L-Functions of Siegel Modular Forms, Their Families and Lifting Conjectures

Alexei PANCHISHKIN
Institut Fourier, Université Grenoble-1
B.P.74, 38402 St.–Martin d'Hères, FRANCE

A talk for the Conference "Modulformen"
October 29 to November 2, 2007 (Oberwolfach, Germany)

October 2007

Introduction

1) *L*- functions of Siegel modular forms
2) Motivic *L*-functions for Sp_g, and their analytic properties
3) Critical values and *p*-adic constructions for the spinor *L*-functions
4) A lifting from $GSp_{2m} \times GSp_{2m}$ to GSp_{4m} (of genus $g = 4m$)
5) Constructions of *p*-adic families of Siegel modular forms

Let $\Gamma = \mathrm{Sp}_g(\mathbb{Z}) \subset \mathrm{SL}_{2g}(\mathbb{Z})$ be the Siegel modular group of genus g. Let p be a prime, $\mathbf{T}(p) = \mathbf{T}(\underbrace{1, \cdots, 1}_{g}, \underbrace{p, \cdots, p}_{g})$ the Hecke p-operator, and

$[\mathbf{p}]_g = p\mathbf{I}_{2g} = \mathbf{T}(\underbrace{p, \cdots, p}_{2g})$ the scalar Hecke operator for Sp_g.

The Fourier expansion of a Siegel modular form.

Let $f = \sum_{\mathcal{T} \in B_n} a(\mathcal{T}) q^{\mathcal{T}} \in \mathcal{M}_k^n$ be a Siegel modular form of weight k and of genus n on the Siegel upper-half plane $\mathbb{H}_n = \{z \in \mathrm{M}_n(\mathbb{C}) \mid \mathrm{Im}(z) > 0\}$.

The formal *Fourier expansion* of f uses the symbol

$$q^{\mathcal{T}} = \exp(2\pi i \,\mathrm{tr}(\mathcal{T} z)) = \prod_{i=1}^{n} q_{ii}^{\mathcal{T}_{ii}} \prod_{i<j} q_{ij}^{2\mathcal{T}_{ij}}$$

$\in \mathbb{C}[\![q_{11}, \ldots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,m}$, where $q_{ij} = \exp(2\pi(\sqrt{-1} z_{i,j}))$, and $\mathcal{T}$ is in the semi-group $B_n = \{\mathcal{T} = {}^t\mathcal{T} \geq 0 | \mathcal{T} \text{ half-integral}\}$.

Satake parameters of an eigenfunction of Hecke operators

Suppose that $f \in \mathcal{M}_k^n$ an eigenfunction of all *Hecke operators* $f \longmapsto f|T$, $T \in \mathcal{L}_{n,p}$ for all primes p, hence $f|T = \lambda_f(T)f$.
Then all the numbers $\lambda_f(T) \in \mathbb{C}$ define a homomorphism $\lambda_f : \mathcal{L}_{n,p} \longrightarrow \mathbb{C}$ given by a $(n+1)$-tuple of complex numbers $(\alpha_0, \alpha_1, \cdots, \alpha_n) = (\mathbb{C}^\times)^{n+1}$ (the Satake parameters of f): $\lambda_f(T) = \mathrm{Satake}(T)$ for $x_0 := \alpha_0, \ldots, x_n := \alpha_n$, where $\mathrm{Satake} : \mathcal{L}_{p,n} \to \mathbb{Q}[x_0^{\pm}, x_1^{\pm}, \ldots, x_n^{\pm}]^{W_n}$ is the Satake isomorphism
Recall that

$$\alpha_0^2 \alpha_1 \cdots \alpha_n = p^{kn - n(n+1)/2}.$$

L-functions, functional equation and motives for Sp_n (see [Pa94], [Yosh01])

One defines

- $Q_{f,p}(X) = (1-\alpha_0 X)\prod_{r=1}^{n}\prod_{1\le i_1<\cdots<i_r\le n}(1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_r}X)$,
- $R_{f,p}(X) = (1-X)\prod_{i=1}^{n}(1-\alpha_i^{-1}X)(1-\alpha_i X) \in \mathbb{Q}[\alpha_0^{\pm 1},\cdots,\alpha_n^{\pm 1}][X]$.

