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Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations

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Abstract

The Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [10] is expanded and extended to several identities for some of the higher order cases.

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1. Introduction

Horadam, in [10], exhibited a plethora of identities for the second order Jacobsthal and Jacobsthal-Lucas numbers. He then went on to explore their relationships and those of a variety of associated and representative sequences. The aim here is to present some additional identities and analogous relationships for numbers arising from some higher order Jacobsthal recurrence relations.

Obtaining properties by extending the Jacobsthal sequence to the third and higher orders depends on the choice of initial conditions. For example, this was done in [3] by taking all of the conditions to be zero, except the last, which was assigned the value 1. The procedure here will be to extend by using other initial values.

2. The second order Jacobsthal case

The second-order recurrence relations for the Jacobsthal numbers, J_n, and for the Jacobsthal-Lucas numbers, j_n, and a few of their relationships are given here for reference. Namely,

Recurrence relations

$$J_{n+2} = J_{n+1} + 2J_n,\ J_0 = 0, J_1 = 1,\ n \geq 0$$

$$j_{n+2} = j_{n+1} + 2j_n,\ j_0 = 2, j_1 = 1,\ n \geq 0$$

Table of values

n	0	1	2	3	4	5	6	7	8	9	10	...
J_n	0	1	1	3	5	11	21	43	85	171	341	...
j_n	2	1	5	7	17	31	65	127	257	511	1025	...

Binet forms

$$J_n = \frac{2^n - (-1)^n}{3} \text{ and } j_n = 2^n + (-1)^n$$

Simson/Cassini/Catalan identities

$$\begin{vmatrix} J_{n+1} & J_n \\ J_n & J_{n-1} \end{vmatrix} = (-1)^n 2^{n-1}, \begin{vmatrix} j_{n+1} & j_n \\ j_n & j_{n-1} \end{vmatrix} = 9(-1)^{n-1}2^{n-1}$$

Ordinary generating functions

$$\sum_{k=0}^{\infty} J_k x^k = \frac{x}{1 - x - 2x^2}$$

$$\sum_{k=0}^{\infty} j_k x^k = \frac{2 - x}{1 - x - 2x^2}$$

Exponential generating functions

$$\sum_{k=0}^{\infty} J_k \frac{x^k}{k!} = \frac{e^{2x} - e^{-x}}{3}$$

$$\sum_{k=0}^{\infty} j_k \frac{x^k}{k!} = e^{2x} + e^{-x}$$

Although these are not given in [10] the exponential generating functions are easily obtained using the Maclaurin series for the exponential function and can be useful in establishing identities. For example, using the method provided in [2, 12,

p. 232ff] the following can be obtained. Let $A = e^x$ and $B = \dfrac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$ where $\alpha = 2$ and $\beta = -1$. Then

$$B = \frac{1}{\alpha - \beta}\left[\frac{(\alpha - \beta)x}{1!} + \frac{(\alpha^2 - \beta^2)x^2}{2!} + \cdots\right] = \sum_{k=0}^{\infty} J_k \frac{x^k}{k!}.$$

Using the well known double sum identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} F(k, n-k)$$

found in [2, 15, p. 56] AB can be written as

$$AB = \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{\infty} J_k \frac{x^k}{k!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} J_k \frac{x^{n+k}}{n!k!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} J_k \frac{x^{(n-k)+k}}{(n-k)!k!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} J_k\right)\frac{x^n}{n!}.$$

In addition AB can also be written as

$$AB = \frac{e^{(\alpha+1)x} - e^{(\beta+1)x}}{\alpha - \beta} = \frac{e^{(2+1)x} - e^{(-1+1)x}}{2-(-1)} = \frac{e^{3x} - 1}{3} = \frac{1}{3}\cdot 0 + \sum_{n=1}^{\infty} 3^{n-1}\frac{x^n}{n!}$$

and so it follows that

$$\sum_{k=0}^{n}\binom{n}{k} J_k = 3^{n-1}.$$

Similarly with $B = \dfrac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$ and $A = e^{-3x}$ it follows that

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^{n-1} J_k = (-3)^{n-1},$$

and if $B = e^{\alpha x + \beta x}$ then

$$\sum_{k=0}^{n}\binom{n}{k} J_k j_{n-k} = 2^n J_n.$$

Other summation identities can be obtained in a similar fashion.

