

INDIVISIBILITY OF CENTRAL VALUES OF L -FUNCTIONS FOR MODULAR FORMS

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ABSTRACT. In this paper, we generalize works of Kohnen and Ono (in *Invent. Math.*, 1999) and James and Ono (in *Math. Ann.*, 1999) on indivisibility of (the algebraic part of) central critical values of L -functions to higher weight modular forms.

1. INTRODUCTION

In this article, we show an indivisibility result on central critical values of L -functions associated to quadratic twists of modular forms using the method of Kohnen-Ono [7] and James-Ono [5].

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. For a fundamental discriminant D with $(D, N)=1$, we define the D -th quadratic twist of f by

$$f \otimes \chi_D = \sum_{n=1}^{\infty} a(n)\chi_D(n)q^n,$$

where χ_D is the quadratic character corresponding to the quadratic extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Then $f \otimes \chi_D$ is a newform of weight $2k$ for $\Gamma_0(D^2N)$. Similarly, the D -th quadratic twist of the L -function $L(f, s)$ is given by

$$L(f \otimes \chi_D, s) = \sum_{n=1}^{\infty} \frac{a(n)\chi_D(n)}{n^s}.$$

Let E be the number field generated by all Fourier coefficients of f and \mathbb{Q} . Then it is known that there exists a period $\Omega_f \in \mathbb{C}^\times$ satisfying that $\frac{L(f \otimes \chi_D, k)D_0^{k-1/2}}{\Omega_f}$ are integers in E for all fundamental discriminants D with $\delta(f) \cdot D > 0$, where $\delta(f) \in \{\pm 1\}$ is the sign defined in Ono-Skinner [10, p. 655] and D_0 is given by

$$D_0 = \begin{cases} |D| & \text{if } D \text{ is odd,} \\ |D|/4 & \text{if } D \text{ is even.} \end{cases}$$

We fix such a period Ω_f .

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For convenience, we denote

$$S(X) = \{D \in \mathbb{Z} \mid |D| < X, D : \text{fundamental discriminant}\},$$

and if functions f, g on \mathbb{R} satisfy that there is a positive constant c such that $f(X) \geq c \cdot g(X)$ for sufficiently large $X > 0$, then we write $f(X) \gg g(X)$.

Theorem 1.1. *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. Then, for all but finitely many primes λ of E , we have*

$$\#\left\{D \in S(X) \mid \delta(f) \cdot D > 0, \lambda \nmid D \text{ and } \frac{L(f \otimes \chi_D, k)D_0^{k-\frac{1}{2}}}{\Omega_f} \not\equiv 0 \pmod{\lambda}\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

This result is a refinement of results of Bruinier [2] and Ono-Skinner [10]. The proof is based on a generalization of a method of Kohlen-Ono [7] and James-Ono [5]. In the above theorem, we do not assume that the Fourier coefficients of f belong to \mathbb{Z} , therefore the surjectivity of the residual Galois representation associated to f for almost all places in general does not hold. This creates some technical difficulty for the proof. To solve this problem, we may use a result of Ribet [12] on the image of Galois representations associated to modular forms. This is an ingredient in our proof. In the last section, we also consider an indivisibility result on non-central critical values of L -functions for higher weight modular forms using congruences of modular forms with different weights.

2. MODULAR FORMS OF HALF-INTEGRAL WEIGHT

We denote the space of modular forms of weight $k + 1/2$, level N with character χ by $M_{k+1/2}(N, \chi)$, and the space of cusp forms of weight $k + 1/2$, level N with character χ by $S_{k+1/2}(N, \chi)$. Then $M_{k+1/2}(N, \chi)$ and $S_{k+1/2}(N, \chi)$ are complex vector spaces.

For a modular form of half-integral weight

$$g(z) = \sum_{n=0}^{\infty} b(n)q^n \in M_{k+1/2}(N, \chi),$$

we define the action of Hecke operator T_{p^2} by

$$T_{p^2}(g)(z) = \sum_{n=0}^{\infty} b'(n)q^n,$$

where $b'(n)$ are given by

$$b'(n) = b(p^2n) + \chi(p) \left(\frac{-1}{p}\right)^k \left(\frac{n}{p}\right) p^{k-1}b(n) + \chi(p^2)p^{2k-1}b(n/p^2)$$

and $b(n/p^2)$ are zero if $p^2 \nmid n$.

