $\mu_j$  are distinct. Our first aim is to show that  $n$  divides each  $n_j$ . To this end consider a row of  $\mu_j$ 's with equal imaginary part, and let  $\mu_{j_0}$  be the one with smallest real part. Since the poles of the  $\gamma$ -factor of  $F(s)^n$  are at the points  $s_{j,k} = -2(\mu_j + k)$  with  $k = 0, 1, \dots$  and  $j = 1, \dots, l$ , the multiplicity of  $s_{j_0,0}$  is  $n_{j_0}$ . Hence  $n | n_{j_0}$ . Let now  $\mu_{j_1}$  be the next one in the same row. If  $\Re(\mu_{j_1} - \mu_{j_0})$  is not an integer, then the same argument shows that  $n | n_{j_1}$ . If  $\Re(\mu_{j_1} - \mu_{j_0})$  is an integer, then the multiplicity of  $s_{j_1,0}$  is  $n_{j_0} + n_{j_1}$ , thus  $n | (n_{j_0} + n_{j_1})$  and hence  $n | n_{j_1}$  in this case as well. Arguing similarly with the other  $\mu_j$ 's we see that  $n | n_j$  for  $j = 1, \dots, l$  as required.

But from (6.3) we have

$$d_{F^n} = 2 \sum_{j=1}^l \frac{n_j}{2} = d,$$

hence  $n|d$ , a contradiction. Therefore  $E \cap \mathbb{Q} = \emptyset$ . Moreover,  $S^\sharp$  is a multiplicative semigroup, hence for every  $d \in E$  and  $d' \in \mathbb{R}^+$  such that  $d + d' \in \mathbb{Q} \setminus \mathbb{N}$  we have  $S_{d'}^\sharp = \emptyset$ . Therefore

$$(6.4) \quad ((\mathbb{Q} \setminus \mathbb{N}) - E) \cap E = \emptyset,$$

and hence  $\mu(E) = 0$ . In fact, given  $A$  and  $B$  with  $\mu(A), \mu(B) > 0$ , the set  $A + B$  contains an open segment, see ex. 19 of Ch. 9 of Rudin [45]. Hence, if  $\mu(E) > 0$  there exist  $r \in \mathbb{Q} \setminus \mathbb{N}$  and  $d_1, d_2 \in E$  such that  $r = d_1 + d_2$ , thus  $r - d_1 = d_2$  contradicting (6.4). ■

Note that the first part of the proof of Theorem 6.6 is a special case of the following result (see Molteni [33]): if the multiplicity of all poles of a  $\gamma$ -factor  $\gamma_1(s)$  is divisible by  $n$ , then there exists another  $\gamma$ -factor  $\gamma_2(s)$  such that  $\gamma_1(s) = \gamma_2(s)^n$ . Note also that the crucial point that  $E \cap \mathbb{Q} = \emptyset$  is a direct consequence of the assumption on the shape of the functional equation for integer degrees. A related problem is

*Problem 6.5. Replace the assumption in Theorem 6.6 by other function-theoretic properties of the functions with integer degree.*

## 7 - Independence

In Section 3 we presented the simplest independence result in the Selberg class, namely the linear independence of the functions in  $S$  (see Theorem 3.8). We also remarked that the unique factorization (UF) and the Selberg orthonormality (SOC) conjectures (see Conjectures 3.2 and 3.3) are stronger forms of independence. In this section we present other independence results, concerning the functions in  $S$  and their zeros. Most such results are conditional, but nevertheless they form a very interesting part of the Selberg class theory.

We start with the following simple consequence of Theorem 3.8, concerning the algebraic independence in  $S$ .

**Corollary 7.1.** ([24])  *$S$  has unique factorization if and only if distinct primitive functions are algebraically independent.*

Clearly, we can't expect that distinct functions in  $\mathcal{S}$  are algebraically independent, since  $\mathcal{S}$  is a semigroup. However, we have

**Theorem 7.1.** ([33]) *Let  $F, G \in \mathcal{S}$  satisfy  $F(s)^a = G(s)^b$  with  $a, b \in \mathbb{N}$ . Then  $F(s) = H(s)^b$  and  $G(s) = H(s)^a$  for some  $H \in \mathcal{S}$ .*

The proof of Theorem 7.1 is based on the characterization of the solubility in  $\mathcal{S}$  of the equation  $X^k = F(s)$  in terms of the multiplicities of the zeros of  $F(s)$  and of its Euler factors  $F_p(s)$ . Note that Theorem 7.1 and the linear independence imply that *pairs of distinct primitive functions in  $\mathcal{S}$  are algebraically independent.*

In the 1940's, Selberg [48], [49] initiated the study of the moments, and hence of the statistical distribution, of  $\log \zeta\left(\frac{1}{2} + it\right)$ . Later, see [50], he outlined the extension of such investigations to the functions in  $\mathcal{S}$ , obtaining their statistical independence as well. Selberg's arguments have been streamlined by Bombieri-Hejhal [3], with the emphasis just on the probabilistic convergence of the relevant measures, and the goal of applications to the distribution of zeros of linear combinations of certain Euler products. Here we present Bombieri-Hejhal's version of Selberg's results on the normal distribution and statistical independence of the values of  $L$ -functions. Note that Bombieri-Hejhal [3] work in a moderately general setting, but their results easily carry over to the Selberg class (see Zanello [57]). Therefore, although Theorems 7.2, 7.3, 7.4 and 7.7 below were originally proved in Bombieri-Hejhal's setting, we state them in the setting of the Selberg class, and refer to [57] for the needed changes in the proofs. Moreover, we refer to the survey paper by Bombieri-Perelli [4] for a discussion of these matters and for a sketch of the proofs of Theorems 7.3, 7.4 and 7.7 below.

We start with two hypotheses needed in the results which follow. The first is a variant of the Selberg orthonormality conjecture (see Conjecture 3.3).