3. The third order Jacobsthal case

First we consider extending the Jacobsthal and Jacobsthal-Lucas numbers to the third order, denoted as $J_n^{(3)}$ and $j_n^{(3)}$ respectively, with the following initial conditions:

Recurrence relations

$$J^{(3)}_{n+3} = J^{(3)}_{n+2} + J^{(3)}_{n+1} + 2J^{(3)}_{n},\ J^{(3)}_{0} = 0, J^{(3)}_{1} = 1, J^{(3)}_{2} = 1\ n \geq 0.$$

$$j^{(3)}_{n+3} = j^{(3)}_{n+2} + j^{(3)}_{n+1} + 2j^{(3)}_{n},\ j^{(3)}_{0} = 2, j^{(3)}_{1} = 1, j^{(3)}_{2} = 5\ n \geq 0.$$

Table of values

n	0	1	2	3	4	5	6	7	8	9	10	...
$J^{(3)}_n$	0	1	1	2	5	9	18	37	73	146	293	...
$j^{(3)}_n$	2	1	5	10	17	37	74	145	293	586	1169	...

Note that we extend to 3^{rd} order using initial conditions $\{0, 1, 1\}$ in the spirit of extending the Fibonacci initial conditions $\{0, 1\}$ to Tribonacci $\{0, 1, 1\}$ and those initial conditions for the Jacobsthal-Lucas numbers in a natural way from the second order case.

Binet forms

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0;\ x = 2, \text{and } x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J^{(3)}_n = \frac{2}{7}2^n - \frac{3 + 2i\sqrt{3}}{21}\omega_1^n - \frac{3 - 2i\sqrt{3}}{21}\omega_2^n,$$

and

$$j^{(3)}_n = \frac{8}{7}2^n + \frac{3 + 2i\sqrt{3}}{7}\omega_1^n + \frac{3 - 2i\sqrt{3}}{7}\omega_2^n. \tag{3.1}$$

Simson's identities

$$\begin{vmatrix} J^{(3)}_{n+2} & J^{(3)}_{n+1} & J^{(3)}_{n} \\ J^{(3)}_{n+1} & J^{(3)}_{n} & J^{(3)}_{n-1} \\ J^{(3)}_{n} & J^{(3)}_{n-1} & J^{(3)}_{n-2} \end{vmatrix} = -2^{n-1}, \quad \begin{vmatrix} j^{(3)}_{n+2} & j^{(3)}_{n+1} & j^{(3)}_{n} \\ j^{(3)}_{n+1} & j^{(3)}_{n} & j^{(3)}_{n-1} \\ j^{(3)}_{n} & j^{(3)}_{n-1} & j^{(3)}_{n-2} \end{vmatrix} = -9 \cdot 2^{n+1}. \tag{3.2}$$

The identities above can be proved using mathematical induction. As an example an inductive proof for the J_n case is provided: For $n = 2, 3, 4$ and 5, the determinants are routinely computed to be $-2, -4, -8, -16$, respectively. So we surmise the general case to be as given in (3.2). Assuming the n^{th} case is true and expanding that determinant by the 3^{rd} column and expanding the $(n + 1)^{th}$ determinant by the 1^{st} column yields the following:

$$\begin{vmatrix} J^{(3)}_{n+3} & J^{(3)}_{n+2} & J^{(3)}_{n+1} \\ J^{(3)}_{n+2} & J^{(3)}_{n+1} & J^{(3)}_{n} \\ J^{(3)}_{n+1} & J^{(3)}_{n} & J^{(3)}_{n-1} \end{vmatrix} = 2 \begin{vmatrix} J^{(3)}_{n+2} & J^{(3)}_{n+1} & J^{(3)}_{n} \\ J^{(3)}_{n+1} & J^{(3)}_{n} & J^{(3)}_{n-1} \\ J^{(3)}_{n} & J^{(3)}_{n-1} & J^{(3)}_{n-2} \end{vmatrix} + C,$$

where

$$C = (J_{n+2}^{(3)} + J_{n+1}^{(3)}) \begin{vmatrix} J_{n+1}^{(3)} & J_n^{(3)} \\ J_n^{(3)} & J_{n-1}^{(3)} \end{vmatrix} - (J_{n+1}^{(3)} + J_n^{(3)}) \begin{vmatrix} J_{n+2}^{(3)} & J_{n+1}^{(3)} \\ J_n^{(3)} & J_{n-1}^{(3)} \end{vmatrix}$$
$$+ (J_n^{(3)} + J_{n-1}^{(3)}) \begin{vmatrix} J_{n+2}^{(3)} & J_{n+1}^{(3)} \\ J_{n+1}^{(3)} & J_n^{(3)} \end{vmatrix}.$$

By expanding C it is easy to see that the expression is 0 and so the conjecture is valid.

Ordinary generating functions

The ordinary generating functions are obtained by standard methods [12, p 237ff] as briefly illustrated here.

Let $g(x) = \sum_{k=0}^{\infty} J_k x^k$ and $h(x) = \sum_{k=0}^{\infty} j_k x^k$. Compute $(1-x-x^2-2x^3)g(x)$ and $(1-x-x^2-2x^3)h(x)$ and apply the initial conditions for the third order Jacobsthal and Jacobsthal-Lucas numbers, respectively, to obtain the following generating functions.

$$\sum_{k=0}^{\infty} J_k^{(3)} x^k = \frac{x}{1-x-x^2-2x^3}.$$

$$\sum_{k=0}^{\infty} j_k x^k = \frac{2-x+2x^2}{1-x-x^2-2x^3}.$$

Exponential generating functions

The exponential generating functions can be obtained from the Maclaurin series for the exponential function as follows. Note that

$$\frac{1}{21}\left(6e^{2x} - (3+2i\sqrt{3})e^{\omega_1 x} - (3+2i\sqrt{3})e^{\omega_2 x}\right) =$$
$$\sum_{k=0}^{\infty} \frac{1}{21}\left(6(2^k) - (3+2i\sqrt{3})\omega_1^k - (3+2i\sqrt{3})\omega_2^k\right)\frac{x^k}{k!} = \sum_{k=0}^{\infty} J_k \frac{x^k}{k!}.$$

Also, since

$$\begin{aligned}(3+2i\sqrt{3})e^{\omega_1 x} + (3+2i\sqrt{3})e^{\omega_2 x} &= e^{-\frac{1}{2}x}\left((3+2i\sqrt{3})e^{\frac{\sqrt{3}}{2}ix} + (3+2i\sqrt{3})e^{\frac{\sqrt{3}}{2}ix}\right) \\ &= e^{-\frac{1}{2}x}\left(6\cos\frac{\sqrt{3}x}{2} + 4\sqrt{3}\sin\frac{\sqrt{3}x}{2}\right),\end{aligned}$$

the exponential generating function for the 3^{rd} order Jacobsthal numbers becomes

$$\sum_{k=0}^{\infty} J_k^{(3)} \frac{x^k}{k!} = \frac{1}{21}\left(6e^{2x} + e^{-\frac{1}{2}x}\left(6\cos\frac{\sqrt{3}x}{2} + 4\sqrt{3}\sin\frac{\sqrt{3}x}{2}\right)\right).$$