Now we give a short review of the theory of the Shimura correspondence. Let N be a positive integer which is divisible by four and χ a Dirichlet character mod N . Then we define a vector space $S_{3/2}^0(N, \chi)$ to be the subspace of $S_{3/2}(N, \chi)$ generated by

$$\left\{f(z) = \sum_{n=1}^{\infty} \psi(n)nq^{tm^2} \mid 4\text{cond}(\psi)^2 \mathfrak{t}/N, \chi = \psi\chi_{-\mathfrak{t}} \text{ and } \psi(-1) = -1\right\}$$

and denote the orthogonal complement by $S'_{3/2}(N, \chi)$. Then we assume

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

if $k \geq 2$, and

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S'_{3/2}(N, \chi)$$

if $k = 1$. Let t be a square-free positive integer. Define a number $A_t(n)$ to be

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\chi(n) \left(\frac{-1}{n}\right)^k \left(\frac{t}{n}\right)}{n^{s-k+1}} \right) \left(\sum_{n=1}^{\infty} \frac{b(tn^2)}{n^s} \right).$$

Then Shimura [14] proved that there is a positive integer M such that $\text{SH}_t(g(z)) = f_t(z) = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}(M, \chi^2)$. (In fact, one can prove that $M = N/2$.) Furthermore for any t, t' , the difference between $\text{SH}_t(g)$ and $\text{SH}_{t'}$ is only constant multiple, so essentially this correspondence is independent of the choice of t . This correspondence between modular forms is called the Shimura correspondence. Moreover if g is an eigenform for all Hecke operators T_{p^2} with $(p, 2N) = 1$, then the image of g under the Shimura correspondence is also an eigenform for all Hecke operators T_p with $(p, 2N) = 1$ and the Hecke eigenvalue of T_{p^2} for g coincides with the Hecke eigenvalue for T_p for $\text{SH}_t(g)$.

We recall the following result which is a useful version of Waldspurger’s formula ([17, Théorém 1]) by Ono-Skinner. This formula gives a relation between the Fourier coefficients of modular forms of half-integral weight and the central values of twisted L -functions for modular forms.

Theorem 2.1 (Ono-Skinner [9], (2a),(2b)). *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$, level M with trivial character. Then there is $\delta(f) \in \{\pm 1\}$, a positive integer N with $4M \mid N$, a Dirichlet character χ modulo N , a period $\Omega_f \in \mathbb{C}^\times$ and a non-zero eigenform*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

with the property that $g(z)$ maps to a twist of f under the Shimura correspondence and for all fundamental discriminants D with $\delta(f)D/ > 0$ we have

$$b(D_0)^2 = \begin{cases} \alpha_D \frac{L(f \otimes \chi_D, k) D_0^{k-1/2}}{\Omega_f} & \text{if } (D, N) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where α_D and $b(n)$ are algebraic integers in some finite extension of \mathbb{Q} . Moreover, there exists a finite set of primes S such that if D is a fundamental discriminant

- (1) $\delta(f)D > 0$,
- (2) $(D, N) = 1$,

then we have $|L(f \otimes \chi_D, k) D_0^{k-1/2} / \Omega_f|_\lambda = |b(|D_0|)^2|_\lambda$ for $\lambda \notin S$.

3. SOME PROPERTIES OF FOURIER COEFFICIENTS OF MODULAR FORMS AND GALOIS REPRESENTATIONS

In this section we generalize some results of Serre [13] and Swinnerton-Dyer [16] using a result of Ribet [12]. These results should be well-known for specialists. However we give a short review of them, since it does not seem to be available in the literature. Let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. Let E be the subfield of \mathbb{C} generated by the Fourier coefficients $a(n)$ of f . Then E is a finite extension of \mathbb{Q} . Let \mathcal{O}_E be the ring of integers of E . For each prime ℓ , we let $\mathcal{O}_{E,\ell} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$ and $E_{\ell} = E \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$.

