Krawtchouk polynomials and Krawtchouk matrices

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Abstract

Krawtchouk matrices have as entries values of the Krawtchouk polynomials for nonnegative integer arguments. We show how they arise as condensed Sylvester-Hadamard matrices via a binary shuffling function. The underlying symmetric tensor algebra is then presented.

To advertise the breadth and depth of the field of Krawtchouk polynomials/matrices through connections with various parts of mathematics, some topics that are being developed into a Krawtchouk Encyclopedia are listed in the concluding section. Interested folks are encouraged to visit the website

`http://chanoir.math.siu.edu/wiki/KravchukEncyclopedia`

which is currently in a state of development.

1 What are Krawtchouk matrices

Of Sylvester-Hadamard matrices and Krawtchouk matrices, the latter are less familiar, hence we start with them.

Definition 1.1 The N^{th}-*order Krawtchouk matrix* $K^{(N)}$ is an $(N+1)\times(N+1)$ matrix, the entries of which are determined by the expansion:

$$(1+v)^{N-j}\,(1-v)^j = \sum_{i=0}^{N} v^i K_{ij}^{(N)} \tag{1.1}$$

Thus, the polynomial $G(v) = (1+v)^{N-j}\,(1-v)^j$ is the *generating function* for the row entries of the j^{th} column of $K^{(N)}$. Expanding gives the explicit values of the matrix entries:

$$K_{ij}^{(N)} = \sum_k (-1)^k \binom{j}{k}\binom{N-j}{i-k}.$$

where matrix indices run from 0 to N.

Here are the Krawtchouk matrices of order zero, one, and two:

$$K^{(0)} = \begin{bmatrix} 1 \end{bmatrix} \qquad K^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad K^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

The reader is invited to see more examples in Table 1 of the Appendix.

The columns of Krawtchouk matrices may be considered *generalized binomial coefficients*. The rows define Krawtchouk *polynomials*: for fixed order N, the i^{th} *Krawtchouk polynomial* takes its corresponding values from the i^{th} row:

$$k_i(j,N) = K_{ij}^{(N)} \tag{1.2}$$

One can easily show that $k_i(j,N)$ can be given as a polynomial of degree i in the variable j. For fixed N, one has a system of $N+1$ polynomials orthogonal with respect to the symmetric binomial distribution.

A fundamental fact is that the square of a Krawtchouk matrix is proportional to the identity matrix.

$$(K^{(N)})^2 = 2^N \cdot I$$

This property allows one to define a Fourier-like *Krawtchouk transform* on integer vectors. For more properties we refer the reader to [12]. In the present article, we focus on Krawtchouk matrices as they arise from corresponding Sylvester-Hadamard matrices. More structure is revealed through consideration of symmetric tensor algebra.

Symmetric Krawtchouk matrices. When each column of a Krawtchouk matrix is multiplied by the corresponding binomial coefficient, the matrix becomes symmetric. In other words, define the **symmetric Krawtchouk matrix** as

$$S^{(N)} = K^{(N)} B^{(N)}$$

where $B^{(N)}$ denotes the $(N+1) \times (N+1)$ diagonal matrix with binomial coefficients, $B^{(N)}_{ii} = \binom{N}{i}$, as its non-zero entries.

Example. For $N = 3$, we have

$$S^{(3)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & 3 & -3 & -3 \\ 3 & -3 & -3 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

Some symmetric Krawtchouk matrices are displayed in Table 2 of the Appendix. A study of the spectral properties of the symmetric Krawtchouk matrices was initiated in work with Fitzgerald [11].

Background note. Krawtchouk's polynomials were introduced by Mikhail Krawtchouk in the late 20's [17, 18]. The idea of setting them in a matrix form appeared in the 1985 work of N. Bose [2] on digital filtering in the context of the Cayley transform on the complex plane. For some further development of this idea, see [12].

The Krawtchouk polynomials play an important rôle in many areas of mathematics. Here are some examples:

- **Harmonic analysis.** As orthogonal polynomials, they appear in the classic work by Szëgo [24]. They have been studied from the point of view of harmonic analysis and special functions, e.g., in work of Dunkl [8, 9]. Krawtchouk polynomials may be viewed as the discrete version of Hermite polynomials (see, e.g., [1]).

- **Statistics.** Among the statistics literature we note particularly Eagleson [10] and Vere-Jones [25].

- **Combinatorics and coding theory.** Krawtchouk polynomials are essential in MacWilliams' theorem on weight enumerators [19, 21], and are a fundamental example in association schemes [5, 6, 7].

- **Probability theory.** In the context of the classical symmetric random walk, it is recognized that Krawtchouk's polynomials are elementary symmetric functions in variables taking values ± 1. It turns out that the generating function (1.1) is a martingale in the parameter N [13].

- **Quantum theory.** Krawtchouk matrices interpreted as operators give rise to two new interpretations in the context of both classical and quantum random walks [12]. The significance of the latter interpretation lies at the basis of quantum computing.

Let us proceed to show the relationship between Krawtchouk matrices and Sylvester-Hadamard matrices.

2 Krawtchouk matrices from Hadamard matrices

Taking the Kronecker (tensor) product of the initial matrix

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

with itself N times defines the family of *Sylvester-Hadamard matrices*.

(For a review of Hadamard matrices, see Yarlagadda and Hershey [27].)