Then the spinor L-function $L(Sp(f),s)$ and the standard L-function $L(St(f),s,\chi)$ of f (for $s\in\mathbb{C}$, and for all Dirichlet characters) χ are defined as the Euler products

- $L(Sp(f),s,\chi) = \prod_p Q_{f,p}(\chi(p)p^{-s})^{-1}$
- $L(St(f),s,\chi) = \prod_p R_{f,p}(\chi(p)p^{-s})^{-1}$

Relations with L-functions and motives for Sp_n

Following [Pa94] and [Yosh01], these functions are conjectured to be motivic for all $k > n$:

$$L(Sp(f), s, \chi) = L(M(Sp(f))(\chi), s), L(St(f), s) = L(M(St(f))(\chi), s), \text{ where}$$

and the motives $M(Sp(f))$ and $M(St(f))$ are *pure* if f is a genuine cusp form (not coming from a lifting of a smaller genus):

- $M(Sp(f))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_f(n))_{n\in\mathbb{N}}$ of rank 2^n, of weight $w = kn - n(n+1)/2$, and of Hodge type $\oplus_{p,q} H^{p,q}$, with

$$\begin{aligned} p &= (k - i_1) + (k - i_2) + \cdots + (k - i_r), \\ q &= (k - j_1) + (k - j_2) + \cdots + (k - i_s), \text{ where } r + s = n, \\ 1 &\le i_1 < i_2 < \cdots < i_r \le n, 1 \le j_1 < j_2 < \cdots < j_s \le n, \\ \{i_1&, \cdots, i_r\} \cup \{j_1, \cdots, i_s\} = \{1, 2, \cdots, n\}; \end{aligned} \tag{3.1}$$

- $M(St(f))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_f(n))_{n\in\mathbb{N}}$ of rank $2n+1$, of weight $w = 0$, and of Hodge type $H^{0,0} \oplus_{i=1}^{n} (H^{-k+i,k-i} \oplus H^{k-i,-k+i})$.

A functional equation

Following general Deligne's conjecture [De79] on the motivic L-functions, the L-function satisfy a functional equation determined by the Hodge structure of a motive:

$$\Lambda(Sp(f), kn - n(n+1)/2 + 1 - s) = \varepsilon(f)\Lambda(Sp(f), s), \text{ where}$$

$\Lambda(Sp(f), s) = \Gamma_{n,k}(s)L(Sp(f), s), \varepsilon(f) = (-1)^{k2^{n-2}}$,

$\Gamma_{1,k}(s) = \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, $\Gamma_{2,k}(s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+2)$, and
$\Gamma_{n,k}(s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h_{p,q}} \Gamma_{\mathbb{R}}(s-(w/2))^{a_+}\Gamma_{\mathbb{R}}(s+1-(w/2))^{a_-}$ for some non-negative integers a_+ and a_-, with $a_+ + a_- = h_{w/2,w/2}$, and
$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, where $h_{p,q} = \dim_{\mathbb{C}} H^{p,q}$.
In particular, for $n = 3$ and $k \geq 5$, $\Lambda(Sp(f), s) = \Lambda(Sp(f), 3k - 5 - s)$,

$$\Lambda(Sp(f), s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+3)\Gamma_{\mathbb{C}}(s-k+2)\Gamma_{\mathbb{C}}(s-k+1)L(Sp(f), s).$$

For $k \geq 5$ the critical values in the sense of Deligne [De79] are :

$$s = k, \cdots, 2k - 5.$$

A study of the analytic properties of $L(Sp(f), s)$

(compare with [Vo]).
One could try to use a link between the eigenvalues $\lambda_f(T)$ and the Fourier coefficients $a_f(\mathcal{T})$, where $T \in \mathcal{D}(\Gamma, S)$ runs through the Hecke operators, and $\mathcal{T} \in B_n$ runs over half-integral symmetric matrices.

$$D(X) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) X^\delta = \frac{E(X)}{F(X)},$$

where

$$\begin{aligned} &E(X) \\ &= 1 - p^2 \left(\mathbf{T}_2(p^2) + (p^2 - p + 1)(p^2 + p + 1)[\mathbf{p}]_3 \right) X^2 + (p+1)p^4 \mathbf{T}(p)[\mathbf{p}]_3 X^3 \\ &- p^7 [\mathbf{p}]_3 \left(\mathbf{T}_2(p^2) + (p^2 - p + 1)(p^2 + p + 1)[\mathbf{p}]_3 \right) X^4 + p^{15} [\mathbf{p}]_3^3 X^6 \ \in \mathcal{L}_{\mathbb{Z}}[X]. \end{aligned}$$

Computing a formal Dirichlet series

Knowing $E(X)$, one computes the following formal Dirichlet series

$$D_E(s) = \sum_{h=1}^{\infty} \mathbf{T}_E(h) h^{-s} = \prod_p D_{E,p}(p^{-s}), \text{ where}$$

$$D_{E,p}(X) = \sum_{\delta=0}^{\infty} \mathbf{T}_E(p^\delta) X^\delta = \frac{D_p(X)}{E(X)} = \frac{1}{F(X)} \in \mathcal{D}(\Gamma, S)[\![X]\!].$$

Hence, $L(Sp(f), s) = \sum_{h=1}^{\infty} \lambda_f(\mathbf{T}_E)(h) h^{-s}$.