Theorem 3.1 (Deligne [3]). *For each prime ℓ , there exists a continuous representation*

$$\rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O}_{E,\ell}) \subset \text{GL}_2(E_{\ell})$$

unramified at all primes $p \nmid N\ell$ such that $\text{trace}_{\rho_{f,\ell}}(\text{Frob}_p) = a(p)$ and $\det \rho_{f,\ell}(\text{Frob}_p) = p^{2k-1}$ for all primes $p \nmid N\ell$, where Frob_p is the arithmetic Frobenius at p .

For each prime ℓ , denote

$$A_{\ell} = \left\{ g \in \text{GL}_2(\mathcal{O}_{E,\ell}) \mid \det(g) \in \mathbb{Z}_{\ell}^{\times(2k-1)} \right\},$$

where $\mathbb{Z}_{\ell}^{\times(2k-1)}$ is the group of $(2k - 1)$ -th powers of elements in $\mathbb{Z}_{\ell}^{\times}$. Replacing $\rho_{f,\ell}$ by an isomorphic representation, we may assume that for almost all $\rho_{f,\ell}$ sends $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to A_{ℓ} . Then Ribet proved the following theorem.

Theorem 3.2 (Ribet [12]). *Assume that f has no complex multiplication. Then for almost all ℓ , we have $\rho_{f,\ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = A_{\ell}$.*

We call the set of primes ℓ with the property $\rho_{f,\ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \neq A_{\ell}$ the exceptional primes for f . Let S be the set of exceptional places for f . Let $\varepsilon_{\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_{\ell}^{\times}$ be the ℓ -adic cyclotomic character. Then by a similar argument to Swinnerton-Dyer [16], one can see that the image of

$$(\rho_{f,\ell}, \varepsilon_{\ell}) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_{\ell}^{\times}$$

is $\{(g, \alpha) \in \text{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_{\ell}^{\times} \mid \det(g) = \alpha^{2k-1}\}$ if ℓ is not exceptional. Since A_{ℓ} contains an element with the form

$$\begin{pmatrix} \text{trace} \rho_{f,\ell}(\sigma) & -1 \\ \det \rho_{f,\ell}(\sigma) & 0 \end{pmatrix},$$

the map $(\text{trace} \rho_{f,\ell}, \varepsilon_{\ell}) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathcal{O}_{E,\ell} \times \mathbb{Z}_{\ell}^{\times}$ is surjective. Moreover by a ramification argument, one can see that the map

$$\prod_{\ell \notin S} (\text{trace} \rho_{f,\ell}, \varepsilon_{\ell}) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \prod_{\ell \notin S} (\mathcal{O}_{E,\ell} \times \mathbb{Z}_{\ell}^{\times})$$

is also surjective. Therefore we have the following result which is a generalization of a result of Serre [13, Théorém 11] using the Chebotarev density theorem.

Theorem 3.3. *Assume that f has no complex multiplication. Let t be a positive integer and α a non-zero integer in E . Fix $\beta \in \mathcal{O}_E/\alpha\mathcal{O}_E$ and $r \in (\mathbb{Z}/t\mathbb{Z})^{\times}$. Suppose that α does not contain a prime divisor which divides an exceptional prime for f . Then the set of primes p with the properties $a(p) \equiv \beta \pmod{\alpha}$ and $p \equiv r \pmod{t}$ has positive density.*

4. INDIVISIBILITY OF FOURIER COEFFICIENTS OF MODULAR FORMS OF HALF-INTEGRAL WEIGHT

In this section, we give a result on modulo ℓ indivisibility of Fourier coefficients of half-integral weight modular forms using a method of Kohlen-Ono [7] and James-Ono [5]. Our result is a refinement of a result of Bruinier [2] and Ono-Skinner [10].

To consider the indivisibility of Fourier coefficients of half-integral weight modular forms, we will use the following results.