Similarly the exponential generating function for the 3^{rd} order Jacobsthal-Lucas numbers can be written as

$$\sum_{k=0}^{\infty} j_k^{(3)} \frac{x^k}{k!} = \frac{1}{7}\left(8e^{2x} + e^{-\frac{1}{2}x}\left(6\cos\frac{\sqrt{3}x}{2} + 4\sqrt{3}\sin\frac{\sqrt{3}x}{2}\right)\right).$$

4. Additional identities for third order Jacobsthal numbers

Summation formulas

$$\sum_{k=0}^{n} J_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & \text{if } n \not\equiv 0 \bmod 3 \\ J_{n+1}^{(3)} - 1 & \text{if } n \equiv 0 \bmod 3 \end{cases}, \quad \sum_{k=0}^{n} j_k^{(3)} = \begin{cases} j_{n+1}^{(3)} - 2 & \text{if } n \not\equiv 0 \bmod 3 \\ j_{n+1}^{(3)} + 1 & \text{if } n \equiv 0 \bmod 3 \end{cases}.$$

Miscellaneous identities

$$3J_n^{(3)} + j_n^{(3)} = 2^{n+1}. \tag{4.1}$$

$$j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)}. \tag{4.2}$$

$$j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+1}^{(3)}.$$

$$\left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+1} j_{n-3}^{(3)}.$$

$$\begin{cases} j_{3n-1}^{(3)} & = J_{3n+1}^{(3)} \\ j_{3n}^{(3)} & = J_{3n+2}^{(3)} + 1 \\ j_{3n+1}^{(3)} & = J_{3n+3}^{(3)} - 1 \end{cases}.$$

$$\begin{cases} j_{3n-1}^{(3)} - 4J_{3n-1}^{(3)} & = 1 \\ j_{3n}^{(3)} - 4J_{3n}^{(3)} & = 2 \\ j_{3n+1}^{(3)} - 4J_{3n+1}^{(3)} & = -3 \end{cases}.$$

$$j_n^{(3)} - 4j_{n-2}^{(3)} = \begin{cases} -3 & \text{if } n \text{ is even} \\ 6 & \text{if } n \text{ is odd} \end{cases}.$$

Squaring both sides of (4.1) and (4.2) and subtracting the results, it follows that

$$J_n^{(3)} j_n^{(3)} = \frac{1}{3}\left(4^n - \left(j_{n-3}^{(3)}\right)^2\right).$$

Note that some observations on generating functions for the Jacobsthal polynomials can be found in [7, 8]. Papers on generating functions for a variety of sequential numbers are abundant. See, for example [1, 4, 5, 6, 9, 13, 14, 16].

As an illustration of how ordinary generating functions can be used to derive identities, we use the technique of Gould, see [4] and used for Fibonacci identities in [2]. Making use of the properties of α and β for Fibonacci numbers as needed, it follows that

$$\begin{aligned}\sum_{k=0}^{\infty} J_k^{(3)} F_k x^k &= \sum_{k=0}^{\infty} J_k^{(3)} \frac{\alpha^k - \beta^k}{\alpha - \beta} x^k \\ &= \frac{\alpha x}{1 - \alpha x - \alpha^2 x^2 - 2\alpha^3 x^3} + \frac{\beta x}{1 - \beta x - \beta^2 x^2 - 2\beta^3 x^3} \\ &= \frac{x - x^3 - 2x^4}{1 - x - 4x^2 - 5x^3 + 4x^5 - 4x^6}.\end{aligned}$$

Similarly if we write (3.1) as $j_n^{(3)} = \frac{8}{7}2^n + \frac{A}{7}\omega_1^n + \frac{B}{7}\omega_2^n$ and make use of the fact that $A\omega_1 = \frac{-9 + i\sqrt{3}}{2}, B\omega_2 = \frac{-9 - i\sqrt{3}}{2}, \omega_1^2 = \omega_2$, and $\omega_2^2 = \omega_1, \omega_1\omega_2 = \omega_1^3 = \omega_2^3 = 1$ then the following generating function is obtained:

$$\begin{aligned}\sum_{k=0}^{\infty} J_k^{(3)} j_k^{(3)} &= \frac{1}{7}\sum_{k=0}^{\infty} J_k^{(3)} \left(8(2x)^k + A(\omega_1 x)^k + B(\omega_2 x)^k\right) \\ &= \frac{13x + 20x^2 + 47x^3 - 16x^4 + 8x^5 - 40x^6 - 32x^7}{7(1 - 2x - 4x^2 - 16x^3)(1 + x + 2x^2 - 5x^3 - x^4 - 2x^5 + 4x^6)}.\end{aligned}$$