**Hypothesis  $S$ .** *The coefficients of  $F_1, \dots, F_N \in \mathcal{S}$  satisfy, for  $x \rightarrow \infty$ ,*

$$\sum_{p \leq x} \frac{a_i(p) \overline{a_j(p)}}{p} = \delta_{ij} n_j \log \log x + c_{ij} + O\left(\frac{1}{\log x}\right),$$

where  $n_j > 0$ ,  $c_{ij} \in \mathbb{C}$ , and  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise.

Note that Hypothesis  $S$  is quantitatively stronger than SOC. However, it does not assume that the functions are primitive (in practice, it requires that  $F_i(s)$  and

$F_j(s)$  are *coprime* if  $i \neq j$ ), and does not require  $n_j = 1$  for primitive functions. Note, however, that for most applications of SOC the requirement  $n_j > 0$  for primitive functions would also suffice.

Writing

$$N_j(\sigma, T) = \#\{\varrho = \beta + i\gamma : F_j(\varrho) = 0, \beta \geq \sigma, |\gamma| \leq T\},$$

the second hypothesis is the following density estimate of Selberg [49] type, which acts as a substitute of GRH in the results below.

*Hypothesis D.* The zeros of  $F_1, \dots, F_N \in \mathcal{S}$  satisfy

$$N_j(\sigma, T) \ll T^{1-a(\sigma-\frac{1}{2})} \log T$$

for some  $0 < a < 1$ , uniformly for  $\sigma \geq \frac{1}{2}$  and  $j = 1, \dots, N$ .

Note that Hypothesis *D* is known for the Riemann zeta function, for the Dirichlet  $L$ -functions and for certain  $GL(2)$   $L$ -functions.

We need further notation. Given  $F_1, \dots, F_N \in \mathcal{S}$  we write

$$V_j(t) = \frac{\log F_j\left(\frac{1}{2} + it\right)}{\sqrt{\pi n_j} \log \log t} \quad j = 1, \dots, N$$

and let

$$\mu_T(\Omega) = \frac{1}{T} |\{t \in [T, 2T] : (V_1(t), \dots, V_N(t)) \in \Omega\}|$$

be the associated probability measure on  $\mathbb{C}^N$ , where  $\Omega \subset \mathbb{C}^N$  is an open set and  $|\mathcal{A}|$  denotes here the Lebesgue measure of  $\mathcal{A}$ . Moreover, let  $e^{-\pi\|z\|^2}$  denote the gaussian measure on  $\mathbb{C}^N$  and  $d\omega$  be the euclidean density on  $\mathbb{C}^N$ .

**Theorem 7.2.** ([50], [3]) *Let  $F_1, \dots, F_N \in \mathcal{S}$  satisfy Hypotheses *S* and *D*. Then, as  $T \rightarrow \infty$ , the measure  $\mu_T$  converges weakly to the gaussian measure with associated density  $e^{-\pi\|z\|^2} d\omega$ .*

By separating real and imaginary parts of the  $V_j(t)$ , Theorem 7.2 may be expressed as follows. If  $F_1, \dots, F_N \in \mathcal{S}$  satisfy Hypotheses  $S$  and  $D$ , then the functions

$$\frac{\log \left| F_1 \left( \frac{1}{2} + it \right) \right|}{\sqrt{\pi n_1 \log \log t}}, \quad \frac{\arg F_1 \left( \frac{1}{2} + it \right)}{\sqrt{\pi n_1 \log \log t}}, \dots,$$

$$\frac{\log \left| F_N \left( \frac{1}{2} + it \right) \right|}{\sqrt{\pi n_N \log \log t}}, \quad \frac{\arg F_N \left( \frac{1}{2} + it \right)}{\sqrt{\pi n_N \log \log t}}$$

become distributed, in the limit of large  $t$ , like independent random variables, each having gaussian density  $e^{-\pi u^2} du$ . It is interesting to observe how an independence hypothesis of SOC type (*i.e.* Hypothesis  $S$ ) implies the *normal distribution* and *statistical independence* of the values of  $\log F_j(s)$  on the critical line (Hypothesis  $D$  is a technical hypothesis in this context).

Theorem 7.2 is a deep result, and here we just mention the main steps in Bombieri-Hejhal's [3] proof. The starting point is the following approximation formula for  $\log F_j \left( \frac{1}{2} + it \right)$ , obtained by Mellin transform techniques: for  $j = 1, \dots, N$  and  $10 \leq X \leq |t|^2$

$$(7.1) \quad \log F_j \left( \frac{1}{2} + it \right) = \Sigma_j(t, X) + R_j(t, X),$$

where  $\Sigma_j(t, X)$  is a suitable Dirichlet polynomial approximation and  $R_j(t, X)$  is a certain sum over zeros of  $F_j(s)$ . In order to derive Theorem 7.2, in [3] the mixed moments of the  $\Sigma_j(t, X)$  are evaluated, and an  $L^1$ -norm bound for the error term  $R_j(t, X)$  is obtained:

$$(7.2) \quad \int_T^{2T} |R_j(t, X)| dt \ll T \frac{\log T}{\log X}$$

for  $10 \leq X \leq T^{a/2}$ , and

$$(7.3) \quad \int_T^{2T} \prod_{j=1}^N \Sigma_j(t, X)^{k_j} \overline{\Sigma_j(t, X)^{l_j}} dt$$

$$= \delta_{kl} \mathbf{k}! T \prod_{j=1}^N (n_j \log \log X)^{k_j} + O \left( (T (\log \log X))^{\frac{K+L-1}{2}} \right)$$

for  $10 \leq X \leq T^{1/(K+L+1)}$ , where  $k_j, l_j$  are non-negative integers,  $\mathbf{k} = (k_1, \dots, k_N)$  and  $K = k_1 + \dots + k_N$  (and similarly for  $\mathbf{l}$  and  $L$ ),  $\mathbf{k}! = \prod_{j=1}^N k_j!$  and  $\delta_{\mathbf{k}\mathbf{l}} = 1$  if  $\mathbf{k} = \mathbf{l}$ ,  $\delta_{\mathbf{k}\mathbf{l}} = 0$  otherwise. Theorem 7.2 follows then from (7.1), (7.2) and (7.3) by a probabilistic argument.

