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A Note on Krawtchouk Polynomials and Riordan Arrays

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Abstract

We study links between Krawtchouk polynomials and Riordan arrays of both the ordinary kind and the exponential kind. We derive summation formulas for values of the Krawtchouk polynomials using the A-sequences of the Riordan arrays.

1 Introduction

The Krawtchouk polynomials play an important role in various areas of mathematics. Notable applications occur in coding theory [11], association schemes [4], and in the theory of group representations [21, 22].

In this note, we explore links between the Krawtchouk polynomials and Riordan arrays, of both ordinary and exponential type, and we study integer sequences defined by evaluating the Krawtchouk polynomials at different values of their parameters.

The link between Krawtchouk polynomials and exponential Riordan arrays is implicitly contained in the umbral calculus approach to certain families of orthogonal polynomials. We shall look at these links explicitly in the following.

The structure of this note is as follows. In the next section, we shall give a brief introduction to the relevant theory of both ordinary and exponential Riordan arrays. We then define the Krawtchouk polynomials, using exponential Riordan arrays, and look at some general properties of these polynomials from this perspective. We then show that for different values of the parameters used in the definition of the Krawtchouk polynomials, there exist interesting families of (ordinary) Riordan arrays.

2 Riordan arrays

The *Riordan group* [12, 17, 19], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1x + g_2x^2 + \dots$ and $f(x) = f_1x + f_2x^2 + \dots$ where $f_1 \neq 0$. We sometimes write $f(x) = xh(x)$ where $h(0) \neq 0$. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by (g, f) or $\mathcal{R}(g, f)$, and is often called the *Riordan array* defined by g and f. When $g_0 = 1$, the array is called a *monic* Riordan array. The group law is given by

$$(g, f) * (u, v) = (g(u \circ f), v \circ f). \tag{1}$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

To each Riordan array as defined above is associated an integer sequence $A = \{a_i\}$ with $a_0 \neq 0$ such that every element $d_{n+1,k+1}$ of the array (not lying in column 0 or row 0) can be expressed as a linear combination with coefficients in A of the elements in the preceding row, starting from the preceding column:

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots$$

$A = \{a_i\}$ is called the A-sequence of the array, and its generating function may be calculated according to

$$A(x) = [h(t)|t = xh(t)^{-1}].$$

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence u_n, is called the *Appell array* (or sometimes the *sequence array*) of the sequence u_n. Its general term is u_{n-k}.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{u} = (u_0, u_1, \dots)'$ is an integer sequence with ordinary generating function $\mathcal{U}(x)$, then the sequence $\mathbf{Mu}$ has ordinary generating function $g(x)\mathcal{U}(f(x))$.

Example 1. The binomial matrix $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$. For this matrix we have $A(x) = 1 + x$, which translates the usual defining relationship for Pascal's triangle

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

Example 2. We let $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ A000108. The array $(1, xc(x))$ is the inverse of the array $(1, x(1-x))$ while the array $(1, xc(x^2))$ is the inverse of the array $(1, \frac{x}{1+x^2})$.

Example 3. The row sums of the matrix (g, f) have generating function $g(x)/(1-f(x))$ while the diagonal sums of (g, f) have generating function $g(x)/(1-xf(x))$. The row sums of the array $(1, xc(x))$, or A106566, are the Catalan numbers C_n since $\frac{1}{1-xc(x)} = c(x)$. The diagonal sums have g.f. $\frac{1}{1-x^2c(x)}$, A132364.

The *exponential Riordan group* [3, 6, 7], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1x + g_2x^2 + \ldots$ and $f(x) = f_1x + f_2x^2 + \ldots$ where $f_1 \neq 0$. The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. It is *monic* if $g_0 = 1$. The group law is then given by

$$[g, f] * [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = \{u_n\}$ is an integer sequence with exponential generating function $\mathcal{U}\ (x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g, f]$ are given by $g(x)e^{f(x)}$ since the sequence $1, 1, 1, \ldots$ has exponential generating function e^x.