Main identity

For all $\mathcal{T}$ one obtains the following identity

$$a_f(\mathcal{T})L(Sp(f),s) = \sum_{h=1}^{\infty} a_f(\mathcal{T},E,h)h^{-s}, \text{ where}$$

$$f|\mathbf{T}_E(h) = \sum_{\mathcal{T}\in B_n} a_f(\mathcal{T},E,h)q^{\mathcal{T}}.$$

Such an identity is an unavoidable step in the problem of analytic continuation of $L(Sp(f),s)$ and in the study of its arithmetical implications: one obtains a mean of computing special values out of Fourier coefficients. We hope also to use the Jacobi forms for the study of L-functions $L(Sp(f),s)$.
For the standard $L(St(f),s)$, (see [Pa94], [CourPa], and [Boe-Schm]).

Critical values, periods and p-adic L-functions for Sp_3

A general conjecture by Coates, Perrin-Riou (see [Co-PeRi], [Co], [Pa94]), predicts that for $n = 3$ and $k > 5$, the motivic function $L(Sp(f), s)$, admits a p-adic analogue.

- The known conjectural condition for the existence of bounded p-adic L-functions in this case takes the form $\mathrm{ord}_p(\alpha_0(p)) = 0$.
  Recall : this condition says that *for a motive M of rank d, Newton p-Polygone at $(d/2)$=Hodge Polygone at $(d/2)$*
- Otherwise, one obtains p-adic L-functions of logarithmic growth $o(\log^h(\cdot))$ with $h = [2\mathrm{ord}_p(\alpha_0(p))] + 1$, $2\mathrm{ord}_p(\alpha_0(p))$ =the difference

  Newton p-Polygone at $(d/2)$–Hodge Polygone at $(d/2)$

For the unitary groups, this condition for the existence of p-adic L-functions was discussed in [Ha-Li-Sk].

Motive of the Rankin product of genus $n = 2$

Let f and g be two Siegel cusp eigenforms of weights k and l, $k > l$, and let $M(Sp(f))$ and $M(Sp(g))$ be the spinor motives of f and g. Then $M(Sp(f))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_f(n))_{n\in\mathbb{N}}$ of rank 4, of weight $w = 2k-3$, and of Hodge type $H^{0,2k-3} \oplus H^{k-2,k-1} \oplus H^{k-1,k-2} \oplus H^{2k-3,0}$, and $M(Sp(g))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_g(n))_{n\in\mathbb{N}}$ of rank 4, of weight $w = 2l-3$, and of Hodge type $H^{0,2l-3} \oplus H^{l-2,l-1} \oplus H^{l-1,l-2} \oplus H^{2l-3,0}$. The tensor product $M(Sp(f)) \otimes M(Sp(g))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_f(n), \lambda_g(n))_{n\in\mathbb{N}}$ of rank 16, of weight $w = 2k + 2l - 6$, and of Hodge type

$$
\begin{aligned}
&H^{0,2k+2l-6} \oplus H^{l-2,2k+l-4} \oplus H^{l-1,2k+l-5} \oplus H^{2l-3,2k-3} \\
&H^{k-2,k+2l-4} \oplus H^{k+l-4,k+l-2} \oplus H_{+}^{k+l-3,k+l-3} \oplus H^{k+2l-5,k-1} \\
&H^{k-1,k+2l-5} \oplus H_{-}^{k+l-3,k+l-3} \oplus H^{k+l-2,k+l-4} \oplus H^{k+2l-4,k-2} \\
&H^{2k-3,2l-3} \oplus H^{2k+l-5,l-1} \oplus H^{2k+l-4,l-2} \oplus H^{2k+2l-6,0}.
\end{aligned}
$$

Motivic L-functions: analytic properties

Following Deligne's conjecture [De79] on motivic L-functions, applied for a Siegel cusp eigenform F for the Siegel modular group $\mathrm{Sp}_4(\mathbb{Z})$ of genus $n = 4$ and of weight $k > 5$, one has $\Lambda(Sp(F), s) = \Lambda(Sp(F), 4k - 9 - s)$, where

$$\Lambda(Sp(F), s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+4)\Gamma_{\mathbb{C}}(s-k+3)\Gamma_{\mathbb{C}}(s-k+2)\Gamma_{\mathbb{C}}(s-k+1)$$
$$\times \Gamma_{\mathbb{C}}(s-2k+7)\Gamma_{\mathbb{C}}(s-2k+6)\Gamma_{\mathbb{C}}(s-2k+5)L(Sp(F), s),$$

(compare this functional equation with that given in [An74], p.115).