Theorem 4.1 (Sturm [15]). *Let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in M_k(N, \chi)$$

be a half-integral or integral weight modular form for which the coefficients $b(m)$ are algebraic integers contained in a number field E . Let v be a finite place of E and let

$$\text{ord}_v(g) = \begin{cases} +\infty & \text{if } b(n) \equiv 0 \pmod v \text{ for all } n, \\ \min\{n \mid b(n) \not\equiv 0 \pmod v\} & \text{otherwise.} \end{cases}$$

Moreover put

$$\mu = \frac{k}{12} [\Gamma_0(1) : \Gamma_0(N)] = \frac{kN}{12} \prod_{p|N} \frac{p+1}{p}.$$

Assume that

$$\text{ord}_v(g) > \mu;$$

then $\text{ord}_v(g) = +\infty$.

Remark 4.2 (cf. [5, Proposition 5]). In [15], Sturm proved this theorem for integral weight modular forms with trivial character, but the general case follows by taking an appropriate power of g .

Lemma 4.3 (Shimura, [14, Section 1]). *Suppose*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

is a half-integral weight cusp form and p is a prime. We define $(U_p g)(z)$, $(V_p g)(z)$ by

$$\begin{aligned} (U_p g)(z) &= \sum_{n=1}^{\infty} u_p(n)q^n = \sum_{n=1}^{\infty} b(pn)q^n, \\ (V_p g)(z) &= \sum_{n=1}^{\infty} v_p(n)q^n = \sum_{n=1}^{\infty} b(n)q^{pn}. \end{aligned}$$

Then

$$(U_p g)(z), (V_p g)(z) \in S_{k+1/2} \left(Np, \chi \left(\frac{4p}{\cdot} \right) \right).$$

Let

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N, \chi)$$

be an integral weight modular form for which the coefficients $a(m)$ are algebraic integers in E . For a prime λ of E and positive integers r, t with $(r, t) = 1$, define $T(r, t)$ and $T(\lambda, r, t)$ by

$$T(r, t) = \{p : \text{prime} \mid a(p) = 0, p \equiv r \pmod t\}$$

and

$$T(\lambda, r, t) = \{p : \text{prime} \mid a(p) \equiv 0 \pmod \lambda, p \equiv r \pmod t\}.$$

For a positive real number X , we also denote $T(r, t, X) = \{p \in T(r, t) \mid p \leq X\}$ and $T(\lambda, r, t, X) = \{p \in T(\lambda, r, t) \mid p \leq X\}$.

For $g = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi) \cap \mathcal{O}_{E, \lambda}[[q]]$, denote $s_{\lambda}(g) = \min\{\text{ord}_{\lambda}(b(n)) \mid n \in \mathbb{Z}_{>0}\}$. The following two lemmas give an estimate for indivisibility of Fourier coefficients of modular forms of half-integral weight.

Lemma 4.4. *Let ℓ be a prime greater than 3. Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigen newform of weight $2k$, level M with trivial character and let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1. Assume that f has complex multiplication in the sense of Ribet [11] and let λ be a prime in E above ℓ . If there exists an integer D' such that $\delta(f)D' > 0$, $(D', N) = 1$, $\varepsilon = \left(\frac{D'}{\ell}\right) \neq 0$ and $\text{ord}_{\lambda}(b(|D'|)) = s_{\lambda}(g)$, then

$$\#\left\{D \in S(X) \mid \left(\frac{D}{\ell}\right) = \varepsilon, \text{ord}_{\lambda}(b(D)) = s_{\lambda}(g)\right\} \gg_{f, \ell} \frac{\sqrt{X}}{\log X}.$$

Proof. By dividing g by $\lambda^{s_{\lambda}(g)}$, we may assume $s_{\lambda}(g) = 0$. If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character χ' . Since f has complex multiplication, then there exists an imaginary quadratic field K such that for every prime p satisfying $p \equiv 3 \pmod 4$, $(p, N) = 1$ and $\left(\frac{\Delta_K}{p}\right) = -1$ we have $a(p) = 0$, where Δ_K is the discriminant of K . Therefore, for such p , using the formulae for the action of Hecke operator T_{p^2} , we find that

$$b(p^2n) + \chi'(p)p^{k-1} \left(\frac{(-1)^kn}{p}\right) b(n) + \chi'(p^2)p^{2k-1}b(n/p^2) = 0.$$