As an element of the group of exponential Riordan arrays, we have $\mathbf{B} = [e^x, x]$. By the above, the exponential generating function of its row sums is given by $e^xe^x = e^{2x}$, as expected.

Riordan group techniques have been used to provide alternative proofs of many binomial identities that originally appeared in works such as [13, 14]. See, for instance, [20, 19].

3 Krawtchouk polynomials

We follow [15] in defining the Krawtchouk polynomials. They form an important family of orthogonal polynomials [5, 16, 23]. Thus the Krawtchouk polynomials will be considered to be the special case $\beta = -N$, $c = \frac{p}{p-1}$, $p + q = 1$ of the Meixner polynomials of the first kind, which form the Sheffer sequence for

$$\begin{aligned} g(t) &= \left(\frac{1-c}{1-ce^t}\right)^{\beta}, \\ f(t) &= \frac{1-e^t}{c^{-1}-e^t}. \end{aligned}$$

Essentially, this says that the Meixner polynomials of the first kind are obtained by operating on the vector $(1, x, x^2, x^3, \ldots)'$ by the exponential Riordan array $[g(t), f(t)]^{-1}$, since

$$[g, f]^{-1} = \left[\frac{1}{g \circ \bar{f}}, \bar{f}\right]$$

and

$$\left[\frac{1}{g \circ \bar{f}}, \bar{f}\right] e^{xt} = \frac{1}{g \circ \bar{f}} e^{x\bar{f}(t)}$$

which is the defining expression for the Sheffer sequence associated to g and f. In order to work with this expression, we calculate $[g, f]^{-1}$ as follows. Firstly,

$$\bar{f} = \log\left(\frac{t-c}{c(t-1)}\right)$$

since

$$\begin{aligned} \frac{1-e^u}{c^{-1}-u} = x \quad &\Longrightarrow \quad e^u = \frac{x-c}{c(x-1)} \\ u = \log\left(\frac{x-c}{c(x-1)}\right) \quad &\Longrightarrow \quad \bar{f}(t) = \log\left(\frac{t-c}{c(t-1)}\right) \end{aligned}$$

Then we have

$$g(\bar{f}(t)) = \left(\frac{1-c}{1-ce^{\bar{f}(t)}}\right)^\beta = \left(\frac{1-c}{1-\frac{t-c}{t-1}}\right)^\beta = (1-t)^\beta.$$

and

$$e^{x\bar{f}(t)} = e^{x\log\left(\frac{t-c}{c(t-1)}\right)} = \left(\frac{t-c}{c(t-1)}\right)^x.$$

Thus we arrive at

$$[g, f]^{-1} = \left[\frac{1}{(1-t)^\beta}, \log\left(\frac{t-c}{c(t-1)}\right)\right]$$

and

$$\begin{aligned} \frac{e^{x\bar{f}(t)}}{g(\bar{f}(t))} &= \frac{1}{(1-t)^\beta}\left(\frac{t-c}{c(t-1)}\right)^x \\ &= \frac{1}{(1-t)^{\beta+x}}\left(\frac{c-t}{c}\right)^x \\ &= (1-t)^{-\beta-x}\left(1-\frac{t}{c}\right)^x. \end{aligned}$$

Specializing to the values $\beta = -N$ and $c = \frac{p}{p-1} = -\frac{p}{q}$, we get

$$\frac{e^{x\bar{f}(t)}}{g(\bar{f}(t))} = (1-t)^{N-x}(1+\frac{q}{p}t)^x.$$

Extracting the coefficient of t^k in this expression, we obtain

$$\begin{aligned} [t^k]\frac{e^{x\bar{f}(t)}}{g(\bar{f}(t))} &= [t^k]\sum_{i=0}\binom{N-x}{i}(-1)^i t^i \sum_{j=0}\binom{x}{j}\left(\frac{q}{p}\right)^j t^j \\ &= \sum_{j=0}^{k}\binom{N-x}{k-j}\binom{x}{j}(-1)^{k-j}q^j p^{-j}. \end{aligned}$$