On the other hand, for $m = 2$ and for two cusp eigenforms f and g for $\mathrm{Sp}_2(\mathbb{Z})$ of weights k, l, $k > l + 1$, $\Lambda(Sp(f) \otimes Sp(g), s) = \varepsilon(f, g)\Lambda(Sp(f) \otimes Sp(g), 2k + 2l - 5 - s)$, $|\varepsilon(f, g)| = 1$, where

$$\Lambda(Sp(f) \otimes Sp(g), s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-l+2)\Gamma_{\mathbb{C}}(s-l+1)\Gamma_{\mathbb{C}}(s-k+2)$$
$$\times \Gamma_{\mathbb{C}}(s-k+1)\Gamma_{\mathbb{C}}(s-2l+3)\Gamma_{\mathbb{C}}(s-k-l+2)\Gamma_{\mathbb{C}}(s-k-l+3)$$
$$\times L(Sp(f) \otimes Sp(g), s).$$

We used here the Gauss duplication formula $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$.
Notice that $a_+ = a_- = 1$ in this case, and the conjectural motive $M(Sp(f)) \otimes M(Sp(g))$ does not admit critical values.

A holomorphic lifting from $GSp_{2m} \times GSp_{2m}$ to GSp_{4m}: a conjecture

Conjecture (on a lifting from $GSp_{2m} \times GSp_{2m}$ to GSp_{4m})

Let f and g be two Siegel modular forms of genus $2m$ and of weights $k > 2m$ and $l = k - 2m$. Then there exists a Siegel modular form F of genus $4m$ and of weight k with the Satake parameters $\gamma_0 = \alpha_0\beta_0, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \cdots, \gamma_{2m} = \alpha_{2m}, \gamma_{2m+1} = \beta_1, \cdots, \gamma_{4m} = \beta_{2m}$ for suitable choices $\alpha_0, \alpha_1, \cdots, \alpha_{2m}$ and $\beta_0, \beta_1, \cdots, \beta_{2m}$ of Satake's parameters of f and g. One readily checks that the Hodge types of $M(Sp(f)) \otimes M(Sp(g))$ and $M(Sp(F))$ are the same (of rank 2^{4m}) (it follows from the above description (3.1), and from Künneth's-type formulas).

An evidence for this version of the conjecture comes from Ikeda-Miyawaki constructions ([Ike01], [Ike06], [Mur02]): let k be an even positive integer, $h \in S_{2k}(\Gamma_1)$ a normalized Hecke eigenform of weight $2k$, $g := F_{2n}(h) \in S_{k+n}(\Gamma_{2n})$ the *Ikeda–Duke-Imamoglu lift* of h of genus $2n$ (we assume $k \equiv n \bmod 2$, $n \in \mathbb{N}$).

Next let $f \in S_{k+n+r}(\Gamma_r)$ be an arbitrary Siegel cusp eigenform of genus r and weight $k+n+r$, with $n, r \geq 1$.
Let us consider the lift $F_{2n+2r}(h) \in S_{k+n+r}(\Gamma_{2n+2r})$, and the integral

$$\mathcal{F}_{h,f} = \int_{\Gamma_r \backslash \mathbb{H}_r} F_{2n+2r}(h)\left(\begin{pmatrix} z & \\ & z' \end{pmatrix}\right) f(z')(\det \mathrm{Im}(z')^{k+n+r-1} dz' \in S_{k+n+r}(\Gamma_{2n+r})$$

which defines the *Ikeda–Miyawaki lift*, where $z \in \mathbb{H}_{2n+r}, z' \in \mathbb{H}_r$.
Under the non-vanishing condition on $\mathcal{F}_{h,f}$, one has the equality

$$L(s, \mathcal{F}_{h,f}, St) = L(s, f, St) \prod_{i=1}^{2n} L(s+k+n+r-i, h)$$

If we take $n = m, r = 2m$, $k+n+r := k+3m$, then an example of the validity of this version of the conjecture is given by

$$(f, g) = (f, F_{2m}(h)) \mapsto \mathcal{F}_{h,f} \in S_{k+3m}(\Gamma_{4m}),$$
$$(f, g) = (f, F_{2m}(h)) \in S_{k+3m}(\Gamma_{2m}) \times S_{k+m}(\Gamma_{2m}).$$

Another evidence comes from Siegel-Eiseinstein series

$$f = E_k^{2m} \text{ and } g = E_{k-2m}^{2m}$$

of even genus $2m$ and weights k and $k-2m$: we have then

$$\alpha_0 = 1, \alpha_1 = p^{k-2m}, \cdots, \alpha_{2m} = p^{k-1},$$
$$\beta_0 = 1, \beta_1 = p^{k-4m}, \cdots, \beta_{2m} = p^{k-2m-1},$$

then we have that

$$\gamma_0 = 1, \gamma_1 = p^{k-4m}, \cdots, \gamma_{2m} = p^{k-1},$$

are the Satake parameters of the Siegel-Eisenstein series $F = E_k^{4m}$.