Notation 2.1 *Denote the Sylvester-Hadamard matrices, tensor (Kronecker) powers of the fundamental matrix H, by*

$$H^{(N)} = H^{\otimes N} = \underbrace{H \otimes H \otimes \cdots \otimes H}_{N \text{ times}}$$

The first three Sylvester-Hadamard matrices are:

$$H^{(1)} = \begin{bmatrix} \bullet & \bullet \\ \bullet & \circ \end{bmatrix} \quad H^{(2)} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \bullet \end{bmatrix} \quad H^{(3)} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ \\ \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ & \bullet \\ \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ \end{bmatrix}$$

where, to emphasize the patterns, we use $\bullet$ for 1 and $\circ$ for -1. See Table 3 of the Appendix for these matrices up to order 5.

For $N = 1$, the Hadamard matrix coincides with the Krawtchouk matrix: $H^{(1)} = K^{(1)}$. Now we wish to see how the two classes of matrices are related for higher N. It turns out that appropriately contracting (condensing) Hadamard-Sylvester matrices yields corresponding symmetric Krawtchouk matrices.

The problem is that the tensor products disperse the columns and rows that have to be summed up to do the contraction. We need to identify the right sets of indices.

Definition 2.2 Define the *binary shuffling function* as the function

$$w\colon \mathbf{N} \to \mathbf{N}$$

giving the "binary weight" of an integer. That is, let $n = \sum_k d_k 2^k$ be the binary expansion of the number n. Then $w(n) = \sum_k d_k$, the number of ones in the representation.

Notice that, as sets,

$$w(\{0, 1, \ldots, 2^N - 1\}) = \{0, 1, \ldots, N\}$$

Here are the first 16 values of w listed for the integers running from 0 through $2^4 - 1 = 15$:

$$0 \quad 1 \quad 1 \quad 2 \quad 1 \quad 2 \quad 2 \quad 3 \quad 1 \quad 2 \quad 2 \quad 3 \quad 2 \quad 3 \quad 3 \quad 4$$

The shuffling function can be defined recursively. Set
$w(0) = 0$ and

$$w(2^N + k) = w(k) + 1 \tag{2.1}$$

for $0 \le k < 2^N$. One can thus create the sequence of values of the shuffling function by starting with 0 and then appending to the current string of values a copy of itself with values increased by 1:

$$0 \;\rightarrow\; 01 \;\rightarrow\; 0112 \;\rightarrow\; 01121223 \;\rightarrow\; \ldots$$

Now we can state the result;

Theorem 2.3 *Symmetric Krawtchouk matrices are reductions of Hadamard matrices as follows:*

$$S_{ij}^{(N)} = \sum_{\substack{w(a)=i \\ w(b)=j}} H_{ab}^{(N)}$$

Example. Let us see the transformation for $H^{(4)} \rightarrow S^{(4)}$ (recall that $\bullet$ stands for 1, and $\circ$ for -1). Applying the binary shuffling function to $H^{(4)}$, mark the rows and columns accordingly:

$$\begin{array}{c} \\ 0\\1\\1\\2\\1\\2\\2\\3\\1\\2\\2\\3\\2\\3\\3\\4 \end{array}
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\begin{array}{cccccccccccccccc} 0&1&1&2&1&2&2&3&1&2&2&3&2&3&3&4 \end{array} \\
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\end{array}\right)
\end{array}$$

The contraction is performed by summing columns with the same index, then summing rows in similar fashion. One checks from the given matrix that indeed this procedure gives the symmetric Krawtchouk matrix $S^{(4)}$:

$$S^{(4)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{rrrrr} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 8 & 0 & -8 & -4 \\ 6 & 0 & -12 & 0 & 6 \\ 4 & -8 & 0 & 8 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \end{array}$$

Now we give a method for transforming the N^{th} (symmetric) Krawtchouk matrix into the $N+1^{\text{st}}$.

Definition 2.4 The *square contraction* $r(M)$ of a $2n \times 2n$ matrix M_{ab}, $1 \le a, b \le 2n$, is the $(n+1) \times (n+1)$ matrix with entries

$$(rM)_{ij} = \sum_{\substack{a=2i,\ 2i+1 \\ b=2j,\ 2j+1}} M_{ab}$$

$0 \le i, j \le n$, where the values of M_{ab} with a or b outside of the range $(1, \ldots, 2n)$ are taken as zero.

Theorem 2.5 *Symmetric Krawtchouk matrices satisfy:*

$$S^{(N+1)} = r(S^{(N)} \otimes H)$$

with $S^{(1)} = H$.

Example. Start with symmetric Krawtchouk matrix of order 2:

$$S^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Take the tensor product with H:

$$S^{(2)} \otimes H = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & -1 & 2 & -2 & 1 & -1 \\ 2 & 2 & 0 & 0 & -2 & -2 \\ 2 & -2 & 0 & 0 & -2 & 2 \\ 1 & 1 & -2 & -2 & 1 & 1 \\ 1 & -1 & -2 & 2 & 1 & -1 \end{bmatrix}$$

surround with zeros and contract:

$$r(S^{(2)}\otimes H) = r\left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ \hline 0 & 1 & -1 & 2 & -2 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 & -2 & -2 & 0 \\ \hline 0 & 2 & -2 & 0 & 0 & -2 & 2 & 0 \\ 0 & 1 & 1 & -2 & -2 & 1 & 1 & 0 \\ \hline 0 & 1 & -1 & -2 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc} 1 & 3 & 3 & 1 \\ 3 & 3 & -3 & -3 \\ 3 & -3 & -3 & 3 \\ 1 & -3 & 3 & -1 \end{array}\right]$$

Corollary 2.6 *Krawtchouk matrices satisfy:*

$$K^{(N+1)} = r(K^{(N)}B^{(N)} \otimes H)(B^{(N+1)})^{-1}$$

where B is the diagonal binomial matrix.