Scaling by p^k, we thus obtain

$$p^k[x^k]\frac{e^{x\bar{f}(t)}}{g(\bar{f}(t))} = \sum_{j=0}^{k} \binom{N-x}{k-j}\binom{x}{j}(-1)^{k-j}q^j p^{k-j}.$$

We use the notation

$$\kappa_n^{(p)}(x,N) = \sum_{j=0}^{n} \binom{N-x}{n-j}\binom{x}{j}(-1)^{n-j}q^j p^{n-j}$$

for the Krawtchouk polynomial with parameters N and p. This can be expressed in hypergeometric form as

$$\kappa_n^{(p)}(x,N) = (-1)^n \binom{N}{n} p^n {}_2F_1(-n,-x;-N;1/p).$$

The form of $[g,f]^{-1}$ allows us to make some interesting deductions. For instance, if we write

$$[g(t),f(t)]^{-1} = \left[\frac{1}{(1-t)^{\beta}}, \log\left(\frac{1-\frac{t}{c}}{1-t}\right)\right]$$

then setting $\beta=-N$ and $c=\frac{p}{p-1}$, we get

$$[g(t),f(t)]^{-1} = \left[\frac{1}{(1-t)^{-N}}, \log\left(\frac{1-\frac{p-1}{p}t}{1-t}\right)\right].$$

Now we let $t=ps$, giving

$$\begin{aligned}
[g(t),f(t)]^{-1} &= \mathrm{Diag}(1/p^n) * \left[(1-ps)^N, \log\left(\frac{1-(p-1)s}{1-ps}\right)\right] \\
&= \mathrm{Diag}(1/p^n) * [(1-ps)^N, s] * \left[1, \frac{s}{1-(p-1)s}\right] * \left[1, \log\left(\frac{1}{1-s}\right)\right] \\
&= \mathrm{Diag}(1/p^n) * \mathbf{P}[p]^{-N} * \mathbf{Lah}[p-1] * \mathbf{s}.
\end{aligned}$$

where we have used the notation of [2] and where for instance $\mathbf{s} = \left[1, \log\left(\frac{1}{1-s}\right)\right]$ is the Stirling array of the first kind.

The matrix $\mathbf{P}[p]^{-N} * \mathbf{Lah}[p-1] * \mathbf{s} = \left[(1-pt)^N, \log\left(\frac{1-(p-1)t}{1-pt}\right)\right]$ is of course a monic exponential Riordan array. If its general term is $T(n,k)$, then that of the corresponding array $[g,f]^{-1}$ is given by $T(n,k)/p^n$.

The above matrix factorization indicates that the Krawtchouk polynomials can be expressed as combinations of the Stirling polynomials of the first kind $1, x, x(x+1), x(x^2+3x+2), x(x^3+6x^2+11x+6), \ldots$.

Example 4. Taking $N=-1$ and $p=2$ we exhibit an interesting property of the matrix $\left[(1-pt)^N, \log\left(\frac{1-(p-1)t}{1-pt}\right)\right]$, which in this case is the matrix $\left[\frac{1}{1-2t}, \log\left(\frac{1-t}{1-2t}\right)\right]$. An easy calculation shows that

$$\left[\frac{1}{1-2t}, \log\left(\frac{1-t}{1-2t}\right)\right]^{-1} = \left[\frac{1}{2e^t-1}, \frac{e^t-1}{2e^t-1}\right].$$

We recall that the Binomial matrix with general term $\binom{n}{k}$ is the Riordan array $[e^t, t]$. Now

$$\left[\frac{1}{1-2t}, \log\left(\frac{1-t}{1-2t}\right)\right][e^t, t]\left[\frac{1}{2e^t-1}, \frac{e^t-1}{2e^t-1}\right] = \left[\frac{1-t}{1-2t}, t\right].$$

Hence the matrices $[e^t, t]$ and $\left[\frac{1-t}{1-2t}, t\right]$ are similar, with $\left[\frac{1}{1-2t}, \log\left(\frac{1-t}{1-2t}\right)\right]$ serving as matrix of change of basis for the similarity.