Also, the compatibility of the conjecture with the formation of Klingen-Eisenstein series was checked after a discussion with Prof. Tetsushi ITO (Kyoto University) in Luminy in July 2007.

Remark

If we compare the L-function of the conjecture (given by the Satake parameters $\gamma_0 = \alpha_0\beta_0, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \cdots, \gamma_{2m} = \alpha_{2m}, \gamma_{2m+1} = \beta_1, \cdots, \gamma_{4m} = \beta_{2m}$ for suitable choices $\alpha_0, \alpha_1, \cdots, \alpha_{2m}$ and $\beta_0, \beta_1, \cdots, \beta_{2m}$ of Satake's parameters of f and g), we see that it corresponds to the tensor product of spinor L-functions, and is not of the same type as that of the Yoshida's lifting [Yosh81], which is a certain product of Hecke's L-functions.

We would like to mention in this context Langlands's functoriality: The denominators of our L-series belong to local Langlands L-factors (attached to representations of L-groups). If we consider the homomorphisms

$$^L GSp_{2m} = GSpin(4m+1) \to GL_{2^{2m}}, \quad ^L GSp_{4m} = GSpin(8m+1) \to GL_{2^{4m}},$$

we see that our conjecture is compatible with the homomorphism of L-groups

$$GL_{2^{2m}} \times GL_{2^{2m}} \to GL_{2^{4m}}, \ (g_1, g_2) \mapsto g_1 \otimes g_2, GL_n(\mathbb{C}) = {}^L GL_n.$$

However, it is unclear to us if Langlands's functoriality predicts a holomorphic Siegel modular form as a lift.

Constructing p-adic L-functions

Together with complex parameter s it is possible to use certain p-adic parameters in order to study the L-functions. We can use such parameters as the twist with Dirichlet character on one hand, and the weight parameter in the theory of families of modular forms on the other.
In several cases one can compute the values of the automorphic L-functions $L(s, \pi \otimes \chi, r)$ related to an automorphic representation π of an algebraic group G over a number field, and the twists with Dirichlet character χ.
A usual tool is to represent these special values as integrals giving both complex-analytic and p-adic-analytic continuation.
In this way one can treat the cases $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$, $G = \mathrm{GL}_2 \times \mathrm{GSp}_{2m}$, and probably $G = \mathrm{GSp}_{2m} \times \mathrm{GSp}_{2m}$ using the doubling method and its p-adic versions. Computations involve differential operators acting on modular forms.

A p-adic approach

Consider Tate's field $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ for prime number p. Let us fix the embedding $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \mathbb{C}_p$ and consider the algebraic numbers as numbers p-adic over i_p. For p-adic family $k \mapsto f_k = \sum_{n=1}^{\infty} a_n(k)q^n \in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!]$ Fourier coefficients $a_n(k)$ of f_k and one of Satake's p-parameters $\alpha(k) := \alpha_p^{(1)}(k)$ are given by the certain p-adic analytic functions $k \mapsto a_n(k)$ for $(n,p) = 1$. A typical example of a p-adic family is given by the Eisenstein series.

$$a_n(k) = \sum_{d|n,(d,p)=1} d^{k-1}, f_k = E_k, \alpha_p^{(1)}(k) = 1, \alpha_p^{(2)}(k) = p^{k-1}.$$

The existence of the p-adic family of cusp forms of positive slope $\sigma > 0$ was shown by Coleman. We define *the slope* $\sigma = \mathrm{ord}_p(\alpha_p^{(1)}(k))$ (and ask it to be constant in a p-adic neighborhood of weight k). An example for $p = 7$, $f = \Delta$, $k = 12$, $a_7 = \tau(7) = -7 \cdot 2392$, $\sigma = 1$ is given by R. Coleman in [CoPB].

Motivation for considering of p-adic families

comes from Birch and Swinnerton-Dyer's conjecture, see [Colm03]. For a cusp eigenform $f = f_2$, corresponding to an elliptic curve E by Wiles [Wi95], we consider a family containing f. One can try to approach $k = 2, s = 1$ from the direction, taking $k \to 2$, instead of $s \to 1$, this leads to a formula linking the derivative over s at $s = 1$ of the p-adic L-function with the derivative over k at $k = 2$ of the p-adic analytic function $\alpha_p(k)$, see in [CST98]: $\boxed{L'_{p,f}(1) = \mathfrak{L}_p(f)L_{p,f}(1)}$ with $\mathfrak{L}_p(f) = -2\frac{d\alpha_p(k)}{dk}\Big|_{k=2}$.
The validity of this formula needs the existence of our two variable L-function!
In order to construct p-adic L-function of two variables (k, s), the theory of p-adic integration is used. The H-admissible measures with integer H related to σ, which appear in this construction, are obtained from H-admissible measures with values in various rings of modular forms, in particular nearly holomorphic modular forms.

