

Jacobsthal ve Jacobsthal-Lucas sayıları üzerine bir çalışma

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Özet

Bu çalışmada, Jacobsthal ve Jacobsthal-Lucas sayılarını inceliyoruz. Onların bazı toplam formüllerinden ve ReSeT teoremi dediğimiz çok ilginç teoremlerden oluşan bazı oldukça önemli özdeşlikleri türetiyoruz. Bu çalışmada sunulan yeni özdeşlikleri kullanarak önceden bilinen bazı toplam formüllerini yeniden oluşturuyoruz.

Anahtar Sözcükler: Jacobsthal Sayılar, Jacobsthal-Lucas Sayılar, Binet Formülü, Toplamalar.

A study on the Jacobsthal and Jacobsthal-Lucas numbers

Abstract

In this study, we investigate the Jacobsthal and Jacobsthal-Lucas numbers. We derive quite important identities, which consist of their certain sums formulas and very curious theorems we call ReSeT theorem. Also using new properties presented in this paper, we reorganize certain foreknown sums formulas.

Key Words: Jacobsthal Numbers, Jacobsthal-Lucas Numbers, Binet Formula, Sums.

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1. Introduction

The Fibonacci sequence is an inexhaustible source of many interesting identities. It is one of the most famous numerical sequences in mathematics and constitutes an integer sequence. If certain fruits are looked at, the number of little bumps around each ring are counted or the sand on the beach and how waves hit it is watched out, the Fibonacci sequence is seen there. Vajda and Koshy present the well-known systematic investigations [1,4]. The same statements can easily be said for the Jacobsthal sequences. For instance, it is well-known that computers use conditional directives to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits,..., which are exactly the Jacobsthal numbers. At first it is studied by Horadam [2]. The usual Jacobsthal sequence is represented with $\{J_n\}$ and defined by the following recurrence:

$$J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2 \tag{1}$$

with, of course, initial conditions $J_0 = 0$ and $J_1 = 1$. Similarly the usual Jacobsthal-Lucas sequence is represented with $\{J_n\}$ and defined by the same recurrence but initial conditions $J_0 = 2$ and $J_1 = 1$. Then the Jacobsthal and Jacobsthal-Lucas sequences are written as

$$J_n = \{0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \dots\}$$

and

$$J_n = \{2, 1, 5, 7, 17, 31, 65, 127, 257, 511, \dots\},$$

respectively. The members of these integer sequences can also be obtained different ways. It appears that this can be done in either of two ways: the Binet formulas or matrix method. The explicit Binet formulas of these numbers are given by Horadam as follows [2]:

$$J_n = \frac{2^n - (-1)^n}{3} \tag{2}$$

and

$$J_n = 2^n + (-1)^n, \tag{3}$$

respectively. As a second way, they can be obtained by a generating matrix, which is called the matrix method:

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} \tag{4}$$

and

$$E^n = \begin{cases} 3^n \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} & \text{if } n \text{ even,} \\ 3^{n-1} \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} & \text{if } n \text{ odd,} \end{cases} \tag{5}$$

Where $F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$ respectively.

The matrices F and E is called the Jacobsthal F -matrix and Jacobsthal-Lucas E -matrix. respectively. Two relationships between these matrices and these sequences are given Köken and Bozkurt as follows [7]:

$$\begin{bmatrix} J_{n+1} \\ J_n \end{bmatrix} = F \begin{bmatrix} J_n \\ J_{n-1} \end{bmatrix} \text{ and } \begin{bmatrix} J_{n+1} \\ J_n \end{bmatrix} = E \begin{bmatrix} J_n \\ J_{n-1} \end{bmatrix}$$

Similar applications can be seen Köken and Bozkurt's paper [8].

Djordjevid and Srivastava present a systematic investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal-Lucas numbers. They give main results involving the generating functions of these incomplete numbers [5].

There exist very miscellaneous properties of the Jacobsthal and Jacobsthal-Lucas numbers. In particular, their Cassini-like and sum formulas consisting of consecutive terms are very nice and quite important. The Cassini-like formulas for these numbers are given by Horadam as follows [2]:

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1} \tag{6}$$

and

$$J_{n+1}J_{n-1} - J_n^2 = 3^2(-2)^{n-1}. \tag{7}$$

Further, sums of their consecutive terms are given by Horadam as follows [2]:

$$\sum_{i=1}^n J_i = \frac{1}{2} (J_{n+2} - 1) \tag{8}$$

and

$$\sum_{i=1}^n J_i = \frac{1}{2} (J_{n+2} - 5) \tag{9}$$

The sums of odd and even terms of the Jacobsthal and Jacobsthal-Lucas sequence are investigated by Köken and Bozkurt [7,8]

$$\sum_{i=0}^n J_{2i+1} = \frac{1}{3} (2J_{2n+2} - n + 1) \tag{10}$$

$$\sum_{i=0}^n J_{2i} = \frac{1}{3} (2J_{2n+2} - n - 1) \tag{11}$$

$$\sum_{i=0}^n J_{2i+1} = 2J_{2n+2} - n - 1 \tag{12}$$

$$\sum_{i=0}^n J_{2i} = J_{2n+2} + n + 1 \tag{13}$$

Cerin considers sums of squares of odd and even terms of the Jacobsthal sequence and sums of their products. Also the author shows that these sums are related to products of appropriate Jacobsthal numbers and several integer sequences [6].