4 Krawtchouk polynomials and Riordan arrays

In this section, we shall use the following notation, where we define a variant on the polynomial family $\kappa_n^{(p)}(x, N)$. Thus we let

$$K(n, k, x, q) = \sum_{j=0}^{k} (-1)^j \binom{x}{j}\binom{n-x}{k-j}(q-1)^{k-j}.$$

We then have

$$K(n, k, x, q) = [t^k](1-t)^x(1+(q-1)t)^{n-x},$$

which implies that

$$K(N, k, N-x, q) = [t^k](1-t)^{N-x}(1+(q-1)t)^x.$$

Letting $P = 1/q$ and thus $(1-P)/P = q-1$ we obtain

$$K(N, k, N-x, q) = \frac{1}{q^n}\kappa_n^{(P)}(x, N).$$

We shall see in the sequel that by varying the parameters n, k, x and q, we can obtain families of (ordinary) Riordan arrays defined by the corresponding Krawtchouk expressions.

Example 5. We first look at the term $K(k, n-k, r, q)$. We have

$$\begin{aligned} K(k, n-k, r, q) &= \sum_{j=0}^{n-k} (-1)^j \binom{r}{j}\binom{k-r}{n-k-j}(q-1)^{n-k-j} \\ &= \sum_{j=0}^{r} (-1)^j \binom{r}{j}\binom{k-r}{n-k-j}(q-1)^{n-k-j}. \end{aligned}$$

But this last term is the general term of the Riordan array

$$\left(\left(\frac{1-x}{1+(q-1)x}\right)^r, x(1+(q-1)x)\right). \tag{2}$$

The term $(-1)^{n-k}K(k, n-k, r, q)$ then represents the general term of the inverse of this Riordan array, which is given by

$$\left(\left(\frac{1+x}{1-(q-1)x}\right)^r, x(1-(q-1)x)\right).$$

The A-sequence of the array (2) is given by

$$A(x) = \frac{1+\sqrt{1+4(q-1)x}}{2}.$$

Thus

$$a_0 = 1, \qquad a_n = (-1)^{n-1}(q-1)^n C_{n-1}.$$

With these values, we therefore have

$$K(k+1,n-k,r,q) = K(k,n-k,r,q)+a_1K(k+1,n-k-1,r,q)+a_2K(k+2,n-k-2,r,q)+\ldots$$

Example 6. We next look at the family defined by $(-1)^k K(n,k,k,q)$. We have

$$\begin{aligned}(-1)^k K(n,k,k,q) &= (-1)^k \sum_{j=0}^{k}(-1)^j \binom{k}{j}\binom{n-k}{k-j}(q-1)^{k-j} \\ &= \sum_{j=0}^{k} \binom{k}{j}\binom{n-k}{k-j}(1-q)^{k-j} \\ &= \sum_{j=0}^{n-k} \binom{k}{j}\binom{n-k}{j}(1-q)^{j}.\end{aligned}$$

Using the results of [1], we see that these represent the family of Riordan arrays

$$\left(\frac{1}{1-x}, \frac{x(1-qx)}{1-x}\right).$$

The A-sequence for this array is given by

$$A(x) = \frac{1+x+\sqrt{1+2x(1-2q)+x^2}}{2}.$$

For example, the matrix with general term $T(n,k) = (-1)^k K(n,k,k,-3)$ is the Riordan array $\left(\frac{1}{1-x}, \frac{x(1+3x)}{1-x}\right)$, A081578 or

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 5 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 9 & 9 & 1 & 0 & 0 & \ldots \\ 1 & 13 & 33 & 13 & 1 & 0 & \ldots \\ 1 & 17 & 72 & 73 & 7 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