Nearly holomorphic modular forms and the method of canonical projection

Let $\mathcal{A}$ be a commutative field. There are several p-adic approaches for studying special values of L-functions using the method of canonical projection (see [PaTV]). In this method the special values and modular symbols are considered as $\mathcal{A}$-linear forms over spaces of modular forms with coefficients in $\mathcal{A}$. Nearly holomorphic ([ShiAr]) modular forms are certain formal series

$$g = \sum_{n=0}^{\infty} a(n; R) q^n \in \mathcal{A}[\![q]\!][R]$$

with the property for $\mathcal{A} = \mathbb{C}$, $z = x + iy \in \mathbb{H}$, $R = (4\pi y)^{-1}$, the series converge to a $\mathcal{C}^\infty$-modular form over $\mathbb{H}$ of given weight k and Dirichlet character ψ. The coefficients $a(n; R)$ are polynomials in $\mathcal{A}[R]$ of bounded degree.

Triple products families

give a recent example of families on the algebraic group of higher rank. This aspect of the project is studied by S. Böcherer and A. Panchishkin [Boe-Pa2006]. The triple product with Dirichlet character χ is defined as a complex L-function (Euler product of degree 8) $L(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \nmid N} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p)p^{-s})$, where

$$L((f_1 \otimes f_2 \otimes f_3)_p, X)^{-1} = \det \left(1_8 - X \begin{pmatrix} \alpha^{(1)}_{p,1} & 0 \\ 0 & \alpha^{(2)}_{p,1} \end{pmatrix} \otimes \begin{pmatrix} \alpha^{(1)}_{p,2} & 0 \\ 0 & \alpha^{(2)}_{p,2} \end{pmatrix} \otimes \begin{pmatrix} \alpha^{(1)}_{p,3} & 0 \\ 0 & \alpha^{(2)}_{p,3} \end{pmatrix} \right).$$

We use the corresponding normalized L-function (see [De79], [Co], [Co-PeRi]) : $\Lambda(f_1 \otimes f_2 \otimes f_3, s, \chi) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k_3+1)\Gamma_{\mathbb{C}}(s-k_2+1)\Gamma_{\mathbb{C}}(s-k_1+1)L(f_1 \otimes f_2 \otimes f_3, s, \chi)$, where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The Gamma-factor determines the *critical values* $s = k_1, \dots, k_2 + k_3 - 2$ of $\Lambda(s)$, which we explicitly evaluate (like in the classical formula $\zeta(2) = \frac{\pi^2}{6}$). *A functional equation* of $\Lambda(s)$ has the form: $s \mapsto k_1 + k_2 + k_3 - 2 - s$.
Let us consider the product of three eigenvalues:
$\lambda = \lambda(k_1, k_2, k_3) = \alpha^{(1)}_{p,1}(k_1)\alpha^{(1)}_{p,2}(k_2)\alpha^{(1)}_{p,3}(k_3)$ with the slope
$\sigma = v_p(\lambda(k_1, k_2, k_3)) = \sigma(k_1, k_2, k_3) = \sigma_1 + \sigma_2 + \sigma_3$ constant and positive
for all triplets (k_1, k_2, k_3) in an appropriate p-adic neighbourhood of the fixed triplet of weights (k_1, k_2, k_3).

The statement of the problem

for triple products is following: *given three p-adic analytic families* $\boldsymbol{f}_j$ *of slope* $\sigma_j \geq 0$*, to construct a four-variable p-adic L-function attached to Garrett's triple product of these families*. We show that this function interpolates the special values $(s, k_1, k_2, k_2) \longmapsto \Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, s, \chi)$ at critical points $s = k_1, \ldots, k_2 + k_3 - 2$ for balanced weights $k_1 \leq k_2 + k_3 - 2$; we prove that these values are algebraic numbers after dividing by certain "periods". However the construction uses directly modular forms, and not the L-values in question, and a comparison of special values of two functions is done *after the construction*.