Hence if $(r, t) = 1$, $4 \mid t$, $r \equiv 3 \pmod 4$, then

$$\#T(r, t, X) = \#\{p \in T(r, t) \mid p \leq X\} \gg_f \frac{X}{\log X}$$

and for any $p \in T(r, t)$ we have

$$(4.1) \quad b(p^2n) = -\chi'(p)p^{k-1} \left(\frac{(-1)^k n}{p} \right) b(n) - \chi'^2(p)p^{2k-1}b(n/p^2).$$

Put $\kappa = (k + \frac{1}{2}) \frac{[\Gamma_0(1) : \Gamma_0(N\ell^2)]}{12} + 1$. Now, we choose (r_0, t_0) satisfying the following properties:

- (1) $N\ell^2 | t_0, (r_0, t_0) = 1, \chi'(r_0) = 1$ and $p \equiv 3 \pmod{4}$.
- (2) If p is a prime with $p \equiv r_0 \pmod{t_0}$, then $\left(\frac{(-1)^k n}{p} \right) = -1$ for any $1 \leq n \leq \kappa$ with $(n, N\ell^2) = 1$.
- (3) For each prime $p \equiv r_0 \pmod{t_0}$ we have $\left(\frac{\Delta_K}{p} \right) = -1$.
- (4) Each prime $p \equiv r_0 \pmod{t_0}$ satisfies $\left| \chi'(p^2)p - \chi'(p) \left(\frac{(-1)^k |D'|}{p} \right) \right|_\lambda = 1$.

If $p \in T(r_0, t_0)$ is a sufficiently large prime, for all $1 \leq n \leq \kappa$

$$u_p(pn) = b_0(p^2n) = -\chi'(p)p^{k-1} \left(\frac{(-1)^k n}{p} \right) b_0(n) - p^{2k-1}\chi'^2(p)b_0(n/p^2).$$

Since $b_0(n/p^2) = 0$, we have $u_p(pn) = \chi'(p)p^{k-1}b_0(n) = p^{k-1}b_0(p) = p^{k-1}v_p(pn)$. By the relation (4.1),

$$v_p(p^3|D'|) = b_0(p^2|D'|) = -\chi'(p)p^{k-1} \left(\frac{(-1)^k |D'|}{p} \right) b_0(|D'|)$$

and

$$u_p(p^3|D'|) = b_0(p^4|D'|) = -p^{2k-1}\chi'(p^2)b_0(|D'|).$$

Therefore by the assumption and the choice of (r_0, t_0) ,

$$\begin{aligned} & \left| u_p(p^3|D'|) - p^{k-1}v_p(p^3|D'|) \right|_\lambda \\ &= \left| \left(\chi'(p^2)p^{2k-1} - \chi'(p)p^{2k-2} \left(\frac{(-1)^k |D'|}{p} \right) \right) b_0(|D'|) \right|_\lambda = 1. \end{aligned}$$

Hence

$$\text{ord}_\lambda(U_p g_0 - p^{k-1}V_p g_0) < +\infty.$$

By Theorem 4.1 and Lemma 4.3, there exists an integer n_p such that

$$1 \leq n_p \leq \left(k + \frac{1}{2} \right) \frac{[\Gamma_0(1) : \Gamma_0(N\ell^2 p)]}{12} = \kappa(p + 1), (n_p, p) = 1$$

and

$$b_0(n_p p) = u_p(n_p) \not\equiv p^{k-1}v_p(n_p) = 0 \pmod{\lambda}.$$

Consequently, let D_{sf} be the square-free part of $D = n_p p$; then

$$|b_0(D_{\text{sf}})|_\lambda = 1.$$

For convenience, let p_i be the primes in $T(r_0, t_0)$ in increasing order, and let D_i be the square-free part of $p_i n_{p_i}$. If $r < s < t$ and $D_r = D_s = D_t$, then $p_r p_s p_t | D_r$. However this can only occur for finitely many r, s and t since $|D_i| < \kappa p_i (p_i + 1)$. Therefore, the number of distinct $|D_i| < X$ is at least half the number of $p \in T(r_0, t_0)$ with $p \leq \sqrt{X/\kappa}$. Therefore the lemma follows from $\#T(r_0, t_0, X) \gg_{f, \lambda} X/\log X$. □