The A-sequence for this array has g.f. $\frac{1+x+\sqrt{1+14x+x^2}}{2}$ which expands to

$$1, 4, -12, 84, -732, 7140, -74604, \ldots$$

Thus

$$\begin{aligned}(-1)^{k+1}K(n+1,k+1,k+1,-3) &= (-1)^k K(n,k,k,-3)\\ &\quad +4(-1)^{k+1}K(n,k+1,k+1,-3)\\ &\quad -12(-1)^{k+2}K(n,k+2,k+2,-3)+\dots\end{aligned}$$

The matrix with general term $(-1)^k K(n,k,k,2)$ is the Riordan array $\left(\frac{1}{1-x}, \frac{x(1-2x)}{1-x}\right)$ or

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & -1 & 1 & 0 & 0 & \dots \\ 1 & -2 & -2 & -2 & 1 & 0 & \dots \\ 1 & -3 & -2 & -2 & -3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The rows of this matrix A098593 are the anti-diagonals (and a signed version of the diagonals) of the so-called Krawtchouk matrices [8, 9] which are defined as the family of $(N+1)\times(N+1)$ matrices with general term

$$K_{ij}^{(N)} = \sum_k (-1)^k \binom{j}{k}\binom{N-j}{i-k}.$$

The matrix with general term $T(n,k) = (-1)^k K(n,k,k,-1)$ is the well-known Delannoy number triangle $\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$ A008288 given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 5 & 5 & 1 & 0 & 0 & \dots \\ 1 & 7 & 13 & 7 & 1 & 0 & \dots \\ 1 & 9 & 25 & 25 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus in particular $(-1)^n K(2n,n,n,-1)$ is the general term of the sequence of Delannoy numbers $1, 3, 13, 63, \dots$ A001850. We have

Proposition 7. *The array with general term $T(n,k) = [k \le n](-1)^k K(n,k,k,q)$ is the Riordan array $\left(\frac{1}{1-x}, \frac{x(1-qx)}{1-x}\right)$.*

Example 8. We now turn our attention to the expression $(-1)^{n-k}K(n-k,n-k,n,q)$.

We have

$$\begin{aligned}
(-1)^{n-k}K(n-k,n-k,n,q) &= (-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{n}{j}\binom{n-k-n}{n-k-j}(q-1)^{n-k-j}\\
&= (-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{n}{j}\binom{-k}{n-k-j}(q-1)^{n-k-j}\\
&= \sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n}{j}\binom{n-j-1}{n-k-j}(q-1)^{n-k-j}\\
&= \sum_{j=0}^{n-k}\binom{n-j-1}{n-k-j}q^{n-k-j}\\
&= [x^n]\frac{1}{1-x}\left(\frac{x}{1-qx}\right)^k.
\end{aligned}$$

Thus the matrix with the general term $T(n,k;q)=(-1)^{n-k}K(n-k,n-k,n,q)$ is the Riordan array $\left(\frac{1}{1-x},\frac{x}{1-qx}\right)$. Taking the q-th inverse binomial transform of this array, we obtain

$$\left(\frac{1}{1+qx},\frac{x}{1+qx}\right)*\left(\frac{1}{1-x},\frac{x}{1-qx}\right)=\left(\frac{1}{1+(q-1)x},x\right).$$

Reversing this equality gives us

$$\left(\frac{1}{1-x},\frac{x}{1-qx}\right)=\left(\frac{1}{1-qx},\frac{x}{1-qx}\right)*\left(\frac{1}{1+(q-1)x},x\right).$$

