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Common Factors of Generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers

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Abstract

In this paper, we present identities involving common factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers. Binet's formula will employ to obtain the identities.

Keywords: *Generalized Fibonacci numbers, Jacobsthal numbers, Binet's formula.*

1 Introduction

It is well-known that the Fibonacci sequence is most prominent examples of recursive sequence. The Fibonacci sequence is famous for possessing wonderful and amazing properties. Fibonacci numbers are a popular topic for mathematical enrichment and popularization. The Fibonacci appear in numerous mathematical problems. The Fibonacci numbers F_n are terms of the sequence $\{0,1,1,2,3,5,...\}$ wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$ and $F_1 = 1$.

2 Generalized Fibonacci Sequences

Generalized Fibonacci sequence [8], is defined as

$$F_k = pF_{k-1} + qF_{k-2}\,,\ k \geq 2\, with\ \ F_0 = a\,,\ F_1 = b\,, \tag{2.1}$$

where p, q, a & b are positive integers.

For different values of p, q, a & b many sequences can be determined.

We focus two cases of sequences $\{V_k\}_{k\geq 0}$ and $\{U_k\}_{k\geq 0}$ which generated in (2.1).

If $p=1$, $q=a=b=2$, we get

$$V_k = V_{k-1} + 2V_{k-2}\quad for\ \ k \geq 2\ \ with\ \ \ V_0 = 2\,,\, V_1 = 2 \tag{2.2}$$

The first few terms of $\{V_k\}_{k\geq 0}$ are 2, 2, 6, 10, 22, 42 and so on.

Its Binet forms is defined by

$$V_k = 2\,\frac{\Re_1^{k+1} - \Re_2^{k+1}}{\Re_1 - \Re_2} \tag{2.3}$$

If $p=1$, $q=a=2$, $b=0$ we get

$$U_k = U_{k-1} + 2U_{k-2}\quad for\ \ k \geq 2\ \ with\ \ \ U_0 = 2\,,\, U_1 = 0 \tag{2.4}$$

The first few terms of $\{U_k\}_{k\geq 0}$ are 2, 0, 4, 4, 12, 20 and so on.

Its Binet forms is defined

$$U_k = 4\,\frac{\Re_1^{k-1} - \Re_2^{k-1}}{\Re_1 - \Re_2} \tag{2.5}$$

The Jacobsthal sequence [1], is defined by the recurrence relation

$$J_k = J_{k-1} + 2j_{k-2}\ ,\ \ k \geq 2\ \ with\ J_0 = 0,\ J_1 = 1 \tag{2.6}$$

Its Binet's formula is defined by

$$J_k = \frac{\Re_1^k - \Re_2^k}{\Re_1 - \Re_2} \tag{2.7}$$

The Jacobsthal-Lucas sequence [1], is defined by the recurrence relation

$$j_k = j_{k-1} + 2j_{k-2}\ ,\ \ k \geq 2\ \ with\ j_0 = 2,\ j_1 = 1 \tag{2.8}$$

Its Binet's formula is defined by

$$j_k = \Re_1^k + \Re_2^k \tag{2.9}$$

where $\Re_1$ & $\Re_2$ are the roots of the characteristic equation $x^2 - x - 2 = 0$.

3 Identities for The Common Factors of Generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas Numbers

There are a lot of identities of Fibonacci and Lucas numbers described in [3]. M. Thongmoon [5], defined various identities of Fibonacci and Lucas numbers. B. Singh, P. Bhadouria and O. Sikhwal [7], present some generalized identities involving common factors of Fibonacci and Lucas numbers. In [8], V. K. Gupta, Y. K. Panwar and O. Sikhwal have defined generalized Fibonacci sequences. In this paper, we present identities of common factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers.

Theorem 3.1 *If V_k and U_k are the generalized Fibonacci numbers and j_k is Jacobsthal-Lucas numbers, then*

(i) $V_{2k+1} j_{2k+1} = V_{4k+2} - 4^{k+1}$ (1)

(ii) $V_{2k+2} j_{2k+1} = V_{4k+3} - 4^{k+1}$ (2)

(iii) $U_{2k+1} j_{2k+1} = U_{4k+4} - 4^{k+1}$ (3)

(iv) $U_{2k+2} j_{2k+1} = U_{4k+1}$ (4)

Proof (i).

$$\begin{aligned} V_{2k+1} j_{2k+1} &= 2\left(\frac{\Re_1^{2k+2} - \Re_2^{2k+2}}{\Re_1 - \Re_2}\right)\left(\Re_1^{2k+1} + \Re_2^{2k+1}\right) \\ &= 2\left(\frac{\Re_1^{4k+3} - \Re_2^{4k+3}}{\Re_1 - \Re_2}\right) + \frac{2}{\left(\Re_1 - \Re_2\right)}\left(\Re_1\Re_2\right)^{2k+1}\left(\Re_1 - \Re_2\right) \\ &= V_{4k+2} - 2\left(\Re_1\Re_2\right)^{2k+1} \\ &= V_{4k+2} - 4^{k+1} \end{aligned}$$

This completes the proof.