For future reference, we state here a *short intervals* version of Theorem 7.2. Let  $M \geq 10$ ,  $h = M/\log T$ ,

$$\tilde{V}_j(t) = \frac{\log F_j\left(\frac{1}{2} + i(t+h)\right) - \log F_j\left(\frac{1}{2} + it\right)}{\sqrt{2\pi n_j \log M}} \quad j = 1, \dots, N$$

and let  $\tilde{\mu}_T$  be the associated probability measure on  $\mathbb{C}^N$  (like  $\mu_T$  above). We have

**Theorem 7.3.** ([5]) *Let  $F_1, \dots, F_N \in \mathcal{S}$  satisfy Hypotheses  $S$  and  $D$  and let  $M = M(T) \rightarrow \infty$  with  $M \leq \frac{\log T}{\log \log T}$ . Then, as  $T \rightarrow \infty$ , the measure  $\tilde{\mu}_T$  converges weakly to the gaussian measure with associated density  $e^{-\pi\|\omega\|^2} d\omega$ .*

The proof of Theorem 7.3 is a short intervals version of the argument in Theorem 7.2.

It is generally accepted that functional equation and Euler product are crucial ingredients for the validity of the Riemann Hypothesis. In the opposite direction, examples of  $L$ -functions without Euler product having zeros *off* the critical line are well known; see *e.g.* Section 10.25 of Titchmarsh [54] and Chapter 9 of Davenport [13]. Usually, such  $L$ -functions are linear combinations of Euler products satisfying a common functional equation, and fairly general methods often show that the number of zeros *on* the critical line up to  $T$  is  $\gg T$  (for small degrees). In some cases, lower bounds of type  $\gg Tf(T)$  with concrete functions  $f(T) \rightarrow \infty$  have been obtained, see Section 1 of [3]. Using Theorem 7.2, Bombieri-Hejhal [3] obtained a very sharp and general result about the zeros on the critical line of linear combinations of Euler products. In order to state such a result we need the following mild condition of well-spacing for zeros.

**Hypothesis  $H_0$ .** *Let  $F_1, \dots, F_N \in \mathcal{S}$  satisfy GRH and, moreover, let the zeros of each  $F_j(s)$  satisfy*

$$\lim_{\varepsilon \rightarrow 0^+} \left\{ \lim_{T \rightarrow \infty} \frac{|\{T \leq \gamma \leq 2T : \gamma' - \gamma \leq \varepsilon/\log T\}|}{T \log T} \right\} = 0,$$

where  $\frac{1}{2} + i\gamma'$  is the successor of  $\frac{1}{2} + i\gamma$ .

We will return on Hypothesis  $H_0$  later on in this section. Given  $F_1, \dots, F_N \in \mathcal{S}$  satisfying the *same* functional equation, let

$$(7.4) \quad F(s) = \sum_{j=1}^N c_j F_j(s),$$

where the coefficients  $c_j$  are such that  $c_j \gamma\left(\frac{1}{2} + it\right) F_j\left(\frac{1}{2} + it\right) \in \mathbb{R}$  for  $t \in \mathbb{R}$  and  $j = 1, \dots, N$ , where  $\gamma(s)$  is the common  $\gamma$ -factor of the  $F_j(s)$ 's.

**Theorem 7.4.** ([2], [3]) *Let  $F_1, \dots, F_N \in \mathcal{S}$  satisfy the same functional equation, GRH and Hypothesis  $H_0$ , and let  $F(s)$  be defined as in (7.4). Then all but  $o(T \log T)$  zeros of  $F(s)$  up to  $T$  are simple and lie on the critical line.*

Roughly speaking, the proof of Theorem 7.4 rests on the fact that on a given interval of the critical line of length roughly  $1/\log T$ , one of the functions  $F_j(s)$  dominates with oscillations larger than the others, thus in that interval the function  $F(s)$  follows fairly closely the behaviour of the  $F_j(s)$  with largest oscillation. The proof is rather complicated because of, among other things, the weak measure theoretic setting in Theorem 7.2.

In view of Theorem 7.4, one may ask about the number of *off-line* zeros up to  $T$  of the function  $F(s)$  in (7.4). We refer to Hejhal [20], [21] for several results in this direction. Here we only recall that there is some expectation that the order of magnitude of the number of such zeros is about

$$\frac{T \log T}{\sqrt{\log \log T}}.$$

By Hadamard's theory, the  $L$ -functions are essentially determined by their zeros, thus one expects that «independent»  $L$ -functions have «independent» zeros. One of the simplest forms of independence of the zeros is asking for **distinct zeros**, in the sense defined below; other forms of independence will be presented later on in this section. In view of SOC and of Theorem 7.2, distinct primitive functions are expected to be independent in several ways, and hence we may expect that they have few common zeros. Therefore, by factorization, distinct functions in  $\mathcal{S}$  should have many distinct zeros. In order to make rigorous and quantitative such heuristic observations, we need some notation.

Given  $F, G \in \mathcal{S}$ , we define two counting functions of the distinct zeros, *with multiplicity*, as follows. The **asymmetric difference** of the zeros of  $F(s)$  and

$G(s)$ , i.e. the number of zeros of  $F(s)$  which are not zeros of  $G(s)$ , is defined by

$$D(T; F, G) = \sum_{|\gamma| \leq T} \max(m_F(\rho) - m_G(\rho), 0),$$

where  $\rho = \beta + i\gamma$  runs over the non-trivial zeros of  $F(s)$  and is counted without multiplicity, and  $m_F(\rho)$  (resp.  $m_G(\rho)$ ) denotes the multiplicity of  $\rho$  as zero of  $F(s)$  (resp. of  $G(s)$ ). The **symmetric difference** is then defined as

$$D_{F,G}(T) = D(T; F, G) + D(T; G, F) = \sum_{|\gamma| \leq T} |m_F(\rho) - m_G(\rho)|,$$

where  $\rho$  runs over the non-trivial zeros of  $F(s)$   $G(s)$  and is counted without multiplicity, and counts the number of zeros and poles of  $F(s)/G(s)$  in the critical strip (excluding the contribution of possible trivial zeros, and of poles at  $s = 1$ , of  $F(s)$  and  $G(s)$ ).