Main result for triple products

1) The function $\mathcal{L}_{\mathbf{f}} : (s, k_1, k_2, k_3) \mapsto \frac{\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\rangle}{\langle \mathbf{f}^0, \mathbf{f}_0\rangle}$ depends p-adic analytically on four variables
$(\chi \cdot y_p^r, k_1, k_2, k_3) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$;
2) Comparison of complex and p-adic values: for all (k_1, k_2, k_3) in an affinoid neighborhood
$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X^3$, satisfying $k_1 \leq k_2 + k_3 - 2$: the values at
$s = k_2 + k_3 - 2 - r$
coincide with the normalized critical special values

$$L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \ldots, k_2 + k_3 - k_1 - 2),$$

for Dirichlet characters χ mod Np^v, $v \geq 1$.
3) Dependence on $x \in X$: let $H = [2\mathrm{ord}_p(\lambda)] + 1$. For any fixed $(k_1, k_2, k_3) \in \mathcal{B}$ and $x = \chi \cdot y_p^r$ then the following linear form (representing modular symbols for triple modular forms)

$$x \longmapsto \frac{\left\langle \mathbf{f}^0, \mathcal{E}(-r, \chi)\right\rangle}{\left\langle \mathbf{f}^0, \mathbf{f}_0\right\rangle},$$

extends to a p-adic analytic function of type $o(\log^H(\cdot))$ of the variable $x \in X$.

A general program

We plan to extend this construction to other sutuations as follows :

1) Construction of modular distributions Φ_j with values in an infinite dimensional modular tower $\mathcal{M}(\psi)$.
2) Application of a canonical projector of type π_α onto a finite dimensional subspace $\mathcal{M}^\alpha(\psi)$ of $\mathcal{M}^\alpha(\psi)$.
3) General admissibility criterium. The family of distributions $\pi_\alpha(\Phi_j)$ with value in $\mathcal{M}^\alpha(\psi)$ give a h-admissible measure $\tilde{\Phi}$ with value in the moduli of finite rank.
4) Application of a linear form ℓ of type of a modular symbol produces distributions $\mu_j = \ell(\pi_\alpha(\Phi_j))$, and an admissible measure from congruences between modular forms $\pi_\alpha(\Phi_j)$.
5) One shows that certain integrals $\mu_j(\chi)$ of the distributions μ_j coincide with certain L-values; however, these integrals are not necessary for the construction of measures (already done at stage 4).
6) One shows a result of uniqueness for the constructed h-admissible measures : they are determined by many of their integrals over Dirichlet characters (not all).
7) In most cases we can prove a functional equation for the constructed measure μ (using the uniqueness in 6), and using a functional equation for the L-values (over complex numbers, computed at stage 5).

This strategy is already applicable in various cases.

Bibliography

- Andrianov, A.N., *Shimura's conjecture for Siegel's modular group of genus 3*, Dokl. Akad. Nauk SSSR 177 (1967), 755-758 = Soviet Math. Dokl. 8 (1967), 1474-1478.

- Andrianov, A.N., *Rationality of multiple Hecke series of a complete linear group and Shimura's hypothesis on Hecke's series of a symplectic group*, Dokl. Akad. Nauk SSSR 183 (1968), 9-11 = Soviet Math. Dokl. 9 (1968), 1295 - 1297. MR

- Andrianov, A.N., *Rationality theorems for Hecke series and zeta functions of the groups GL_n and SP_n over local fields*, Izv. Akad. Nauk SSSR, Ser. Mat., Tom 33 (1969), No. 3, (Math. USSR – Izvestija, Vol. 3 (1969), No. 3, pp. 439–476).

- Andrianov, A.N., *Spherical functions for GL_n over local fields and summation of Hecke series*, Mat. Sbornik, Tom 83 (125) (1970), No 3, (Math. USSR Sbornik, Vol. 12 (1970), No. 3, pp. 429–452).

Andrianov, A.N., *Euler products corresponding to Siegel modular forms of genus 2*, Russian Math. Surveys, 29:3 (1974), pp. 45–116, (Uspekhi Mat. Nauk 29:3 (1974) pp. 43–110).

Andrianov, A.N., *Quadratic Forms and Hecke Operators*, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987.

Andrianov, A.N., Zhuravlev, V.G., *Modular Forms and Hecke Operators*, Translations of Mathematical Monographs, Vol. 145, AMS, Providence, Rhode Island, 1995.

Böcherer, S., Panchishkin, A.A., *Admissible p-adic measures attached to triple products of elliptic cusp forms*, accepted in Documenta Math. in March 2006 (a special volume dedicated to John Coates).

Böcherer, S., and Schmidt, C.-G., *p-adic measures attached to Siegel modular forms*, Ann. Inst. Fourier 50, N°5, 1375-1443 (2000).

Coates, J. *On p–adic L–functions.* Sem. Bourbaki, 40eme annee, 1987-88, n° 701, Asterisque (1989) 177–178.