Note that starting with the 2×2 identity matrix, I, set $I^{(1)} = I$, $I^{(N+1)} = r(I^{(N)} \otimes I)$. Then, in fact, $I^{(N)} = B^{(N)}$.

Next, we present the algebraic structure underlying these remarkable properties.

3 Krawtchouk matrices and symmetric tensors

Given a d-dimensional vector space V over $\mathbf{R}$, one may construct a d^N-dimensional space $V^{\otimes N}$, the N-fold tensor product of V, and, as well, a $\binom{d+N-1}{N}$-dimensional *symmetric tensor space* $V^{\otimes_s N}$. There is a natural map

$$\mathtt{symm}\colon V^{\otimes N} \longrightarrow V^{\otimes s N}$$

which, for homogeneous tensors, is defined via

$$\mathtt{symm}\,(v \otimes w \otimes \ldots) = \text{symmetrization of}\,(v \otimes w \otimes \ldots)$$

For computational purposes, it is convenient to use the fact that the symmetric tensor space of order N of a d-dimensional vector space is isomorphic to the space of polynomials in d variables homogeneous of degree N.

Let $\{e_1, e_2, \ldots e_d\}$ be a basis of V. Map e_i to x_i, replace tensor products by multiplication of the variables, and extend by linearity. For example,

$$2e_1 \otimes e_2 + 3e_2 \otimes e_1 - 7e_3 \otimes e_2 \;\longrightarrow\; 5x_1x_2 - 7x_2x_3$$

thus identifying basis (elementary) tensors in $V^{\otimes N}$ that are equivalent under any permutation.

This map induces a map on certain linear operators. Suppose $A \in \mathrm{End}(V)$ is a linear transformation on V. This induces a linear transformation $A_N = A^{\otimes N} \in \mathrm{End}(V^{\otimes N})$ defined on elementary tensors by:

$$A_N(v \otimes w \otimes \ldots) = A(v) \otimes A(w) \otimes \ldots$$

Similarly, a linear operator on the symmetric tensor spaces is induced so that the following diagram commutes:

$$\begin{array}{ccc} V^{\otimes N} & \xrightarrow{A_N} & V^{\otimes N} \\ \mathtt{symm}\big\downarrow & & \mathtt{symm}\big\downarrow \\ V^{\otimes_s N} & \xrightarrow{\overline{A}_N} & V^{\otimes_s N} \end{array}$$

This can be understood by examining the action on polynomials. We call $\overline{A}_N$ the *symmetric representation of A in degree N*. Denote the matrix elements of $\overline{A}_N$ by $\overline{A}_{mn}$. If A has matrix entries A_{ij}, let

$$y_i = \sum_j A_{ij}\, x^j$$

It is convenient to label variables with indices from 0 to $\delta = d - 1$. Then the matrix elements of the symmetric representation are defined by the expansion:

$$y_0^{m_0} \cdots y_\delta^{m_\delta} = \sum_n \overline{A}_{mn}\, x_0^{n_0} \cdots x_\delta^{n_\delta}$$

with multi-indices m and n homogeneous of degree N.

Mapping to the symmetric representation is an algebra homomorphism, i.e.,

$$\overline{AB} = \overline{A}\,\overline{B}$$

Explicitly, in matrix notation, $\overline{(AB)}_{mn} = \sum_r (\overline{A})_{mr} (\overline{B})_{rn}$.

Now we are ready to state our result

Proposition 3.1 *For each $N > 0$, the symmetric representation of the $N^{\rm th}$ Sylvester-Hadamard matrix equals the transposed $N^{\rm th}$ Krawtchouk matrix:*

$$(\overline{H}_N)_{ij} = K^{(N)}_{ji} \ .$$

Proof: Writing (x, y) for (x_0, x_1), we have in degree N for the $k^{\rm th}$ component:

$$(x+y)^{N-k}(x-y)^k = \sum_l \overline{H}_{kl}\, x^{N-l} y^l$$

Substituting $x = 1$ yields the generating function (1.1) for the Krawtchouk matrices with the coefficient of y^l equal to $K^{(N)}_{lk}$. Thus the result. □

Insight into these correspondences can be gained by splitting the fundamental Hadamard matrix H $(= K^{(1)})$ into two special symmetric 2×2 operators:

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

so that

$$H = F + G = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

One can readily check that

$$F^2 = G^2 = I$$
$$FH = HG \qquad \text{and} \qquad GH = HF \tag{3.1}$$

The first of the second pair of equations may be viewed as the spectral decomposition of F and we can interpret the Hadamard matrix as diagonalizing F into G. Taking transposes gives the second equation of (3.1).