Lemma 4.5. *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigen newform of weight $2k$, level M with trivial character. Denote $E = \mathbb{Q}(\{a(n)|n \geq 1\})$ and let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1. We fix a prime number ℓ greater than 3 and let λ be a prime in E above ℓ . Assume that f does not have complex multiplication and the image of the Galois representation associated to f

$$\rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O}_{E,\ell})$$

coincides with A_ℓ . If there exists an integer D' such that $\delta(f)D' > 0$, $(D', N) = 1$, $\varepsilon = \left(\frac{D'}{\ell}\right) \neq 0$ and $\text{ord}_\lambda(b(|D'|)) = s_\lambda(g)$, then

$$\#\left\{D \in S(X) \mid \left(\frac{D}{\ell}\right) = \varepsilon, \text{ord}_\lambda(b(D)) = s_\lambda(g)\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

Proof. First, we may assume $\text{ord}_\lambda(g) = 0$. If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character χ' . If $a(p) \equiv 0 \pmod{\lambda}$, by the formula for the action of Hecke operator T_{p^2} we find that

$$b(p^2n) + \chi'(p)p^{k-1} \left(\frac{(-1)^kn}{p}\right) b(n) + \chi'^2(p)p^{2k-1}b(n/p^2) \equiv 0 \pmod{\lambda}.$$

By the assumption, ℓ is not exceptional. Hence Theorem 3.3 implies

$$\#T(\lambda, r, t, X) = \#\{p \in T(\lambda, r, t) \mid p \leq X\} \gg_{f,\lambda} \frac{X}{\log X}$$

and for each $p \in T(\lambda, r, t)$

$$(4.2) \quad b(p^2n) \equiv -\chi'(p)p^{k-1} \left(\frac{(-1)^kn}{p}\right) b(n) - \chi'^2(p)p^{2k-1}b(n/p^2) \pmod{\lambda}.$$

Let κ be the number as in the proof of Lemma 3.4. Now, we choose (r_0, t_0) satisfying the following properties:

- (1) $N\ell^2|t_0$, $(r_0, t_0) = 1$, $\chi'(r_0) = 1$.
- (2) If p is a prime with $p \equiv r_0 \pmod{t_0}$, then $\left(\frac{(-1)^kn}{p}\right) = -1$ for any $1 \leq n \leq \kappa$ with $(n, N\ell^2) = 1$.
- (3) For each prime $p \equiv r_0 \pmod{t_0}$ we have $\left(\frac{(-1)^k|D'|}{p}\right) = -1$.
- (4) Each prime $p \equiv r_0 \pmod{t_0}$ has the property that $1 + p \not\equiv 0 \pmod{\lambda}$.

If $p \in T(\lambda, r_0, t_0)$ is a sufficiently large prime, for all $1 \leq n \leq \kappa$ with $(n, N\ell^2) = 1$, then

$$\begin{aligned} u_p(pn) &= b_0(p^2n) \equiv -p^{k-1} \left(\frac{(-1)^{kn}}{p} \right) b_0(n) - p^{2k-1} b_0(n/p^2) \\ &= p^{k-1} b_0(n) = p^{k-1} v_p(pn) \pmod{\lambda}. \end{aligned}$$

By the relation (4.2), we have

$$v_p(p^3|D'|) = b_0(p^2|D'|) \equiv p^{k-1} b_0(|D'|) \pmod{\lambda},$$

also

$$u_p(p^3|D'|) = b_0(p^4|D'|) \equiv -p^{2k-1} b_0(|D'|) \pmod{\lambda}.$$

Therefore by assumption and the choice of (r_0, t_0) ,

$$p^{k-1} v_p(p^3|D'|) - u_p(p^3|D'|) \equiv p^{2k-2}(1+p)b_0(|D'|) \not\equiv 0 \pmod{\lambda}.$$