- Coates, J. and Perrin-Riou, B., *On p-adic L-functions attached to motives over* $\mathbb{Q}$, Advanced Studies in Pure Math. 17, 23–54 (1989)
- R. Coleman, *p-adic Banach spaces and families of modular forms*, Invent. Math. 127, N° 3 (1997), 417–479.
- R. Coleman, G. Stevens, J. Teitelbaum, *Numerical experiments on families of p-adic modular forms*, in Computational perspectives in Number Theory, ed. by D.A. Buell, J.T. Teitelbaum, Amer. Math. Soc. (1998), 143–158.
- P. Colmez *La conjecture de Birch et Swinnerton-Dyer p-adique*, Séminaire Bourbaki, exposé n° 919, juin 2003.
- Courtieu,M., Panchishkin ,A.A., *Non-Archimedean L-Functions and Arithmetical Siegel Modular Forms*, Lecture Notes in Mathematics 1471, Springer-Verlag, 2004 (2nd augmented ed.)
- Deligne P., *Valeurs de fonctions L et périodes d'intégrales*, Proc.Sympos.Pure Math. vol. 55. Amer. Math. Soc., Providence, RI, 1979 , 313-346

Evdokimov, S. A., *Dirichlet series, multiple Andrianov zeta-functions in the theory of Euler modular forms of genus* 3, (Russian) Dokl. Akad. Nauk SSSR 277 (1984), no. 1, 25–29.

Hecke, E., *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktenwickelung, I, II.* Math. Annalen 114 (1937), 1-28, 316-351 (Mathematische Werke. Göttingen: Vandenhoeck und Ruprecht, 1959, 644-707).

Harris, M., Li, Jian-Shu., Skinner, Ch.M., *p-adic L-functions for unitary Shimura varieties.* Preprint, 2006.

Ikeda, T., *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) 154 (2001), 641-681.

Ikeda, T., *Pullback of the lifting of elliptic cusp forms and Miyawaki's Conjecture* Duke Mathematical Journal, **131**, 469-497 (2006)

Kurokawa, Nobushige, *Analyticity of Dirichlet series over prime powers.* Analytic number theory (Tokyo, 1988), 168–177, Lecture Notes in Math., 1434, Springer, Berlin, 1990.

Maass, H. *Indefinite Quadratische Formen und Eulerprodukte.* Comm. on Pure and Appl. Math, 19, 689-699 (1976)

Murokawa, K., *Relations between symmetric power L-functions and spinor L-functions attached to Ikeda lifts*, Kodai Math. J. 25, 61-71 (2002)

Panchishkin, A., *Admissible Non-Archimedean standard zeta functions of Siegel modular forms*, Proceedings of the Joint AMS Summer Conference on Motives, Seattle, July 20–August 2 1991, Seattle, Providence, R.I., 1994, vol.2, 251 – 292

Panchishkin, A.A., *A new method of constructing p-adic L-functions associated with modular forms*, Moscow Mathematical Journal, 2 (2002), Number 2, 1-16

Panchishkin, A.A., *Two variable p-adic L functions attached to eigenfamilies of positive slope*, Invent. Math. v. 154, N3 (2003), pp. 551 - 615

Panchishkin,A.A., *Triple products of Coleman's families and their periods (a joint work with S.Boecherer)* Proceedings of the 8th Hakuba conference "Periods and related topics from automorphic forms", September 25 - October 1, 2005

Panchishkin,A.A., *p-adic Banach modules of arithmetical modular forms and triple products of Coleman's families*, (for a special volume of Quarterly Journal of Pure and Applied Mathematics dedicated to Jean-Pierre Serre), 2006.

Panchishkin, A., Vankov, K. *On the numerator of the symplectic Hecke series of degree three.* Arxiv, math.NT/0604602 (2006).

Rhodes, J.A. and Shemanske, T.R. *Rationality theorems for Hecke operators on* GL_n, J. Number Theory 102 (2003), no. 2, 278-297.

Shimura, G., *On modular correspondences for Sp(n, Z) and their congruence relations*, Proc. Nat. Acad. Sci. U.S.A. 49 (1963), 824-828.

Shimura G., *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press, 1971.

G. Shimura, *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs. 82. Providence, RI: American Mathematical Society (AMS). x, 302 p. (2000)

Tamagawa T., *On the ζ-function of a division algebra*, Ann. of Math. 77 (1963), 387-405

Tilouine, J. and Urban, E. , *Several variable p-adic families of Siegel-Hilbert cusp eigenforms and their Galois representations*, Ann. scient. Éc. Norm. Sup. 4^{e} série, 32 (1999) 499–574.

Vankov, K. *Explicit formula for the symplectic Hecke series of genus four.* Arxiv, math.NT/0606492, (2006).

S. Vo, The spin L-function on the symplectic group GSp(6), Israel Journal of Mathematics 101 (1997), 1-71.

A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Ann. Math., II. Ser. 141, No.3 (1995), 443–55.

Yoshida, H., *Siegel's Modular Forms and the Arithmetic of Quadratic Forms*, Inventiones math. 60, 193–248 (1980)

Yoshida, H., *Motives and Siegel modular forms*, American Journal of Mathematics, 123 (2001), 1171–1197.