Thus

$$(-1)^{n-k}K(n-k,n-k,n,q)=\sum_{j=k}^{n}\binom{n}{j}q^{n-j}(1-q)^{j-k}.$$

The row sums of the Riordan array $\left(\frac{1}{1-x},\frac{x}{1-qx}\right)$ have generating function

$$\frac{\frac{1}{1-x}}{1-\frac{x}{1-qx}}=\frac{1-qx}{(1-x)(1-(q+1)x)}.$$

This is thus the generating function of the sum

$$\sum_{k=0}^{n}(-1)^{n-k}K(n-k,n-k,n,q)=\sum_{k=0}^{n}\sum_{j=k}^{n}\binom{n}{j}q^{n-j}(1-q)^{j-k}=\frac{(1+q)^n-(1-q)}{q}.$$

We remark that $(-1)^kK(k,k,n,q)$ is a triangle given by the reverse of the Riordan array $\left(\frac{1}{1-x},\frac{x}{1-qx}\right)$, and will thus have the same row sums and central coefficients.

The A-sequence of this array is simply $1+qx$, which implies that

$$K(n-k,n-k,n+1,q)=-K(n-k-1,n-k-1,n,q)+qK(n-k-2,n-k-2,n,q).$$

Example 9. We now consider the expression $(-1)^{n-k}K(n-k,n-k,k,q)$. We have

$$\begin{aligned}(-1)^{n-k}K(n-k,n-k,k,q) &= (-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{k}{j}\binom{n-k-k}{n-k-j}(q-1)^{n-k-j}\\ &= \sum_{j=0}^{n-k}\binom{k}{j}\binom{n-2k}{n-k-j}(1-q)^{n-k-j}.\end{aligned}$$

This is the (n,k)-th element $T(n,k;q)$ of the Riordan array

$$\left(\frac{1}{1+(q-1)x}, x(1+qx)\right).$$

Other expressions for $T(n,k;q)$ include

$$\begin{aligned}T(n,k;q) &= \sum_{j=0}^{n-k}\binom{k}{n-k-j}(1-q)^j q^{n-k-j}\\ &= \sum_{j=0}^{n-k}\sum_{i=0}^{k}\binom{k}{i}\binom{k-i}{n-k-i-j}(-1)^j(q-1)^{i+j},\end{aligned}$$

hence these provide alternative expressions for $(-1)^{n-k}K(n-k,n-k,k,q)$.

We note that for $q=1$, we obtain the Riordan array $(1,x(1+x))$ whose inverse is the array $(1,xc(x))$. The row sums of $(1,x(1+x))$ are $F(n+1)$, thus giving us

$$\sum_{k=0}^{n}(-1)^{n-k}K(n-k,n-k,k,1) = F(n+1).$$

Similarly, we find

$$\sum_{k=0}^{n}(-1)^{n-k}K(n-k,n-k,k,0) = n+1.$$

$\sum_{k=0}^{n}(-1)^{n-k}K(n-k,n-k,k,-1)$ is the sequence $1,3,6,11,21,42,\ldots$ A024495 with generating function $\frac{1}{(1-x)^3-x^3}$.

These matrices have the interesting property that $T(2n,n;q)=1$. This is so since

$$\begin{aligned}T(2n,n;q) &= \sum_{j=0}^{2n-n}\binom{n}{2n-n-j}(1-q)^j q^{2n-n-j}\\ &= \sum_{j=0}^{n}\binom{n}{n-j}(1-q)^j q^{n-j}\\ &= \sum_{j=0}^{n}\binom{n}{j}(1-q)^j q^{n-j}\\ &= (1-q+q)^n = 1.\end{aligned}$$

Thus we have

$$K(n,n,n,q)=(-1)^n.$$

The A-sequence for these arrays has generating function

$$A(x)=\frac{1+\sqrt{1+4qx}}{2}$$

and thus we have

$$a_0=1, \qquad a_n=(-1)^{n-1}q^n C_{n-1}, \qquad n>0.$$

With these values we therefore have

$$\begin{aligned}(-1)^{n-k}K(n-k,n-k,k+1,q) &= (-1)^{n-k}K(n-k,n-k,k,q)\\ &\quad +a_1(-1)^{n-k-1}K(n-k-1,n-k-1,k+1,q)+\ldots\end{aligned}$$