Hence

$$\text{ord}_\lambda(U_p g_0 - p^{k-1} V_p g_0) < +\infty.$$

By Theorem 4.1 and Lemma 4.3, there exists an integer n_p such that

$$1 \leq n_p \leq (k + 1/2)[\Gamma_0(1) : \Gamma_0(N\ell^2 p)]/12 = \kappa(p + 1), (n_p, p) = 1$$

and

$$b_0(n_p p) = u_p(n_p) \not\equiv v_p(n_p) = 0 \pmod{\lambda}.$$

In particular, let D_{sf} be the square-free part of $D = n_p p$; then

$$|b_0(D_{\text{sq}})|_\lambda = 1.$$

Now the lemma follows from the same argument as in the proof of the previous lemma. □

Proof of Theorem 1.1. Let

$$g(z) = \sum_{n=1}^\infty b(n)q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1 for f .

By replacing f by a suitable quadratic twist of f if necessary, we may assume that $\varepsilon = \delta(f)$, where ε is the sign of the functional equation of $L(f, s)$. By the result of Friedberg and Hoffstein [4], we can take an integer D' such that $\delta(f)D' > 0$, $(D', 2N) = 1$ and $b(D') \neq 0$. In particular, for almost all finite places λ of E we have

$$|b(D')|_\lambda = 1.$$

Thus by Lemmas 4.4, 4.5, Theorem 2.1 and Theorem 3.3, for all but finitely many primes λ we have

$$\begin{aligned} \# \left\{ D \in S(X) \mid \delta(f) \cdot D > 0, (\ell, D) = 1 \text{ and } \left| \frac{L(f \otimes \chi_D, k) D_0^{k-1/2}}{\Omega_f} \right|_\lambda = 1 \right\} \\ \gg_{f, \lambda} \frac{\sqrt{X}}{\log X}. \end{aligned}$$

This completes the proof. □

5. INDIVISIBILITY FOR THE NON-CENTRAL CRITICAL VALUES

In this section, we consider a special case for non-central values of L -functions for modular forms. We fix a prime ℓ greater than 7 and let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigenform of weight $\ell + 1$ for $SL_2(\mathbb{Z})$. Let λ be a prime in a number field E . We assume that the integer ring of E contains all Fourier coefficients of f and choose a period Ω_f^{\pm} as in Ash-Stevens [1, Theorem 4.5]. Then for any Dirichlet character χ , the quotient $\tau(\chi^{-1}) \frac{L(f \otimes \chi, 1)}{(2\pi i)\Omega_f^{\pm}}$ is an integer in $E_{\lambda}(\chi)$ where τ is the Gauss sum and $\pm = \chi(-1)$.

Theorem 5.1. *Let λ be a prime in E above ℓ . We assume the following conditions.*

- (1) *There exists a unique eigenform F of weight 2 for $\Gamma_0(\ell)$ such that*

$$F \equiv f \pmod{\lambda}.$$

- (2) *ℓ is not exceptional.*

- (3) *There exists a square-free negative integer d_0 such that $(d_0, 2\ell) = 1$, $\left(\frac{d_0}{p}\right) \chi_{d_0}(\ell) = -\varepsilon(F)$, where $\varepsilon(F)$ is the sign of functional equation of $L(F, s)$ and*

$$\frac{L(f \otimes \chi_{d_0}, 1)\sqrt{d_0}}{(2\pi i)\Omega_f^{\pm}} \not\equiv 0 \pmod{\lambda}.$$

Then we have

$$\#\left\{D \in S(X) \mid \frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}} \not\equiv 0 \pmod{\lambda}\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

For the proof, we recall a result of Ash and Stevens.