Of course, the asymmetric difference is more difficult to study than the symmetric difference since, in general, we cannot expect a positive lower bound for  $D(T; F, G)$ , as the example  $G(s) = F(s)^2$  shows. In the symmetric case we expect that  $D_{F,G}(T) \gg T \log T$  as soon as  $F(s) \neq G(s)$ ; for example, this is trivially the case when  $d_F \neq d_G$  (see (7.5) below). Recalling that  $f(x) = \Omega(g(x))$  is the negation of  $f(x) = o(g(x))$ , in the general case we have

**Theorem 7.5.** ([38]) *Let  $F, G \in \mathcal{S}$  be distinct. Then  $D_{F,G}(T) = \Omega(T)$ .*

The proof follows by a comparison of Landau's formula, expressing the von Mangoldt function in terms of an exponential sum over the zeros, for  $F(s)$  and  $G(s)$ . In particular, the Euler products of  $F(s)$  and  $G(s)$  are used. We remark that the same  $\Omega$ -estimate has been obtained by Bombieri-Perelli [6] for the number of zeros and poles of a class of exponential sums  $f(s)$  assuming only certain *function theoretic properties*, disregarding their arithmetic aspects. In this case the proof is more involved and uses Nevanlinna's theory. Choosing  $f(s) = F(s)/G(s)$ , Theorem 2 of [6] yields  $\Omega(T)$  distinct zeros in the case of two Dirichlet series  $F(s)$  and  $G(s)$  satisfying the *same* functional equation, without any assumption on the Euler product.

Clearly, in order to get non-trivial lower bounds for the asymmetric difference  $D(T; F, G)$  we have to assure at least that  $F(s) \not\sim G(s)$ . A convenient way to do this is to assume that  $d_F \geq d_G$ . Since clearly (see Section 2)

$$(7.5) \quad D(T; F, G) \geq N_F(T) - N_G(T) \gg T \log T \quad \text{if } d_F > d_G,$$

as far as we are not concerned with *asymptotic formulae* for  $D(T; F, G)$  we may simply assume that  $d_F = d_G$ . An unconditional result in this direction has been obtained by Srinivas [52].

**Theorem 7.6.** ([52]) *Let  $F, G \in \mathcal{S}$  be distinct and  $d_F = d_G$ . Then for  $T$  sufficiently large there exists a zero  $\varrho$  of  $F(s)$  with  $m_F(\varrho) > m_G(\varrho)$  such that  $|T - \varrho| \ll \log \log T$ . Hence, in particular,*

$$D(T; F, G) \gg \frac{T}{\log \log T}.$$

Of course, the same holds switching the roles of  $F(s)$  and  $G(s)$ . The proof is based on a contour integration argument applied to the quotient  $G(s)/F(s)$ . The order  $\log \log T$  of the short intervals in Theorem 7.6 should be compared with the order  $1/\log \log \log T$  of the short intervals containing a zero of a given  $F \in \mathcal{S}$  (extension to  $\mathcal{S}$  of Littlewood's theorem, see Theorem 9.12 of [54], obtained by Anirban-Srinivas, unpublished).

**Problem 7.1.** *Replace the estimates  $\Omega(T)$  in Theorem 7.5 and  $\gg T/\log \log T$  in Theorem 7.6 by the lower bound  $\gg T$ .*

Theorem 7.3 allows to obtain the best possible lower bound for  $D(T; F, G)$ , under Hypothesis  $S$  and the technical Hypothesis  $D$ .

**Theorem 7.7.** ([5]) *Let  $F, G \in \mathcal{S}$  be distinct and satisfy Hypotheses  $S$  and  $D$ , and let  $d_F = d_G$ . Then*

$$D(T; F, G) \gg T \log T.$$

Again, the same result holds switching  $F(s)$  and  $G(s)$ . The proof of Theorem 7.7 rests on a similar phenomenon as in Theorem 7.4, *i.e.* usually one of the functions is dominating on short intervals of length roughly  $1/\log T$ . Also, the weak measure theoretic setting of Theorem 7.3 gives rise to some complications in the proof. A consequence is that the argument is by contradiction, thus the constant in the  $\gg$ -symbol is not effectively computable. However, as remarked in Section 4 of Bombieri-Perelli [4], an effective constant can be obtained at the cost of substantial additional complications in the proof. Further, we already remarked that Hypothesis  $D$  is known for many classical  $L$ -functions of degree  $\leq 2$ . Since such  $L$ -functions satisfy SOC as well, Theorem 7.7 is unconditional in those cases. Again, we refer to the survey paper [4] for a discussion of the distinct zeros problem.

We remark at this point that the first result on distinct zeros has been obtained by Fujii [15]. He proved, by Selberg's moments method, that  $D(T; F, G) \gg T \log T$  in the case of two Dirichlet  $L$ -functions. Moreover, Raghunathan [43], [44] obtained that  $D(T; F, G) \rightarrow \infty$  for certain classical  $L$ -functions  $F(s)$  and  $G(s)$ , by a method based on converse theorems of Hecke type.