5. Higher order Jacobsthal numbers

As seen in [3] one way to generalize the Jacobsthal recursion is as follows.

$$J_{n+k}^{(k)} = \sum_{j=1}^{k-1} J_{n+k-j}^{(k)} + 2J_n^{(k)}$$

with $n \geq 0$ and initial conditions $J_i^{(k)} = 0$, for $i = 0, 1, \ldots k-2$ and $J_{k-1}^{(k)} = 1$, has characteristic equation $(x-2)(x^{k-1} + x^{k-2} + \cdots + x^2 + x + 1) = 0$ with eigenvalues 2 and $\omega_j = e^{\frac{2\pi i m}{k}}$ for $j = 1, 2, \ldots, k-1$, which yields the Binet form:

$$J_n^{(k)} = \frac{1}{\prod_{j=1}^{k-1}(2 - \omega_j)} \left(2^n - \sum_{j=1}^{k-1} \prod_{m \neq j}^{k-1} \frac{2 - \omega_m}{\omega_j - \omega_m} \omega_j^n\right).$$

In this paper we generalize the Jacobsthal recursion as

$$J_{n+k}^{(k)} = \sum_{j=1}^{k-1} J_{n+k-j}^{(k)} + 2J_n^{(k)},$$

with $n \geq 0$ and initial conditions $J_0^{(k)} = 0$ and $J_i^{(k)} = 1$ for $i = 1, \ldots k-1$. For the k^{th} order Jacobsthal -Lucas numbers $j_n^{(k)}$ we use the same recursion with initial conditions $j_i^{(k)} = j_i^{(k-1)}$ for $i = 0 \ldots k-1$. With the change of initial conditions a similar compact form for k^{th} order Binet formulae appears to be unobtainable as indicated in the examples below.

Ordinary generating function

A formula for the ordinary generating function for all generalized Fibonacci numbers has been addressed in other papers. For example, that given in [11] for the recurrence

$$a_n = b_{k-1}a_{n-1} + b_{k-2}a_{n-2} + \cdots + b_0 a_{n-k}$$

with arbitrary constant coefficients, b_j, and with arbitrary initial conditions is

$$g(x) = \frac{a_0 + \sum_{i=1}^{k-1} \left(a_i - \sum_{j=0}^{i} b_{k-i+j} a_j \right) x^i}{1 - \sum_{i=1}^{k} b_{k-i} x^i}. \tag{5.1}$$

Here we exhibit (5.1) for the k^{th} order Jacobsthal case (which could also be obtained by using the same procedure used in deriving the generating function for the 3^{rd} order case) namely

$$\sum_{i=0}^{\infty} J_i^{(k)} x^i = \frac{J_0^{(k)} + (J_1^{(k)} - J_0^{(k)})x + \cdots + (J_{k-1}^{(k)} - J_{k-2}^{(k)} - \cdots 2J_0^{(k)})x^{k-1}}{1 - x - x^x - \cdots - 2x^k}.$$

Examples

(1) The Fourth Order Jacobsthal and Jacobsthal–Lucas numbers

Recurrence relations

$$J_{n+4}^{(4)} = J_{n+3}^{(4)} + J_{n+2}^{(4)} + J_{n+1}^{(4)} + 2J_n^{(4)},$$

where $n \geq 0$ and $J_0^{(4)} = 0, J_1^{(4)} = J_2^{(4)} = J_3^{(4)} = 1$.

$$j_{n+4}^{(4)} = j_{n+3}^{(4)} + j_{n+2}^{(4)} + j_{n+1}^{(4)} + 2j_n^{(4)},$$

where $n \geq 0$ and $j_0^{(4)} = 2, j_1^{(4)} = 1, j_2^{(4)} = 5, j_3^{(4)} = 10$.