Now we proceed to the interpretation leading to a symmetric Bernoulli quantum random walk ([12]). For this interpretation, the Hilbert space of states is represented by the N^{th} tensor power of the original 2-dimensional space V, that is, by the 2^N-dimensional Hilbert space $V^{\otimes N}$. Define the following linear operator on $V^{\otimes N}$:

$$\begin{aligned} X_F &= F \otimes I \otimes \cdots \otimes I \\ &\quad + I \otimes F \otimes I \otimes \cdots \otimes I \\ &\quad + \ldots \\ &\quad + I \otimes I \otimes \cdots \otimes F \\ &= f_1 + f_2 + \ldots + f_i + \ldots + f_N \end{aligned}$$

each term describing a "flip" at the i^{th} position (cf. [14, 22]). Analogously, we define:

$$\begin{aligned} X_G &= G \otimes I \otimes \cdots \otimes I \\ &\quad + I \otimes G \otimes I \otimes \cdots \otimes I \\ &\quad + \ldots \\ &\quad + I \otimes I \otimes \cdots \otimes G \\ &= g_1 + g_2 + \ldots + g_i + \ldots + g_N \end{aligned}$$

From equations (3.1) we see that our X-operators intertwine the Sylvester-Hadamard matrices:

$$X_F H^{(N)} = H^{(N)} X_G \qquad \text{and} \qquad X_G H^{(N)} = H^{(N)} X_F$$

Since products are preserved in the process of passing to the symmetric tensor space, we get

$$\overline{X}_F\, \overline{H}_N = \overline{H}_N\, \overline{X}_G \qquad \text{and} \qquad \overline{X}_G\, \overline{H}_N = \overline{H}_N\, \overline{X}_F \tag{3.2}$$

the bars indicating the corresponding induced maps.

We have seen in Proposition 3.1 how to calculate $\overline{H}_N$ from the action of H on polynomials in degree N. For symmetric tensors we have the components in degree N, namely $x^{N-k}y^k$, for $0 \le k \le N$, where for convenience we write x for x_0 and y for x_1. Now consider the generating function for the elementary symmetric functions in the quantum variables f_j. This is the N-fold tensor power

$$\mathcal{F}_N(t) = (I + tF)^{\otimes N} = I^{\otimes N} + t\, X_F + \cdots$$

noting that the coefficient of t is X_F. Similarly, define

$$\mathcal{G}_N(t) = (I + tG)^{\otimes N} = I^{\otimes N} + tX_G + \cdots$$

From $(I + tF)H = H(I + tG)$ we have

$$\mathcal{F}_N H^{(N)} = H^{(N)}\mathcal{G}_N \qquad \text{and} \qquad \overline{\mathcal{F}}_N\, \overline{H}_N = \overline{H}_N\, \overline{\mathcal{G}}_N$$

The difficulty is to calculate the action on the symmetric tensors for operators, such as X_F, that are not pure tensor powers. However, from $\mathcal{F}_N(t)$ and $\mathcal{G}_N(t)$ we can recover X_F and X_G via

$$X_F = \frac{d}{dt}\bigg|_{t=0} (I + tF)^{\otimes N}, \qquad X_G = \frac{d}{dt}\bigg|_{t=0} (I + tG)^{\otimes N}$$

with corresponding relations for the barred operators. Calculating on polynomials yields the desired results as follows.

$$I + tF = \begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}, \qquad I + tG = \begin{bmatrix} 1+t & 0 \\ 0 & 1-t \end{bmatrix}$$

In degree N, using x and y as variables, we get the k^{th} component for $\overline{X}_F$ and $\overline{X}_G$ via

$$\frac{d}{dt}\bigg|_{t=0} (x + ty)^{N-k}(tx + y)^k \;=\; (N - k)\, x^{N-(k+1)}y^{k+1} + k\, x^{N-(k-1)}y^{k-1}$$

and since $I + tG$ is diagonal,

$$\frac{d}{dt}\bigg|_{t=0} (1 + t)^{N-k}(1 - t)^k\, x^{N-k}y^k \;=\; (N - 2k)\, x^{N-k}y^k\,.$$

For example, calculations for $N = 4$ result in

$$\overline{X}_F = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} \tag{3.3}$$

$$\overline{X}_G = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \tag{3.4}$$

$$\overline{H}_4 = \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \tag{3.5}$$

Since $\overline{X}_G$ is the result of diagonalizing $\overline{X}_F$, we observe that

Corollary 3.2 *The spectrum of $\overline{X}_F$ is $N, N-2, \ldots, 2-N, -N$, coinciding with the support of the classical random walk.*

Remark on the shuffling map. Notice that the top row of $(I + tF)^{\otimes N}$ is exactly $t^{w(k)}$, where $w(k)$ is the binary shuffling function of section §2. Each time one tensors with $I + tF$, the original top row is reproduced, then concatenated with a replica of itself modified in that each entry picks up a factor of t (compare with equation (2.1)). And, collapsing to the symmetric tensor space, the top row will have entries $\binom{N}{k} t^k$. This follows as well by direct calculation of the 0^{th} component matrix elements in degree N, namely by expanding $(x + ty)^N$.

We continue with some areas where Krawtchouk polynomials/matrices play a rôle, very often not explicitly recognized in the original contexts.

4 Ehrenfest urn model

In order to explain how the apparent irreversibility of the second law of thermodynamics arises from reversible statistical physics, the Ehrenfests introduced a so-called urn model, variations of which have been considered by many authors ([15, 16, 26]).