The problem of the **strongly-distinct zeros**, *i.e.* the zeros placed at different points, appears to be more difficult. Conrey-Ghosh-Gonek [11], [12] dealt with the case of two Dirichlet  $L$ -functions, by considering the more difficult problem of getting simple zeros of  $L(s, \chi_1) L(s, \chi_2)$ . They obtained  $\gg T^{6/11}$  such zeros up to  $T$  and, assuming the Riemann Hypothesis for one of the two functions, got a positive proportion of strongly distinct zeros. However, apparently the analytic techniques in [11] and [12] do not extend to higher degree  $L$ -functions.

**Problem 7.2.** *Deal with the strongly-distinct zeros problem for the functions in  $S$ .*

We conclude the discussion of distinct zeros with the following conjectures.

**Conjecture 7.1.** (simple zeros conjecture) *Let  $F \in S$  be primitive. Then all but  $o(T \log T)$  non-trivial zeros of  $F(s)$  up to  $T$  are simple.*

**Conjecture 7.2.** (distinct zeros conjecture) *Let  $F, G \in S$  be distinct primitive functions. Then all but  $o(T \log T)$  non-trivial zeros of  $F(s)$  and  $G(s)$  are strongly-distinct.*

Although apparently Conjecture 7.1 cannot be regarded as an independence statement, we included it here in view of its relevance in the problem of the strongly-distinct zeros. Conjectures 7.1 and 7.2 are not at all the strongest conjectures of this type. In fact, it is generally expected that a non-trivial zero with multiplicity greater than 1 of a primitive function can occur only at the point  $s = \frac{1}{2}$ . The same phenomenon is expected to hold for the distinct zeros as well, but one has to be more careful here. In fact, the *shifts* of the primitive functions are expected to be primitive (see Section 3) and GRH is expected to hold, hence it is likely that distinct primitive functions have common zeros other than  $s = \frac{1}{2}$ . However, if  $F, G \in S$  are primitive and **normalized**, *i.e.* the shifts  $\theta_F$  and  $\theta_G$  vanish (as for the classical  $L$ -functions), then it is expected that the only common zero can occur at  $s = \frac{1}{2}$ . The above expectations take into account the Birch and Swinnerton-Dyer conjecture.

Another form of independence is the **functional independence of the zeros**. By this we mean, roughly speaking, the following problem: given  $F_1, \dots, F_N \in \mathcal{S}$  and a holomorphic function  $H(\mathbf{z}, s)$ ,  $\mathbf{z} = (z_1, \dots, z_N)$ , can

$$H(\log F_1(s), \dots, \log F_N(s), s) \quad \text{or} \quad H\left(\frac{F_1'}{F_1}(s), \dots, \frac{F_N'}{F_N}(s), s\right)$$

have only finitely many singularities in the half-plane  $\sigma \geq \frac{1}{2}$ ?

We consider first the problem with the logarithms. It is clear that one needs to impose some natural restrictions on the functions  $F_1, \dots, F_N \in \mathcal{S}$  and on  $H(\mathbf{z}, s)$  in order to get infinitely many singularities. To this end, we consider a region  $\mathcal{O}$  containing the half-plane  $\sigma \geq \frac{1}{2}$  and holomorphic functions  $H(\mathbf{z}, s)$  on  $\mathbb{C}^N \times \mathcal{O}$  such that for every  $s_0 \in \mathcal{O}$  and  $\varepsilon > 0$

$$H(\mathbf{z}, s_0) \ll e^{\varepsilon \|\mathbf{z}\|} \quad \|\mathbf{z}\| \rightarrow \infty;$$

we denote by  $\mathcal{H}$  the ring of such functions. Moreover, we say that  $\deg H = 0$  if for every  $s_0 \in \mathcal{O}$ ,  $H(\mathbf{z}, s_0)$  is constant as a function of  $\mathbf{z}$ . Further, for  $H \in \mathcal{H}$  we write

$$h(s) = H(\log F_1(s), \dots, \log F_N(s), s),$$

which is holomorphic in the region obtained by suitably cutting  $\mathcal{O}$  at the singularities of the  $\log F_j(s)$ 's. Note that the growth condition on  $H(\mathbf{z}, s)$  cannot be significantly relaxed, as shown by the example with  $N = 1$ ,  $F(s) = \zeta(s)^k$ ,  $H(\mathbf{z}, s) = e^{z/k}$  and  $k \in \mathbb{N}$ , where  $h(s)$  has only the singularity at  $s = 1$ . Moreover, the  $\log F_1(s), \dots, \log F_N(s)$  must be linearly independent over  $\mathbb{Q}$ , otherwise there are simple examples of  $H \in \mathcal{H}$  such that  $h(s)$  has no singularities at all. The following result shows that  $h(s)$  has always infinitely many singularities in the half-plane  $\sigma \geq \frac{1}{2}$ , unless there are obvious reasons for the cancellation of the singularities.

**Theorem 7.8.** ([25]) *Let  $F_1, \dots, F_N \in \mathcal{S}$  be such that  $\log F_1(s), \dots, \log F_N(s)$  are linearly independent over  $\mathbb{Q}$ , and let  $H \in \mathcal{H}$  with  $\deg H \neq 0$ . Then  $h(s)$  has infinitely many singularities in the half-plane  $\sigma \geq \frac{1}{2}$ .*

Since the polynomials belong to  $\mathcal{H}$ , Theorem 7.8 provides, in particular, a kind of *algebraic independence* of the zeros, in the sense explained above. Moreover, if the unique factorization conjecture holds (see Conjecture 3.2), then  $h(s)$  has infini-

tely many singularities in  $\sigma \geq \frac{1}{2}$  for any distinct primitive functions  $F_1, \dots, F_N \in \mathcal{S}$  and  $H \in \mathcal{H}$  with  $\deg H \neq 0$ . We also remark that Theorem 7.8 allows to describe the structure of the functions  $H \in \mathcal{H}$  such that  $h(s)$  is holomorphic on  $\mathcal{O}$ , depending on the number of  $F_1, \dots, F_N \in \mathcal{S}$  with linearly independent logarithms, see [25]. As a consequence, one obtains the following corollary of Theorem 7.8 (see [25]):

*if  $F_1, \dots, F_N \in \mathcal{S}$  and  $H \in \mathcal{H}$ , then  $h(s)$  is either holomorphic on  $\mathcal{O}$  or has infinitely many singularities on  $\sigma \geq \frac{1}{2}$ .*

Moreover, non-trivial examples of vanishing  $\mathbb{Q}$ -linear forms of logarithms, and hence of holomorphic  $h(s)$ , can be obtained in the framework of Artin and Hecke  $L$ -functions (see Kaczorowski-Perelli [25]).