Example 10. We next look at the expression $(-1)^{n-k}K(n,n-k,k,q)$. We have

$$\begin{aligned}(-1)^{n-k}K(n,n-k,k,q) &= (-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{k}{j}\binom{n-k}{n-k-j}(q-1)^{n-k-j}\\ &= \sum_{j=0}^{n-k}\binom{k}{j}\binom{n-k}{n-k-j}(1-q)^{n-k-j}.\end{aligned}$$

This is the general term $T(n,k;q)$ of the Riordan array

$$\left(\frac{1}{1+(q-1)x}, \frac{x(1+qx)}{1+(q-1)x}\right).$$

Expressing $T(n,k;q)$ differently allows us to write

$$\sum_{j=0}^{n-k}\binom{k}{j}\binom{n-k}{n-k-j}(1-q)^{n-k-j}=\sum_{j=0}^{k}\binom{n}{j}\binom{n-j}{n-k-j}q^j(1-q)^{n-k-j}.$$

The central coefficients of these arrays, $T(2n,n;q)$, have generating function $e^{(2-q)x}I_0(2\sqrt{1-q}x)$ and represent the n-th terms in the expansion of $(1+(2-q)x+(1-q)x^2)^n$.

The A-sequence for this family of arrays has generating function

$$A(x)=\frac{1+(1-q)x+\sqrt{1+2x(1+q)+(q-1)^2x^2}}{2}.$$

Expanding this as $a_0,a_1,a_2,\ldots$ we thus obtain

$$(-1)^{n-k}K(n+1,n-k,k+1,q)=a_0(-1)^{n-k}K(n,n-k,k,q)+a_1(-1)^{n-k-1}K(n,n-k,k+1,q)\ldots$$

Example 11. The expression $K(n, n-k, N, q)$ is the general term of the Riordan array

$$\left(\frac{(1-qx)^N}{1-(q-1)x}, \frac{x}{1-(q-1)x}\right).$$

This implies that

$$\sum_{j=0}^{n-k}\binom{N}{j}\binom{n-N}{n-k-j}(-1)^j(q-1)^{n-k-j} = \sum_{j=0}^{n-k}\binom{N}{j}\binom{n-j}{n-k-j}(-1)^j q^j (q-1)^{n-k-j}.$$

The A-sequence for this family of arrays is given by $1+(q-1)x$. Thus we obtain

$$K(n+1, n-k, N, q) = K(n, n-k, N, q) + (q-1)K(n, n-k-1, N, q).$$

Example 12. In this example, we indicate that summing over one of the parameters can still lead to a Riordan array. Thus the expression

$$\sum_{i=0}^{n-k}(-1)^i K(n-k, i, n, q)$$

is equivalent to the general term of the Riordan array

$$\left(\frac{1}{1-2x}, \frac{x}{1-qx}\right)$$

while the expression

$$\sum_{i=0}^{n-k} K(n-k, i, n, q)$$

is equivalent to the general term of the Riordan array

$$\left(1, \frac{x}{1+qx}\right).$$

Thus

$$\begin{aligned}\sum_{i=0}^{n-k}(-1)^i K(n-k, i, n, q) &= \sum_{i=0}^{n-k}\sum_{j=0}^{i}\binom{n}{j}\binom{k+i-j-1}{i-j}(q-1)^{i-j} \\ &= \sum_{j=0}^{n-k}\binom{j+k-1}{j}2^{n-k-j}q^j\end{aligned}$$

and

$$\sum_{i=0}^{n-k} K(n-k, i, n, q) = \binom{n-1}{n-k}(-q)^{n-k}.$$

The A-sequence for this example is given by $1+qx$, and so for example we have

$$\sum_{i=0}^{n-k}(-1)^i K(n-k, i, n+1, q) = \sum_{i=0}^{n-k}(-1)^i K(n-k, i, n, q) + q\sum_{i=0}^{n-k-1}(-1)^i K(n-k-1, i, n, q).$$