It is the object of this article to investigate the corresponding new elementary identities associated with the usual Jacobsthal and Jacobsthal-Lucas numbers. Organization of material is as follows: in section 2, new elementary properties related to the Jacobsthal numbers are discovered, while in section 3 the Jacobsthal-Lucas numbers is dealt with.

The Usual Jacobsthal Numbers

In this section our goal is to investigate the Jacobsthal numbers and give certain new formu-

las, especially sum formula with their variable index terms.

Then we start with the following theorem.

Theorem 1. Let J_n be n th Jacobsthal number. For arbitrary positive n and k integers, the following equality is satisfied:

$$J_n J_{n+k} = \frac{1}{3} (J_{2n+k} + (-1)^{n+1} J_{n+k} + (-1)^{n+k+1} J_n) \tag{14}$$

Proof. The following statements can be written by using the Binet formula of Jacobsthal numbers,

$$\begin{aligned} J_n J_{n+k} &= \left(\frac{2^n - (-1)^n}{3} \right) \left(\frac{2^{n+k} - (-1)^{n+k}}{3} \right) \\ &= \frac{2^{2n+k} - (-1)^{n+k} 2^n - (-1)^n 2^{n+k} + (-1)^{2n+k}}{3^2} \end{aligned}$$

more simply

$$J_n J_{n+k} = \frac{1}{3} \left(\frac{2^{2n+k} - (-1)^{2n+k}}{3} + (-1)^{n+1} \frac{2^{n+k} - (-1)^{n+k}}{3} + (-1)^{n+k+1} \frac{2^n - (-1)^n}{3} \right)$$

Thus, the proof is completed.

As an example of what could happen, consider $n = 3$ and $k = 2$. Then the following equalities can be seen:

$$J_3 J_5 = 3 \cdot 11 = 33$$

and

$$\frac{1}{3} (J_{2 \cdot 3 + 2} + (-1)^{3+1} J_{2+3} + (-1)^{2+3+1} J_3) = \frac{1}{3} (85 + 11 + 3) = 33.$$

Then

$$J_3 J_5 = \frac{1}{3} (J_8 + J_5 + J_3)$$

In addition, taking especially $k=0$ and $k=1$, then

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$$J_n^2 = \frac{1}{3} (J_{2n} + 2(-1)^{n+1} J_n) \tag{15}$$

and

$$J_n J_{n+1} = \frac{1}{3} (J_{2n+1} + (-1)^{n+1} J_{n-1}), \tag{16}$$

respectively.

Then we will give certain elementary properties.

Theorem 2. For the usual Jacobsthal number, the following equality is held:

$$J_{n+1}^2 - J_{n-1}^2 + 2J_n J_{n+1} = \frac{1}{3} (J_{2n+3} - J_{2n-2} + (-1)^n J_n - 1) \quad (17)$$

Proof. From the equations (15) and (16), we can write

$$\begin{aligned} J_{n+1}^2 - J_{n-1}^2 + 2J_n J_{n+1} &= \frac{1}{3} \left(\begin{array}{c} J_{2n+2} - J_{2n-2} + J_{2n+1} \\ + 2(-1)^n (J_{n+1} - J_{n-1} - J_n) - 2 \end{array} \right) \\ &= \frac{1}{3} \left(\begin{array}{c} J_{2n+3} - J_{2n-2} - 2 \\ + 2(-1)^n \left(\frac{J_n - (-1)^{n-1}}{2} \right) \end{array} \right) \\ &= \frac{1}{3} (J_{2n+3} - J_{2n-2} + (-1)^n J_n - 1), \end{aligned}$$

which is desired.

The proofs of the next three theorems are analogous to the proof of Theorem 2, so it will be omitted.