The proof of Theorem 7.8 is based on the following lemma, which is of some independent interest. Given  $\varrho \in \mathbb{C}$  we write  $\mathbf{m}(\varrho) = (m_1(\varrho), \dots, m_N(\varrho))$ , where  $m_j(\varrho)$  denotes, as usual, the multiplicity of  $\varrho$  as zero of  $F_j(s)$ . Moreover,  $\varrho_j$  denotes a non-trivial zero of  $F_j(s)$  in the half-plane  $\sigma \geq \frac{1}{2}$ .

**Lemma 7.1.** ([25]) *Let  $F_1, \dots, F_N \in \mathcal{S}$  be as in Theorem 7.8. Then there exist infinitely many  $N$ -tuples  $(\varrho_1, \dots, \varrho_N)$  such that the vectors  $\mathbf{m}(\varrho_1), \dots, \mathbf{m}(\varrho_N)$  form a basis of  $\mathbb{R}^N$ .*

Very likely, Lemma 7.1 can be made quantitative in the sense that many such  $N$ -tuples  $(\varrho_1, \dots, \varrho_N)$  up to  $T$  can be obtained. Therefore, quantitative versions of Theorem 7.8 are within reach. Accordingly, we raise the following

**Problem 7.3.** *Get a quantitative version of Theorem 7.8, with at least  $\Omega(T)$  singularities.*

The analogous problem with the logarithmic derivatives in place of the logarithms is more delicate, since poles are easier to cancel than logarithmic singularities. In this case we cannot even expect as general results as before. In fact, let  $F_1, \dots, F_N \in \mathcal{S}$ ,  $m_{F_j}$  be the polar order of  $F_j(s)$ ,  $P(z)$  be a polynomial of degree  $k \geq 1$ , and let

$$h(s) = \left( \prod_{j=1}^N (s-1)^{m_{F_j}+1} F_j(s) \right)^k P \left( \frac{F_1'}{F_1}(s), \dots, \frac{F_N'}{F_N}(s) \right);$$

clearly,  $h(s)$  is an entire function. We have

**Theorem 7.9.** ([25]) *Let  $P(z)$  be a polynomial with degree  $> 0$ ,  $F_1, \dots, F_N \in \mathcal{S}$ ,  $k_j, \alpha_j \in \mathbb{N}$  and  $\eta_j \in \mathbb{C}$  for  $j = 1, \dots, N$ . Then*

$$h(s) = P\left(\frac{F_1'}{F_1}(\alpha_1 s + \eta_1)^{(k_1)}, \dots, \frac{F_N'}{F_N}(\alpha_N s + \eta_N)^{(k_N)}\right)$$

*is either constant or has infinitely many poles.*

Note that Theorem 7.9 is weaker than the above stated corollary of Theorem 7.8, in the sense that the function  $h(s)$  is of less general type and there is no non-trivial lower bound for the real part of the singularities. The proof of Theorem 7.9 is based on a Mellin transform argument. Similarly to Problem 7.3, one may ask for explicit lower bounds for the number of poles up to  $T$  in Theorem 7.9.

Clearly, the functional independence of the zeros is closely related, at least *morally*, to the distinct zeros; we therefore raise the following

**Problem 7.4.** *Are there direct implications between the results on the functional independence and on the distinct zeros?*

We finally turn to a very strong form of independence of the zeros, namely the **pair correlation**. Following Montgomery [34], given  $F, G \in \mathcal{S}$  and  $\alpha \in \mathbb{R}$ , Murty-Perelli [39] defined the (*asymmetric and normalized*) *pair correlation function*

$$\mathcal{F}(\alpha; F, \overline{G}) = \frac{\pi}{d_F T \log T} \sum_{|\gamma_F|, |\gamma_G| \leq T} T^{i\alpha d_F(\gamma_F - \gamma_G)} w(\gamma_F - \gamma_G),$$

where  $\gamma_F, \gamma_G$  are the imaginary parts of the non-trivial zeros of  $F(s), G(s)$  and  $w(u) = 4/(4 + u^2)$ , and studied the behaviour of  $\mathcal{F}(\alpha; F, \overline{G})$  under GRH. Writing

$$A_F(n, x) = \begin{cases} A_F(n) \left(\frac{n}{x}\right)^{1/2} & \text{if } n \leq x \\ A_F(n) \left(\frac{x}{n}\right)^{3/2} & \text{if } n > x \end{cases} \quad \text{and} \quad \Psi_{F \times \overline{G}}(x) = \sum_{n=1}^{\infty} A_F(n, x) \overline{A_G(n, x)}$$

they obtained

**Proposition 7.1.** ([39]) *Assume GRH and let  $F, G \in \mathcal{S}$ ,  $\varepsilon > 0$  and  $X = T^{ad_F}$ . Then, uniformly for  $0 \leq \alpha d_F \leq 1 - \varepsilon$ , as  $T \rightarrow \infty$  we have*

$$\mathcal{F}(\alpha; F, \overline{G}) = \frac{1}{d_F X \log T} \Psi_{F \times \overline{G}}(X) + (1 + o(1)) d_G T^{-2ad_F} \log T + o(1).$$

We may consider only  $\alpha \geq 0$  since  $\mathcal{F}(-\alpha; F, \overline{G}) = \overline{\mathcal{F}(\alpha; F, \overline{G})}$ . Note that  $\Psi_{F \times \overline{G}}(x)$  is related by partial summation to

$$\psi_{F \times \overline{G}}(x) = \sum_{n \leq x} A_F(n) \overline{A_G(n)},$$

for which SOC suggests the asymptotic formula

$$(7.6) \quad \psi_{F \times \overline{G}}(x) = (\delta_{F, G} + o(1)) x \log x,$$

provided  $F(s)$  and  $G(s)$  are primitive. Therefore, assuming GRH and (7.6), Proposition 7.1 yields

$$\mathcal{F}(\alpha; F, \overline{G}) = \delta_{F, G} \alpha + (1 + o(1)) d_G T^{-2ad_F} \log T + o(1)$$

uniformly for  $0 \leq \alpha d_F \leq 1 - \varepsilon$ . This is the analog of the Theorem in Montgomery [34], and still in analogy with [34], the following conjecture has been stated in [39].