We have an urn with N balls. Each ball can be in two states represented by, say, being lead or gold. At each time $k \in \mathbb{N}$, a ball is drawn at random, changed by a Midas-like touch into the opposite state (gold $\leftrightarrow$ lead) and placed back in the urn. The question is of course about the distribution of states — and this leads to Krawtchouk matrices.

Represent the states of the model by vectors in $\mathbb{R}^{n+1}$, namely by the state of k gold balls by

$$\mathbf{v}_k = [\,0 \quad 0 \quad \cdots \quad \underset{\substack{\uparrow \\ k^{\text{th}} \text{ position}}}{1} \quad \cdots \quad 0\,]^\top \tag{4.1}$$

In the case of, say, $N = 3$, we have 4 states

$$\begin{matrix}\textsf{0 gold balls}\\ \textsf{3 lead balls}\end{matrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad \begin{matrix}\textsf{1 gold ball}\\ \textsf{2 lead balls}\end{matrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \qquad \cdots \qquad \begin{matrix}\textsf{3 gold balls}\\ \textsf{0 lead balls}\end{matrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

It is easy to see that the matrix of elementary state change in this case is

$$\begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \frac{1}{3}A^{(3)} \,,$$

and in general, we have the **Kac matrix** with off-diagonals in arithmetic progression $1, 2, 3, \ldots$ descending and ascending, respectively:

$$A^{(N)} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ N & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & N-1 & 0 & 3 & \vdots & 0 & 0 \\ 0 & 0 & N-2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & N \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

It turns out that the spectral properties of the Kac matrix involve Krawtchouk matrices, namely, the *collective solution* to the eigenvalue problem $Av = \lambda v$ is

$$A^{(N)}K^{(N)} = K^{(N)}\Lambda^{(N)}$$

where $\Lambda^{(N)}$ is the $(N{+}1)\times(N{+}1)$ diagonal matrix with entries $\Lambda^{(N)}_{ii} = N-2i$

$$\Lambda^{(N)} = \begin{bmatrix} N & & & & & \\ & N-2 & & & (*) & \\ & & N-4 & & & \\ & & & \ddots & & \\ & (*) & & & 2-N & \\ & & & & & -N \end{bmatrix}$$

the $(*)$'s denoting blocks of zeros.

To illustrate, for $N = 3$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

To see this in general, we note that, cf. equations (3.3–3.5), these are the same operators appearing in the quantum random walk model, namely, we discover that $\Lambda^{(N)} = \overline{X}_G$, $A^{(N)} = \overline{X}_F^{\top}$. Now, recalling $K^{(N)} = \overline{H}_N^{\top}$, taking transposes in equation (3.2) yields

$$A^{(N)}\,K^{(N)} = K^{(N)}\,\Lambda^{(N)} \qquad \text{and} \qquad K^{(N)}\,A^{(N)} = \Lambda^{(N)}\,K^{(N)}$$

which is the spectral analysis of $A^{(N)}$ from both the left and the right. Thus, e.g., the columns of the Krawtchouk matrix are eigenvectors of the Ehrenfest model with N balls where the k^{th} column $\mathbf{v}_k := (K_{\cdot k})$ has corresponding eigenvalue $\lambda_k = (N - 2k)/N$.

Remarks

1. Clearly, the Ehrenfest urn problem can be expressed in other terms. For instance, it can be reformulated as a random walk on an N-dimensional cube. Suppose an ant walks on the cube, choosing at random an edge to progress to the next vertex. Represent the states by vectors in $Z = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, N factors. The equivalence of the two problems comes via the correspondence of states

$$Z \ni [\ a_1\ a_2 \ldots a_N\ ] \longrightarrow \mathbf{v}_w \in \mathbb{R}^{N+1}$$

   where $w = \sum a_i$ is the weight of the vector calculated in $\mathbb{N}$, see (4.1).

2. The urn model in the appropriate limit as $N \to \infty$ leads to a diffusion model on the line, the discrete distributions converging to the diffusion densities. See Kac' article ([15]).

3. There is a rather unexpected connection of the urn model with finite-dimensional representations of the Lie algebra $sl(2) \cong so(2,1)$. Indeed, introduce a new matrix by the commutator:

$$\overline{A} = \frac{1}{2}\,[A, \Lambda]$$

   The matrix $\overline{A}$ is a skew-symmetric version of A. For $N = 3$, it is

$$\overline{A} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   It turns out that the triple A, $\overline{A}$ and Λ is closed under commutation, thus forms a Lie algebra, namely

$$\text{span}\ \{\ A,\ \overline{A},\ \Lambda\ \} \cong so(2,1) \cong sl(2,\mathbb{R})$$

with commutation relations

$$[A,\overline{A}] = 2\Lambda\,, \qquad [\overline{A},\Lambda] = 2A\,, \qquad [\Lambda, A] = -2\overline{A}$$

5 Krawtchouk matrices and classical random walks

In this section we will give a probabilistic meaning to the Krawtchouk matrices and illustrate some connections with classical random walks.