Proof (ii). It can be proved same as **Theorem1: (i)**

Proof (iii).

$$U_{2k+1}j_{2k+1} = 4\left(\frac{\Re_1^{2k} - \Re_2^{2k}}{\Re_1 - \Re_2}\right)\left(\Re_1^{2k+1} + \Re_2^{2k+1}\right)$$

$$= 4\left(\frac{\Re_1^{4k+3} - \Re_2^{4k+3}}{\Re_1 - \Re_2}\right) + \frac{4}{\left(\Re_1 - \Re_2\right)}\left(\Re_1\Re_2\right)^{2k}\left(\Re_2 - \Re_1\right)$$

$$= U_{4k+4} - 4\left(\Re_1\Re_2\right)^{2k}$$

$$= U_{4k+4} - 4^{k+1}$$

This completes the proof.

Proof (iv). It can be proved same as **Theorem1: (iii)**

Corollary 3.2:

(i) $V_{2k+1}j_{2k+1} = 2J_{4k+3} - 4^{k+1}$ (5)

(ii) $V_{2k+2}j_{2k+1} = 2J_{4k+4} - 4^{k+1}$ (6)

(iii) $U_{2k+1}j_{2k+1} = 4\left(J_{4k+3} - 4^k\right)$ (7)

(iv) $U_{2k+2}j_{2k+1} = 4J_{4k}$ (8)

Following theorems can be solved by Binet's formulae **(2.3), (2.5), (2.7)** and **(2.9)**

Theorem 3.3: *If* V_k & U_k *are the generalized Fibonacci numbers and* J_k & j_k *are Jacobsthal and jacobsthal-Lucas numbers, then*

(i) $V_{2k+1}j_{2k+2} = 2J_{4k+4}$ (9)

(ii) $V_{2k+1}j_{2k} = 2\left(J_{4k+2} + 4^k\right)$ (10)

(iii) $U_{2k+1}j_{2k} = 4J_{4k}$ (11)

(iv) $U_{2k+1}j_{2k+2} = 4\left(J_{4k+2} - 4^k\right)$ (12)

Corollary 3.4:

(i) $V_{2k+1}j_{2k+2} = V_{4k+2}$ (13)

(ii) $V_{2k+2}j_{2k} = V_{4k+1} + 2^{2k+1}$ (14)

(iii) $U_{2k+1}j_{2k} = U_{4k+1}$ (15)

(iv) $U_{2k+2}j_{2k+2} = U_{4k+3} - 4^{k+1}$ (16)

Theorem 3.5:

$(i)\quad V_{2k+2}j_{2k} = V_{4k+2} + 3\left(2^{2k+1}\right)$ (17)

$(ii)\quad V_{2k}j_{2k} = V_{4k} + 2^{2k+1}$ (18)

$(iii)\quad U_{2k+2}j_{2k} = U_{4k+2} + 4^{k+1}$ (19)

$(iv)\quad U_{2k}j_{2k} = U_{4k} + 2^{2k+1}$ (20)

Corollary 3.6:

$(i)\quad V_{2k+2}j_{2k} = 2\left(J_{4k+3} + 3\left(4^k\right)\right)$ (21)

$(ii)\quad V_{2k}j_{2k} = 2\left(J_{4k+1} - 4^k\right)$ (22)

$(iii)\quad U_{2k+2}j_{2k} = 4\left(J_{4k+1} + 4^k\right)$ (23)

$(iv)\quad U_{2k}j_{2k} = 2\left(2J_{4k-1} + 4^k\right)$ (24)

Theorem 3.7:

$(i)\quad V_{2k-1}j_{2k+1} = V_{4k} - 2^{2k+1}$ (25)

$(ii)\quad V_{2k-1}j_{2k-1} = V_{4k-2} - 4^k$ (26)

$(iii)\quad U_{2k-1}j_{2k+1} = U_{4k} - 3\left(4^k\right)$ (27)

$(iv)\quad U_{2k-1}j_{2k-1} = U_{4k-2} - 4^k$ (28)

Corollary 3.6:

$(i)\quad V_{2k-1}j_{2k+1} = 2\left(J_{4k+1} - 4^k\right)$ (29)

$(ii)\quad V_{2k-1}j_{2k-1} = 2J_{4k-1} - 4^k$ (30)

$(iii)\quad U_{2k-1}j_{2k+1} = 4J_{4k-1} - 3(4^k)$ (31)

$(iv)\quad U_{2k-1}j_{2k-1} = 4J_{4k-3} - 4^k$ (32)

Theorem 3.8:

$(i)\quad V_{2k}j_{2k+1} = V_{4k+1}$ (33)

$(ii)\quad V_{2k-1}j_{2k} = V_{4k-1}$ (34)

$(iii)\quad U_{2k}j_{2k+1} = U_{4k+1} + 2^{2k+1}$ (35)

$(iv)\quad U_{2k-1}j_{2k} = U_{4k-1} - 4^k$ (36)

Corollary 3.6:

(*i*) $V_{2k} j_{2k+1} = 2J_{4k+2}$ (37)

(*ii*) $V_{2k-1} j_{2k} = 2J_{4k}$ (38)

(*iii*) $U_{2k} j_{2k+1} = 2\left(2J_{4k} + 4^k\right)$ (39)

(*iv*) $U_{2k-1} j_{2k} = 4J_{4k-2} - 4^k$ (40)

4 Conclusion

In this paper we have stated and derived many identities of common factors of generalized Fibonacci and Jacobsthal and Jacobsthal-Lucas numbers with the help of their Binet's formula. The concept can be executed for generalized Fibonacci sequences as well as polynomials.

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