Table of values

n	0	1	2	3	4	5	6	7	8	9	10	...
$J_n^{(4)}$	0	1	1	1	3	7	13	25	51	103	205	...
$j_n^{(4)}$	2	1	5	10	20	37	77	154	308	613	1229	...

Binet form

The auxiliary equation, and its roots are given by

$$x^4 - x^3 - x^2 - x - 2 = 0, \;\; x_1 = 2, x_2 = -1, x_3 = i, x_4 = -i,$$

and the Binet formulas can be written as

$$J_n^{(4)} = \frac{1}{8+i}\left(2^n - \frac{1}{2}(1+8i)i^n + \frac{1}{2}(3+i)(-1)^n - \frac{1}{2}(4-7i)(-i)^n\right)$$

and

$$j_n^{(4)} = \frac{104(1-3i)2^n - 15(11+3i)i^n - 6(6+17i)(-1)^n - 15(7+9i)(-i)^n}{4(16-63i)}.$$

Rewriting these in terms of the roots of unity, ω_j does not suggest a pattern when compared with the 2^{nd} and 3^{rd} order cases.

Simson's identity

$$\begin{vmatrix} J_{n+3}^{(4)} & J_{n+2}^{(4)} & J_{n+1}^{(4)} & J_n^{(4)} \\ J_{n+2}^{(4)} & J_{n+1}^{(4)} & J_n^{(4)} & J_{n-1}^{(4)} \\ J_{n+1}^{(4)} & J_n^{(4)} & J_{n-1}^{(4)} & J_{n-2}^{(4)} \\ J_n^{(4)} & J_{n-1}^{(4)} & J_{n-2}^{(4)} & J_{n-3}^{(4)} \end{vmatrix} = 0, \quad \begin{vmatrix} j_{n+3}^{(4)} & j_{n+2}^{(4)} & j_{n+1}^{(4)} & j_n^{(4)} \\ j_{n+2}^{(4)} & j_{n+1}^{(4)} & j_n^{(4)} & j_{n-1}^{(4)} \\ j_{n+1}^{(4)} & j_n^{(4)} & j_{n-1}^{(4)} & j_{n-2}^{(4)} \\ j_n^{(4)} & j_{n-1}^{(4)} & j_{n-2}^{(4)} & j_{n-3}^{(4)} \end{vmatrix} = 2^{n-2}\cdot 3^5.$$

Summation formulas

$$\sum_{k=0}^{n} J_k^{(4)} = \begin{cases} J_{n+1}^{(4)} & \text{if } n \equiv \pm 1 \bmod 4 \\ J_{n+1}^{(4)} - 1 & \text{if } n \equiv 0 \bmod 4 \\ J_{n+1}^{(4)} + 1 & \text{if } n \equiv 2 \bmod 4 \end{cases}, \sum_{k=0}^{n} j_k^{(4)} = \begin{cases} j_{n+1}^{(4)} - 2 & \text{if } n \not\equiv 0 \bmod 4 \\ j_{n+1}^{(4)} + 1 & \text{if } n \equiv 0 \bmod 4 \end{cases}.$$