5.1 Bernoulli random walk

Let X_i be independent symmetric Bernoulli random variables taking values ± 1. Let $x_N = X_1 + \cdots + X_N$ be the associated random walk starting from 0. Now observe that the generating function of the elementary symmetric functions in the X_i is a martingale, in fact a discrete exponential martingale:

$$M_N = \prod_{i=1}^{N}(1 + vX_i) = \sum_k v^k a_k(X_1, \ldots, X_N)$$

where a_k denotes the k^{th} elementary symmetric function. The martingale property is immediate since each X_i has mean 0. Refining the notation by setting $a_k^{(N)}$ to denote the k^{th} elementary symmetric function in the variables $X_1, \ldots, X_N$, multiplying M_N by $1 + vX_{N+1}$ yields the recurrence

$$a_k^{(N+1)} = a_k^{(N)} + a_{k-1}^{(N)}\, X_{N+1}$$

which, with the boundary conditions $a_k^{(0)} = 0$, for $k > 0$, $a_0^{(n)} = 1$ for all $n \geq 0$, yields, for $k > 0$,

$$a_k^{(N+1)} = \sum_{j=0}^{N} a_{k-1}^{(j)}\, X_{j+1}$$

that is, these are discrete or *prototypical iterated stochastic integrals* and thus the simplest example of Wiener's homogeneous chaoses.

Suppose that at time N, the number of the X_i that are equal to -1 is j_N, with the rest equal to $+1$. Then $j_N = (N - x_N)/2$ and M_N can be expressed solely in terms of N and x_N, or, equivalently, of N and j_N

$$M_N = (1+v)^{N-j_N}(1-v)^{j_N} = (1+v)^{(N+x_N)/2}(1-v)^{(N-x_N)/2}$$

From the generating function for the Krawtchouk matrices, equation (1.1), follows

$$M_N = \sum_i v^i K^{(N)}_{i,j_N}$$

so that as functions on the Bernoulli space, each sequence of random variables $K^{(N)}_{i,j_N}$ is a martingale.

Now we can derive two basic recurrences. From a given column of $K^{(N)}$, to get the corresponding column in $K^{(N+1)}$, we have the Pascal's triangle recurrence:

$$K^{(N)}_{i-1\,j} + K^{(N)}_{i\,j} = K^{(N+1)}_{i\,j}$$

This follows in the probabilistic setting by writing $M_{N+1} = (1 + vX_N)M_N$ and remarking that for j to remain constant, X_N must take the value $+1$. The martingale property is more interesting in the present context. We have

$$K^{(N)}_{i\,j_N} = E(K^{(N)}_{i\,j_{N+1}}|X_1,\ldots,X_N) = \frac{1}{2}\left(K^{(N+1)}_{i\,j_N+1} + K^{(N+1)}_{i\,j_N}\right)$$

since half the time X_{N+1} is -1, increasing j_N by 1, and half the time j_N is unchanged. Thus, writing j for j_N,

$$K^{(N)}_{ij} = \frac{1}{2}\left(K^{(N+1)}_{i\,j+1} + K^{(N+1)}_{ij}\right)$$

which may be considered as a 'reverse Pascal'.

5.1.1 Orthogonality

As noted above — here with a slightly simplified notation — it is natural to use variables (x, N), with x denoting the position of the random walk after N steps. Writing $K_\alpha(x, N)$ for the Krawtchouk polynomials in these variables, cf. equation (1.2), we have the generating function

$$G(v) = \sum_{\alpha=0}^{N} v^\alpha K_\alpha(x, N) = (1+v)^{(N+x)/2}(1-v)^{(N-x)/2}$$

The expansion

$$(1-v)^{y-a}(1-(1-R)v)^{-y} = \sum_{n=0}^{\infty} \frac{v^n}{n!}\,(a)_n\, {}_2F_1\left(\begin{matrix}-n, y\\ a\end{matrix}\,\middle|\, R\right) \tag{5.1}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$, yields the identification as hypergeometric functions

$$K_\alpha(x,N) = \binom{N}{\alpha}\, {}_2F_1\left(\begin{matrix}-\alpha, (x-N)/2\\ -N\end{matrix}\,\middle|\, 2\right)$$

The calculation

$$\langle G(v)\, G(w)\rangle = \prod \langle 1 + (v+w)X_j + vwX_j^2\rangle = (1+vw)^N$$

exhibits the orthogonality of the K_α if one observes that after taking expectations only terms in the product vw remain. Thus, the K_α are notable for two important features:

1. They are the iterated integrals (sums) of the Bernoulli process.
2. They are orthogonal polynomials with respect to the binomial distribution.

5.2 Multivariate Krawtchouk polynomials

The probabilistic approach may be carried out for general finite probability spaces. Fix an integer $d > 0$ and d values $\{\xi_0, \ldots, \xi_\delta\}$, with the convention $\delta = d-1$. Take a sequence of independent identically distributed random variables having distribution $P(X = \xi_j) = p_j$, $0 \le j \le \delta$. Denote the mean and variance of the X_i by μ and σ^2 as usual.

For $N > 0$, we have the martingale

$$M_N = \prod_{j=1}^{N} (1 + v(X_j - \mu))$$

We now switch to the multiplicities as variables. Set

$$n_j = \sum_{k=1}^{N} \mathbf{1}_{\{X_k = \xi_j\}}$$

the number of times the value ξ_j is taken. Thus the generating function

$$G(v) = \prod_{j=0}^{\delta}(1 + v(\xi_j - \mu))^{n_j} = \sum_{\alpha=0}^{N} v^\alpha K_\alpha(n_0, \ldots, n_\delta)$$

defines our generalized Krawtchouk polynomials. One quickly gets

Proposition 5.1 *Denoting the multi-index* $\mathbf{n} = (n_0, \ldots, n_\delta)$ *and by* $\mathbf{e_j}$ *the standard basis on* $\mathbb{Z}^d$*, Krawtchouk polynomials satisfy the recurrence*

$$K_\alpha(\mathbf{n} + \mathbf{e_j}) = K_\alpha(\mathbf{n}) + (\xi_j - \mu)K_{\alpha-1}(\mathbf{n})$$

We also find by binomial expansion

Proposition 5.2

$$K_\alpha(n_0, \ldots, n_\delta) = \sum_{|\mathbf{k}|=\alpha} \prod_j \binom{n_j}{k_j} (\xi_j - \mu)^{k_j}$$

where $|\mathbf{k}| = \sum_{j=0}^{\delta} k_j$.