**Conjecture 7.3.** (pair correlation conjecture, PC) *Let  $F, G \in \mathcal{S}$  be primitive. Then*

$$\mathcal{F}(\alpha; F, \overline{G}) = \begin{cases} \delta_{F, G} |\alpha| + (1 + o(1)) d_G T^{-2|\alpha|d_F} \log T + o(1) & \text{if } |\alpha| \leq 1 \\ \delta_{F, G} + o(1) & \text{if } |\alpha| \geq 1 \end{cases}$$

as  $T \rightarrow \infty$ , uniformly for  $\alpha$  in any bounded interval.

Clearly, Conjecture 7.3 can be suitably modified if  $F, G \in \mathcal{S}$  are not primitive, see (3.5) of [39]. As already remarked above, Conjecture 7.3 is a very strong independence statement. For example, by convolution with suitable kernels (see [34]) PC yields

**Theorem 7.10.** ([39]) *PC implies Conjectures 7.1 and 7.2. Moreover, if the asymptotic formula of PC holds for some  $\alpha_0 > 0$ , then the unique factorization conjecture follows.*

We remark that the UF conjecture is deduced in Theorem 7.10 using the properties of the pair correlation function  $\mathcal{F}(\alpha; F, \overline{G})$  in a direct way. However, it is

quite clear from Proposition 7.1 that PC has implications on the following version of SOC

$$(7.7) \quad \Psi_{F \times \overline{G}}(x) = (\delta_{F,G} + o(1)) x \log x .$$

Precisely, (7.7) follows from GRH and

$$(7.8) \quad \mathcal{F}(\alpha_0; F, \overline{G}) = \delta_{F,G} \alpha_0 + o(1)$$

for a suitably small  $0 < \alpha_0 = \alpha_0(F, G)$ . But (7.7) can be used as a substitute of SOC in results like Theorems 3.6 and 3.7. Hence, in turn, GRH and (7.8) imply results of such type, in particular the Artin conjecture.

The pair correlation conjecture may also be formulated, more explicitly, in terms of the differences of the imaginary parts of the zeros, as in (12) of [34]. In that way one sees that Hypothesis  $H_0$  is in fact a weak version of PC, dealing only with the behaviour at  $\alpha = 0$ .

The above treatment of PC follows Montgomery's [34] approach, thus assumes GRH. However, Rudnik-Sarnak [46], [47] investigated suitably weighted versions of the  $n$ -level correlation of the zeros of automorphic  $L$ -functions (assuming a mild form of the Ramanujan conjecture when the degree is  $> 4$ ) without assuming the Riemann Hypothesis for such  $L$ -functions, thus getting interesting general results. This point of view has been carried over by Murty-Zaharescu [40] to the framework of the Selberg class.

To this end, Murty-Zaharescu [40] defined the pair correlation function in the following slightly modified way (here  $w(z) = \frac{4}{4-z^2}$ )

$$\tilde{\mathcal{F}}(\alpha; F, G) = \frac{\pi}{d_F T \log T} \sum_{|\gamma_F|, |\gamma_G| \leq T} T^{ad_F(\varrho_F + \varrho_G - 1)} w(\varrho_F + \varrho_G - 1),$$

so that  $\tilde{\mathcal{F}}(\alpha; F, \overline{G}) = \mathcal{F}(\alpha; F, \overline{G})$  under GRH. Observing that an off-line zero  $\varrho = \beta + i\gamma$  of a primitive  $F \in \mathcal{S}$  gives rise to a term of order  $T^{ad_F(2\beta-1)}/T \log T$  in  $\tilde{\mathcal{F}}(\alpha; F, \overline{F})$ , and that such a term tends to infinity as  $T \rightarrow \infty$  provided  $\alpha$  is suitably large, it is clear that Conjecture 7.3, with  $\tilde{\mathcal{F}}(\alpha; F, \overline{G})$  in place of  $\mathcal{F}(\alpha; F, \overline{G})$ , *morally* (at least) assumes GRH. In order to have a GRH-free form of PC, in [40] the following conjecture has been formulated.

**Conjecture 7.4.** (weak pair correlation conjecture, WPC) *Let  $F, G \in \mathcal{S}$  be primitive. Then there exists a constant  $c_{F,G} > 0$  such that for any  $0 < \alpha < c_{F,G}$*

$$\tilde{\mathcal{F}}(\alpha; F, \bar{G}) = \delta_{F,G} \alpha + o(1) \quad \text{as } T \rightarrow \infty.$$

As pointed out in [40], results of this type are indeed *proved* in Rudnik-Sarnak [47] in the framework of automorphic  $L$ -functions. Moreover, it is clear from the second statement of Theorem 7.10 and from the discussion after it that WPC has interesting consequences. In fact, the following unconditional version of Proposition 7.1 holds (recall that  $\vartheta_F$  appears in axiom (v) of the Selberg class).