Theorem 5.2 (Ash-Stevens, [1]). *Let k be a positive integer less than $\ell + 2$ and $f = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(1))$ an eigenform satisfying the assumptions of Theorem 5.1. We fix a prime λ above ℓ in a number field E which contains all Fourier coefficients of f . Assume that*

- (1) *There exists a prime q satisfying $a(q) \not\equiv q^{k-1} + 1 \pmod{\lambda}$.*
- (2) *There exists a unique eigenform $F \in S_2(\Gamma_1(\ell))$ such that $f \equiv F \pmod{\lambda}$.*

Then there exists a complex number Ω_F^{\pm} such that for any Dirichlet character χ satisfying $(\text{cond } \chi, p) = 1$, we have

$$\frac{\tau(\chi^{-1})L(f \otimes \chi, 1)}{(2\pi i)\Omega_f^{\pm}} \equiv \frac{\tau(\chi^{-1})L(F \otimes \chi, 1)}{(2\pi i)\Omega_F^{\pm}} \pmod{\lambda}.$$

Now we prove Theorem 5.1. By the Kohnen-Zagier formula [6], there exists an eigenform

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{3/2}(\Gamma_0(4\ell))$$

such that for any negative square-free integer D satisfying $\left(\frac{D}{\ell}\right) = -\varepsilon(F)$,

$$|b(|D|)|^2 = 2 \cdot \frac{\sqrt{D}}{\pi} \cdot \frac{\langle g, g \rangle}{\langle F, F \rangle} L(F \otimes \chi_D, 1),$$

where $\langle \cdot, \cdot \rangle$ is the Petersson inner product. We can normalize g by the relation $\frac{\langle F, F \rangle}{\langle g, g \rangle} = \Omega_f^\pm$. Taking a linear combination of twists of g , one may assume $b(|D|) = 0$ if $(\frac{D}{\ell}) \neq -\varepsilon(F)$ and $D < 0$. From the assumptions of the theorem, ℓ is not exceptional. This implies the existence of a prime q satisfying $a(q) \not\equiv q^{k-1} + 1 \pmod{\lambda}$, therefore the assumptions of Theorem 5.1 imply the assumptions of Theorem 5.2. Since $\tau(\chi_D)^{-1} = \pm 1/\sqrt{D}$, one can see that

$$\frac{L(f \otimes \chi, 1)\sqrt{D}}{(2\pi i)\Omega_f^\pm} \equiv \frac{L(F \otimes \chi, 1)\sqrt{D}}{(2\pi i)\Omega_F^\pm} = |b(|D|)|^2 \cdot c \pmod{\lambda}$$

with a λ -adic unit c . By the assumption (3), we have

$$\text{ord}_\lambda \left(\frac{L(f \otimes \chi_{d_0}, 1)\sqrt{d_0}}{(2\pi i)\Omega_f^\pm} \right) = 0$$

therefore $\text{ord}_\lambda(b(d_0)) = \min\{\text{ord}_\lambda(b(n)) \mid n : \text{square-free}, \chi_{d_0}(\ell) = -\varepsilon(f)\}$. Hence Lemma 4.5 implies

$$\#\{D \in S(X) \mid \chi_D(\ell) = -\varepsilon(f), \text{ord}_\lambda(b(D)) = s\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X},$$

thus we have

$$\#\left\{D \in S(X) \mid \frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_f^\pm} \not\equiv 0 \pmod{\lambda}\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

This completes the proof.

Remark 5.3. Lemma 4.5 is stated only for g given in Theorem 2.1, but one can show a similar result for any eigenform $g \in S_{k+1/2}(N, \chi)$ if $k \geq 2$ ($S'_{\frac{3}{2}}(N, \chi)$ if $k = 1$) corresponding to some eigenform $f \in S_{2k}(\Gamma_0(M))$ under the Shimura correspondence.

Example 5.4. Let

$$f = \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\Gamma_0(1))$$

and

$$F = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 \in S_2(\Gamma_0(11)).$$

Then it is well known that $f \equiv F \pmod{11}$, $\dim S_2(\Gamma_0(11)) = 1$ and the mod 11 Galois representation associated to f is surjective. Moreover one can check that

$$\frac{L(\Delta \otimes \chi_{-3}, 1)}{\Omega_{\Delta \otimes \chi_{-3}}^+} = 36741600 \not\equiv 0 \pmod{11}$$

by using MAGMA. So the assumptions of Theorem 5.1 are satisfied for $f = \Delta$. Hence we have

$$\#\left\{D \in S(X) \mid \frac{L(\Delta \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_\Delta^\pm} \not\equiv 0 \pmod{11}\right\} \gg \frac{\sqrt{X}}{\log X}.$$

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