There is an interesting connection with the multivariate hypergeometric functions of Appell and Lauricella. The Lauricella polynomials F_B are defined by

$$F_B\left(\begin{matrix}-\mathbf{r}, \mathbf{b} \\ t\end{matrix} \;\middle|\; \mathbf{s}\right) = \sum_{\mathbf{k}\in\mathbb{N}^\delta} \frac{(-\mathbf{r})_\mathbf{k}(\mathbf{b})_\mathbf{k}}{(t)_{|\mathbf{k}|}\mathbf{k}!} \mathbf{s}^\mathbf{k}$$

with, e.g., $\mathbf{r} = (r_1, \ldots, r_\delta)$, $(\mathbf{r})_\mathbf{k} = (r_1)_{k_1}(r_2)_{k_2} \cdots (r_\delta)_{k_\delta}$ for multi-index $\mathbf{k}$, also $\mathbf{s}^\mathbf{k} = s_1^{k_1} \cdots s_\delta^{k_\delta}$, and $\mathbf{k}! = k_1! \cdots k_\delta!$. Note that t is a single variable. The generating function of interest here is

$$(1 - \sum v_i)^{\sum b_j - t} \prod_j (1 - \sum v_i + s_j v_j)^{-b_j} = \sum_{\mathbf{r}\in\mathbb{N}^\delta} \frac{\mathbf{v}^\mathbf{r}(t)_{|\mathbf{r}|}}{\mathbf{r}!} F_B\left(\begin{matrix}-\mathbf{r}, \mathbf{b} \\ t\end{matrix} \;\middle|\; \mathbf{s}\right) \tag{5.2}$$

a multivariate version of (5.1).

Proposition 5.3 *Let* $N = |\mathbf{n}|$*. If* $\xi_0 = 0$*, then,*

$$K_\alpha(\mathbf{n}) = (-N)_\alpha \sum_{|\mathbf{r}|=\alpha} \frac{\prod (p_j\xi_j)^{r_j}}{\mathbf{r}!} F_B\left(\begin{matrix}-\mathbf{r}, -\mathbf{n} \\ -N\end{matrix} \;\middle|\; \frac{1}{p_1}, \ldots, \frac{1}{p_\delta}\right)$$

Proof Let $v_j = vp_j\xi_j$, $b_j = -n_j$, $t = -N$, $s_j = p_j^{-1}$ in (5.2), for $1 \le j \le \delta$. Note that $\sum v_j = v\mu$, $\sum b_j - t = N - (\sum_{1\le j\le\delta} n_j) = n_0$. $\square$

Orthogonality follows similar to the binomial case:

Proposition 5.4 *The Krawtchouk polynomials $K_\alpha(n_0,\ldots,n_\delta)$ are orthogonal with respect to the induced multinomial distribution. In fact, with $N = |\mathbf{n}|$,*

$$\langle K_\alpha K_\beta\rangle = \delta_{\alpha\beta}\,\sigma^{2\alpha}\binom{N}{\alpha}$$

Proof

$$\begin{aligned}\langle G(v)\,G(w)\rangle &= \sum \binom{N}{n_0,\ldots,n_\delta} p_0^{n_0}\cdots p_\delta^{n_\delta}\prod\left(1+(v+w)(\xi_j-\mu)+vw(\xi_j-\mu)^2\right)^{n_j} \\ &= \left(\sum\left(p_j+(v+w)p_j(\xi_j-\mu)+vwp_j(\xi_j-\mu)^2\right)\right)^N\end{aligned}$$

Thus, $\langle G(v)\,G(w)\rangle = (1+vw\sigma^2)^N$. This shows orthogonality and yields the squared norms as well. $\square$

6 "Kravchukiana" or the World of Krawtchouk Polynomials

About the year 1995, we held a seminar on Krawtchouk polynomials at Southern Illinois University. As we continued, we found more and more properties and connections with various areas of mathematics.

Eventually, by the year 2000 the theory of quantum computing had been developing with serious interest in the possibility of implementation, at the present time of MUCH interest. Sure enough, right in the middle of everything there are our flip operators, su(2), etc., etc. — same ingredients making up the Krawtchouk universe. Well, we can only report that how this all fits together is still quite open. Of special note is the idea of a hardware implementation of a Krawtchouk transform. A beginning in this direction may be found in the just-published article with Schott, Botros, and Yang [3].

At any rate, for the present we list below the topics which are central to our program. They are the basis of the **Krawtchouk Encyclopedia**, still in development; we are in the process of filling in the blanks. An extensive web resource for Krawtchouk polynomials we recommend is Zelenkov's site:

`http://www.geocities.com/orthpol/`

Note that we do not mention work in areas less familiar to us, notably that relating to q-Krawtchouk polynomials, such as in [23].