**Proposition 7.2.** ([40]) *Let  $F, G \in \mathcal{S}$ ,  $\varepsilon > 0$  and  $X = T^{ad_F}$ . Then for  $\varepsilon \leq ad_F \leq 1 - \varepsilon$ , as  $T \rightarrow \infty$  we have*

$$\tilde{\mathcal{F}}(\alpha; F, \bar{G}) = \frac{1}{d_F X \log T} \Psi_{F \times \bar{G}}(X) + O(T^{-\delta}),$$

where  $\delta = \varepsilon \min\left(\frac{1}{2}, 1 - \vartheta_F - \vartheta_G\right)$ .

In view of the discussion after Theorem 7.10, WPC implies (7.7) and hence

**Corollary 7.2.** ([40]) *WPC implies UF, the Artin conjecture and the Langlands reciprocity conjecture for solvable extensions of  $\mathbb{Q}$ .*

It is natural to raise at this point the following

**Problem 7.5.** *Prove that some form of PC implies SOC.*

The proof of Proposition 7.2 is based on a suitable version of Landau's formula, see Proposition 1 of [40]. Such a technique allows to introduce weight functions, thus giving more general explicit formulae for the pair correlation of the zeros of functions in  $\mathcal{S}$ , of type

$$(7.9) \quad \sum_{|\gamma_F|, |\gamma_G| \leq T} f(\varrho_F + \varrho_G) = \sum_{n=1}^{\infty} A_F(n) A_G(n) g(n) + \text{error},$$

where  $g(u)$  is a suitable weight function and  $f(s)$  is its Mellin transform (see Theorems 3 and 4 and Corollaries 1 and 2 in Murty-Zaharescu [40]).

In Section 6 we already met an extension to  $\mathcal{S}^\sharp$  of the classical Rankin-Selberg convolution, namely  $F \times \bar{F}(s)$ . Of course, the same type of convolution can be defi-

ned more generally for two functions  $F, G \in \mathcal{S}^\sharp$  as

$$(7.10) \quad F \times G(s) = \sum_{n=1}^{\infty} a_F(n) a_G(n) n^{-s}.$$

Another type of extension to  $\mathcal{S}$  of the Rankin-Selberg convolution appears in Narayanan [41], and is defined as follows (see also [40]). For  $F, G \in \mathcal{S}$  define

$$(7.11) \quad F \otimes G(s) = \prod_p (F \otimes G)_p(s),$$

where for all but finitely many primes  $p$

$$\log (F \otimes G)_p(s) = \sum_{m=1}^{\infty} \frac{mb_F(p^m) b_G(p^m)}{p^{ms}}.$$

Note that in view of the multiplicity one property of  $\mathcal{S}$  (see Theorem 3.4) the definition of the Euler factors at a finite number of primes is not critical. Note also that the convolutions (7.10) and (7.11) are closely related, and that (7.11) is expected to have better analytic properties (see Chapter II of Moroz [36]).

Assuming that  $F \otimes G \in \mathcal{S}$ , from (7.11) we see that (apart from  $n = p^m$  with  $p$  in a finite set)

$$(7.12) \quad \Lambda_F(n) \Lambda_G(n) = \Lambda(n) \Lambda_{F \otimes G}(n),$$

where  $\Lambda(n)$  denotes the classical von Mangoldt function associated with  $\zeta(s)$ . Hence the explicit formulae of type (7.9) and Proposition 7.2 imply

**Corollary 7.3.** ([40]) *Let  $F, G \in \mathcal{S}$  be such that  $F \otimes G \in \mathcal{S}$ . Then for  $f(s)$  as in (7.9)*

$$\sum_{|\gamma_F|, |\gamma_G| \leq T} f(\varrho_F + \varrho_G) = \sum_{|\gamma|, |\gamma_{F \otimes G}| \leq T} f(\varrho + \varrho_{F \otimes G}) + \text{error},$$

where  $\varrho$  and  $\varrho_{F \otimes G}$  denote the non-trivial zeros of  $\zeta(s)$  and  $F \otimes G(s)$ . Moreover, for  $\varepsilon > 0$  and  $\varepsilon \leq ad_F \leq 1 - \varepsilon$ , as  $T \rightarrow \infty$

$$\tilde{\mathcal{F}}(\alpha; F, \bar{G}) = \tilde{\mathcal{F}}(\alpha; \zeta, F \otimes \bar{G}) + O(T^{-\delta}),$$

where  $\delta = \varepsilon \min\left(\frac{1}{2}, 1 - \vartheta_F - \vartheta_G\right)$ .

Corollary 7.3 reflects once again the *universality* of the pair correlation of the  $L$ -functions, as well as the expectation that  $F, G \in \mathcal{S}$  are coprime if and only if the Rankin-Selberg convolution  $F \otimes \bar{G}(s)$  is holomorphic at  $s = 1$  (or, equivalently, if

and only if  $F \otimes \overline{G}(s)$  is coprime with  $\zeta(s)$ ). Another consequence of Proposition 7.2 pointed out in [40] is that, in analogy to Rudnik-Sarnak [47], if  $F, G \in \mathcal{S}$  are primitive and  $F \otimes \overline{G}(s)$  has nice analytic properties, then WPC holds for  $\tilde{\mathcal{F}}(\alpha; F, \overline{G})$ . In fact, by standard arguments in prime number theory one gets, via (7.12), the expected asymptotic formula (7.7) from mild information on the analytic continuation, polar structure at  $s = 1$  and zero-free regions of  $F \otimes \overline{G}(s)$ .

We finally state a natural problem on the extension of the pair correlation problems (see Section 7 of [40]).

**Problem 7.6.** *Investigate the  $N$ -level correlation of the zeros of functions in  $\mathcal{S}$  (or the correlation of the zeros of  $N$ -tuples of functions in  $\mathcal{S}$ ).*

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#### Abstract

*This is the second part of a survey of the Selberg class  $S$  of  $L$ -functions. This part contains the following topics: linear and non-linear twists of the  $L$ -functions in  $S$ ; classification of the  $L$ -functions with degree between 1 and 2; independence problems for the  $L$ -functions in  $S$ .*

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