Theorem 3. If n is any positive integer, then

$$2J_n J_{n-1} + J_{n-1}^2 - J_n^2 = \frac{1}{3} (J_{2n-2} + 4(-1)^n J_n + 1). \quad (18)$$

Theorem 4. If n is any positive integer, then

$$J_{n+1}^2 + J_{n-1}^2 - 2J_n J_{n+1} = \frac{1}{3} (J_{2n-2} + 7(-1)^n J_n + 4). \quad (19)$$

Theorem 5. Let J_n be nth Jacobsthal number. Then

$$J_{n+1}^2 - J_n^2 = J_{2n} + 2(-1)^n J_n + 1. \quad (20)$$

We now give very private properties of the Jacobsthal numbers. We call this identity as the ReSeT theorem, which are very curious identities and convert difficult forms to more simply structures. But the following two theorems are presented without the proof because the proofs are quite exhausting, complicated and very tedious.

Theorem 6. (ReSeT theorem for the Jacobsthal

numbers) Let R, S and T be an arbitrary positive integer. Then,

$$J_R J_S J_T = \frac{1}{9} \left(\begin{array}{c} J_{R+S+T} + (-1)^{T+1} J_{R+S} \\ + (-1)^{S+1} J_{R+T} + (-1)^{R+1} J_{S+T} \\ + (-1)^{R+T} J_S + (-1)^{S+T} J_R + (-1)^{R+S} J_T \end{array} \right) \quad (21)$$

We now handle the sums formulas involving the Jacobsthal numbers. Then we start with the following theorem.

Theorem 7. Let J_n be nth Jacobsthal number. Then for all $n, k \geq 0$ integers,

$$\sum_{i=1}^n J_i J_{i+k} = \frac{1}{9} \left(\begin{array}{c} J_{2n+k+2} + (-1)^{n+1} J_{n+k+1} \\ + (-1)^{n+k+1} J_{n+1} - J_{k+1} + (-1)^k (n+2) \end{array} \right) \quad (22)$$

Proof. Writing particularly the equation (14) from 1 to n, counting them up and considering the equations (8), (10), and (11) and after some mathematical operations, then the desired result can readily be obtained.

Taking separately $k = 0$ and $k = 1$, we have the following corollaries.

Corollary 8. Let J_n be nth Jacobsthal number. Then

$$\sum_{k=1}^n J_k^n = \frac{1}{9} (J_{2n+2} + (-1)^{n+1} J_{n+2} + n). \quad (23)$$

Corollary 9. Let J_n be nth Jacobsthal number. Then

$$\sum_{k=1}^n J_k J_{k+1} = \frac{1}{9} (J_{2n+3} + (-1)^{n+1} J_{n+1} - (n+2)). \quad (24)$$

The Usual Jacobsthal-Lucas Numbers

In this section we give thought to the Jacobsthal-Lucas numbers. We will give new identities involving them as those in previous section. We also present certain interrelationships identities. But most of the presented identities in this section are similar to those in previous section. Therefore in this section we avoid long proofs.

Then we start with the following lemma without proof. But proof can easily seen by the Binet formula of the Jacobsthal-Lucas numbers.

Theorem 10. Let J_n be n th Jacobsthal-Lucas number. Then

$$J_n J_{n+k} = J_{2n+k} + (-1)^n J_{n+k} + (-1)^{n+k} J_n + 2(-1)^{k+1} \quad (25)$$

Proof. From the Binet formula of the Jacobsthal-Lucas numbers, the proof can easily be done.

Consider separately $k = 0$ and $k = 1$. Then

$$J_n^2 = J_{2n} + 2(-1)^n J_n - 2 \quad (26)$$

and

$$J_n J_{n+1} = J_{2n+1} + 2(-1)^n J_{n-1} + 2. \quad (27)$$

Now we present the following theorem without proof. However proofs are similar to theorem (4) and theorem (5) involving elementary identities for the Jacobsthal numbers,.

Theorem 11. Let J_n be n th Jacobsthal-Lucas number. Then for all n positive integers,

$$J_{n+1}^2 - J_{n-1}^2 + 2J_n J_{n+1} = J_{2n+3} - J_{2n-2} + 5(-1)^n J_n - 5 \quad (28)$$

and

$$J_{n+1}^2 + J_{n-1}^2 - 2J_n J_{n+1} = J_{2n-2} - 2(-1)^n J_n - 5 \quad (29)$$

We will give a new identity, which is inter-relationship, involving the Jacobsthal and Jacobsthal-Lucas numbers.

Theorem 12. Let J_n and J_n be n th Jacobsthal and Jacobsthal-Lucas number, respectively. Then

$$J_m J_n = \begin{cases} J_{n+m} + (-1)^{m+1} 2^m J_{n-m} & \text{if } n \geq m \\ J_{m+n} + (-1)^n 2^n J_{m-n} & \text{if } m \geq n \end{cases} \quad (30)$$

Proof. In order to prove this theorem, we shall consider two cases.