We welcome contributions. If you wish either to send a reference to your paper(s) on Krawtchouk polynomials or contribute an article, please contact one of us !

Our email: `pfeinsil@math.siu.edu` or `jkocik@math.siu.edu`.

6.1 Krawtchouk Encyclopedia

Here is a list of topics currently in the Krawtchouk Encyclopedia.

1. Pascal's Triangle
2. Random Walks
   - Path integrals
   - A, K, and Λ
   - Nonsymmetric Walks
   - Symmetric Krawtchouk matrices and binomial expectations
3. Urn Model
   - Markov chains
   - Initial and invariant distributions
4. Symmetric Functions. Energy

- Elementary symmetric functions and determinants
- Traces on Grassman algebras

5. Martingales

    - Iterated integrals
    - Orthogonal functionals
    - Krawtchouk polynomials and multinomial distribution

6. Lie algebras and Krawtchouk polynomials

    - so(2,1) explained
    - so(2,1) spinors
    - Quaternions and Clifford algebras
    - S and so(2,1) tensors
    - Three-dimensional simple Lie algebras

7. Lie Groups. Reflections

    - Reflections
    - Krawtchouk matrices as group elements

8. Representations

    - Splitting formula
    - Hilbert space structure

9. Quantum Probability and Tensor Algebra

    - Flip operator and quantum random walk
    - Krawtchouk matrices as eigenvectors
    - Trace formulas. MacMahon's Theorem
    - Chebyshev polynomials

10. Heisenberg Algebra

- Representations of the Heisenberg algebra
- Raising and velocity operator. Number operator
- Evolution structure. Hamiltonian.
- Time-zero polynomials

11. Central Limit Theorem

    - Hermite polynomials
    - Discrete stochastic differential equations

12. Clebsch-Gordan Coefficients

    - Clebsch-Gordan coefficients and Krawtchouk polynomials
    - Racah coefficients

13. Orthogonal Polynomials

    - Three-term recurrence in terms of A, K, Lambda
    - Nonsymmetric case

14. Krawtchouk Transforms

    - Orthogonal transformation associated to K
    - Exponential function in Krawtchouk basis
    - Krawtchouk transform

15. Hypergeometric Functions

    - Krawtchouk polynomials as hypergeometric functions
    - Addition formulas

16. Symmetric Krawtchouk Matrices

    - The matrix T
    - S-squared and trace formulas
    - Spectrum of S

17. Gaussian Quadrature
    - Zeros of Krawtchouk polynomials
    - Gaussian-Krawtchouk summation
18. Coding Theory
    - MacWilliams' theorem
    - Association schemes
19. Appendices
    - K and S matrices for N from 1 to 14
    - Krawtchouk polynomials in the variables x,N/i,j/j,N for N from 1 to 20
    - Eigenvalues of S
    - Remarks on the multivariate case
    - Time-zero polynomials
    - Mikhail Philippovitch Krawtchouk: a biographical sketch

7 Appendix

7.1 Krawtchouk matrices

$$K^{(0)} = \begin{bmatrix} 1 \end{bmatrix}$$

$$K^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$K^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$K^{(3)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$K^{(4)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

$$K^{(5)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 3 & 1 & -1 & -3 & -5 \\ 10 & 2 & -2 & -2 & 2 & 10 \\ 10 & -2 & -2 & 2 & 2 & -10 \\ 5 & -3 & 1 & 1 & -3 & 5 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

$$K^{(6)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & -20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Table 1

7.2 Symmetric Krawtchouk matrices

$$S^{(0)} = \begin{bmatrix} 1 \end{bmatrix}$$

$$S^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$S^{(3)} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & 3 & -3 & -3 \\ 3 & -3 & -3 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

$$S^{(4)} = \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 8 & 0 & -8 & -4 \\ 6 & 0 & -12 & 0 & 6 \\ 4 & -8 & 0 & 8 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

$$S^{(5)} = \begin{bmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ 5 & 15 & 10 & -10 & -15 & -5 \\ 10 & 10 & -20 & -20 & 10 & 10 \\ 10 & -10 & -20 & 20 & 10 & -10 \\ 5 & -15 & 10 & 10 & -15 & 5 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

$$S^{(6)} = \begin{bmatrix} 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ 6 & 24 & 30 & 0 & -30 & -24 & -6 \\ 15 & 30 & -15 & -60 & -15 & 30 & 15 \\ 20 & 0 & -60 & 0 & 60 & 0 & -20 \\ 15 & -30 & -15 & 60 & -15 & -30 & 15 \\ 6 & -24 & 30 & 0 & -30 & 24 & -6 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix}$$

Table 2

7.3 Sylvester-Hadamard matrices

$$H^{(0)} = \begin{bmatrix} \bullet \end{bmatrix}$$

$$H^{(1)} = \begin{bmatrix} \bullet & \bullet \\ \bullet & \circ \end{bmatrix}$$

$$H^{(3)} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \bullet \end{bmatrix}$$

$$H^{(4)} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ \\ \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ & \bullet \\ \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ \end{bmatrix}$$

$$H^{(5)} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ \\ \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ & \bullet \\ \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet \\ \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \bullet & \circ & \circ & \bullet \end{bmatrix}$$

Table 3

Replace $\bullet$ with 1 and $\circ$ with -1 to obtain Sylvester-Hadamard matrices.

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