Example 13. The Riordan arrays encountered so far have all been of an elementary nature. The next example indicates that this is not always so. We make the simple change of $2n$ for n in the third parameter in the previous example. We then find that $\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,q)$ is the general term of the Riordan array

$$\left(\frac{1-2x-q(2-q)x^2}{1+qx}, \frac{x}{(1+qx)^2}\right)^{-1}.$$

For instance, $\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,1)$ represents the general term of the Riordan array

$$\left(\frac{1}{2}\left(\frac{1}{1-4x}+\frac{1}{\sqrt{1-4x}}\right), \frac{1-2x-\sqrt{1-4x}}{2x}\right)$$

while $\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,2)$ represents the general term of

$$\left(\frac{1}{\sqrt{1-8x}}, \frac{1-4x-\sqrt{1-8x}}{2x}\right).$$

The A-sequence for the first array above is $(1+x)^2$, so that we obtain

$$\begin{aligned}\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2(n+1),1) &= \sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,1)\\ &+2\sum_{i=0}^{n-k-1}(-1)^i K(n-k-1,i,2n,1)\\ &+\sum_{i=0}^{n-k-2}(-1)^i K(n-k-2,i,2n,1)\end{aligned}$$

while that of the second array is $(1+2x)^2$ and so

$$\begin{aligned}\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2(n+1),2) &= \sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,2)\\ &+4\sum_{i=0}^{n-k-1}(-1)^i K(n-k-1,i,2n,2)\\ &+4\sum_{i=0}^{n-k-2}(-1)^i K(n-k-2,i,2n,2).\end{aligned}$$

We summarize these examples in the following table.

Table 1. Summary of Riordan arrays

Krawtchouk expression	Riordan array	g.f. for A-sequence
$K(k,n-k,r,q)$	$\left(\left(\frac{1-x}{1+(q-1)x}\right)^r, x(1+(q-1)x)\right)$	$\frac{1+\sqrt{1+4(q-1)x}}{2}$
$(-1)^{n-k}K(k,n-k,r,q)$	$\left(\left(\frac{1+x}{1-(q-1)x}\right)^r, x(1-(q-1)x)\right)$	$\frac{1+\sqrt{1-4(q-1)x}}{2}$
$(-1)^k K(n,k,k,q)$	$\left(\frac{1}{1-x}, \frac{x(1-qx)}{1-x}\right)$	$\frac{1+x+\sqrt{1+2x(1-2q)+x^2}}{2}$
$(-1)^{n-k}K(n-k,n-k,k,q)$	$\left(\frac{1}{1-x}, \frac{x}{1-qx}\right)$	$1+qx$
$(-1)^{n-k}K(n-k,n-k,k,q)$	$\left(\frac{1}{1+(q-1)x}, x(1+qx)\right)$	$\frac{1+\sqrt{1+4x}}{2}$
$(-1)^{n-k}K(n,n-k,k,q)$	$\left(\frac{1}{1+(q-1)x}, \frac{x(1+qx)}{1+(q-1)x}\right)$	$\frac{1+(1-q)x+\sqrt{1+2x(1+q)+(q-1)^2x^2}}{2}$
$K(n,n-k,N,q)$	$\left(\frac{(1-qx)^N}{1-(q-1)x}, \frac{x}{1-(q-1)x}\right)$	$1+(q-1)x$
$\sum_{i=0}^{n-k}(-1)^i K(n-k,i,n,q)$	$\left(\frac{1}{1-2x}, \frac{x}{1-qx}\right)$	$1+qx$
$\sum_{i=0}^{n-k} K(n-k,i,n,q)$	$\left(1, \frac{x}{1+qx}\right)$	$1-qx$
$\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,q)$	$\left(\frac{1-2x-q(2-q)x^2}{1+qx}, \frac{x}{(1+qx)^2}\right)^{-1}$	$(1+qx)^2$

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