Case 1 $n \geq m$

Considering the Binet formulas of them, we can write

$$\begin{aligned} J_m J_n &= \left(\frac{2^m - (-1)^m}{3} \right) (2^n + (-1)^n) \\ &= \underbrace{\frac{2^{m+n} - (-1)^{m+n}}{3}}_{j_{m+n}} - (-1)^m 2^m \underbrace{\frac{2^{n-m} - (-1)^{n-m}}{3}}_{j_{n-m}} \\ &= J_{m+n} + (-1)^{m+1} 2^m J_{n-m} \end{aligned}$$

Case 2 $m \geq n$

Similarly considering the Binet formula, we can write

$$\begin{aligned} J_m J_n &= \left(\frac{2^m - (-1)^m}{3} \right) (2^n + (-1)^n) \\ &= \underbrace{\frac{2^{m+n} - (-1)^{m+n}}{3}}_{j_{m+n}} + (-1)^n 2^n \underbrace{\frac{2^{m-n} - (-1)^{m-n}}{3}}_{j_{n-m}} \\ &= J_{m+n} + (-1)^n 2^n J_{m-n} \end{aligned}$$

which is desired. Thus the prof is completed. As an example, consider $n=0$ and $m=5$. In appearance, $m \geq n$. Then

$$J_5 J_0 = 11.17 = 187$$

and

$$J_{5+0} + (-1)^4 2^4 J_{5-0} = 171+16 = 187$$

The proofs of next two theorems will be omitted due to previous reasons.

Theorem 13. (ReSeT theorem for the Jacobsthal-Lucas numbers) Let r, s and t be an arbitrary positive integer. Then,

$$\begin{aligned} J_r J_s J_t &= J_{r+s+t} + (-1)^r J_{s+t} + (-1)^s J_{r+t} \\ &\quad + (-1)^{s+t} J_r + (-1)^{r+t} + (-1)^t J_{r+s} J_s \quad (31) \\ &\quad + (-1)^{r+s} J_t + 6(-1)^{r+s+t+1} \end{aligned}$$

Finally, we present a bit of sum formulas for the Jacobsthal-Lucas numbers. The next theorem is given without prof due to similar to that in the proof of the Theorem 7.

Theorem 14. Let J_n and J_n be n th Jacobsthal and Jacobsthal-Lucas number, respectively. Then for all $n, k \geq 0$ integers,

$$\sum_{i=1}^n J_i J_{i+k} = J_{2n+k+2} + (-1)^{n+1} J_{n+k+1} + (-1)^{n+k} J_{n+1} - 3J_{k+1} + (-1)^k n \tag{32}$$

Taking especially $k=0$ and $k=1$, we obtain the following corollaries.

Corollary 15. Let J_n and J_n be n th Jacobsthal and Jacobsthal-Lucas number, respectively. Then

$$\sum_{k=1}^n J_k^2 = J_{2n+2} + 4(-1)^n J_{n+n-1} \tag{33}$$

Corollary 16. Let J_n and J_n be n th Jacobsthal and Jacobsthal-Lucas number, respectively. Then

$$\sum_{k=1}^n J_k J_{k+1} = J_{2n+3} + (-1)^n J_{n+1} - (n+4). \tag{34}$$

Findings and Discussion

In this paper, very important results involving the Jacobsthal and Jacobsthal-Lucas numbers are obtained. Also we simply reconstitute certain foreknown sums formulas of them by using different ways. Therefore these formulas are accomplished to constitute very nice forms. However, we do not succeed to find equalities of the following sums: for any $p, k \in \mathbb{Z}^+$

$$\sum_{i=1}^n J_{i+p} J_{i+k}, \sum_{i=1}^n J_{i+p}^2, \sum_{i=1}^n J_{i+p} J_{i+k}, \text{ and } \sum_{i=1}^n J_{i+p}^2$$

We consider that there exist anymore inter-relationships between them.

Over the years, several articles have appeared in many journals relating the integer sequences to growth patterns in plants. Recently, Jacobsthal sequence is very important research area because of related to computer science. Quite apart from pursuing the discovery of additional formulas by the matrix techniques indicated, we can introduce different matrices to obtain new results. Naturally, the consequences of the use of matrix methods in developing combinatorial number theory are by no means exhausted in our paper. This is

one of the outstanding research areas. Among the opportunities available for exploration are, at least, the following three:

1. Are there any relationships between the Jacobsthal and Fibonacci numbers?
2. Are there any relationships between them and matrices? (Probably the answer is positive)
3. Can these identities be generalized formally further complicated?

Conclusion

In this study, we consider the usual Jacobsthal and Jacobsthal-Lucas sequences. We derive some identities involving the terms of these sequences, some relations between them and sums formulas of them.

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