Miscellaneous fourth order identities

$$6J_n^{(4)} + j_n^{(4)} = \begin{cases} j_{n+1}^{(4)} + 1 & \text{if } n \equiv 0 \bmod 4 \\ j_{n+1}^{(4)} + 2 & \text{if } n \equiv 1 \bmod 4 \\ j_{n+1}^{(4)} + 1 & \text{if } n \equiv 2 \bmod 4 \\ j_{n+1}^{(4)} - 4 & \text{if } n \equiv 3 \bmod 4 \end{cases}.$$

$$j_n^{(4)} - 6J_n^{(4)} = \begin{cases} 2 & \text{if } n \equiv 0 \bmod 4 \\ -5 & \text{if } n \equiv 1 \bmod 4 \\ -1 & \text{if } n \equiv 2 \bmod 4 \\ 4 & \text{if } n \equiv 3 \bmod 4 \end{cases}.$$

$$J_n^{(4)} + j_n^{(4)} = \begin{cases} J_{n+2}^{(4)} & \text{if } n \equiv 0 \bmod 4 \\ J_{n+2}^{(4)} + 2 & \text{if } n \equiv 1 \bmod 4 \\ J_{n+2}^{(4)} - 1 & \text{if } n \equiv 2 \bmod 4 \\ J_{n+2}^{(4)} - 1 & \text{if } n \equiv 3 \bmod 4 \end{cases}.$$

In this case the product of the Jacobsthal and Jacobsthal–Lucas functions is somewhat less appealing than in previous cases:

$$24J_n^{(4)}j_n^{(4)} = \begin{cases} (j_{n+1}^{(4)}+1)^2-4 & \text{if } n \equiv 0 \bmod 4 \\ (j_{n+1}^{(4)}+2)^2-25 & \text{if } n \equiv 1 \bmod 4 \\ (j_{n+2}^{(4)}+1)^2-1 & \text{if } n \equiv 2 \bmod 4 \\ (j_{n+2}^{(4)}-4)^2-16 & \text{if } n \equiv 3 \bmod 4 \end{cases}.$$

(2) The Fifth Order Jacobsthal and Jacobsthal–Lucas numbers
Recurrence relations

$$J_{n+5}^{(5)} = J_{n+4}^{(4)} + J_{n+3}^{(5)} + J_{n+2}^{(5)} + J_{n+1}^{(5)} + 2J_n^{(5)},$$

where $n \geq 0$ and $J_0^{(5)} = 0, J_1^{(5)} = J_2^{(5)} = J_3^{(5)} = J_4^{(5)} = 1$.

$$j_{n+5}^{(5)} = j_{n+4}^{(5)} + j_{n+3}^{(5)} + j_{n+2}^{(5)} + j_{n+1}^{(5)} + 2j_n^{(5)},$$

where $n \geq 0$ and $j_0^{(5)} = 2, j_1^{(5)} = 1, j_2^{(5)} = 5, j_3^{(5)} = 10, j_4^{(5)} = 20$.
Table of values

n	0	1	2	3	4	5	6	7	8	9	10	...
$J_n^{(5)}$	0	1	1	1	1	4	9	17	33	65	132	...
$j_n^{(5)}$	2	1	5	10	20	40	77	157	314	628	1256	...

Binet form

The auxiliary equation, and its roots are given by

$$x^5 - x^4 - x^3 - x^2 - x - 2 = 0, x_1 = 2, x_2 = \omega_1, x_3 = \omega_2, x_4 = \omega_3, x_5 = \omega_4,$$

where for $m = 1, 2, 3, 4$, $\omega_m = \exp\left(\frac{2\pi i m}{5}\right)$. The Binet formulas can be written as

$$\begin{aligned} J_n^{(5)} &= \frac{-4}{33}2^n - \frac{24+43\omega_1+37\omega_2-59\omega_3-45\omega_4}{155}\omega_1^n \\ &+ \frac{24-59\omega_1+43\omega_2-45\omega_3+37\omega_4}{155}\omega_2^n + \frac{24+37\omega_1-45\omega_2+43\omega_3-59\omega_4}{155}\omega_3^n \\ &- \frac{24-45\omega_1-59\omega_2+37\omega_3+43\omega_4}{155}\omega_4^n, \end{aligned}$$

and similarly

$$\begin{aligned} j_n^{(5)} &= \frac{42}{33}2^n + \frac{3(14-24\omega_1-12\omega_2+25\omega_3-3\omega_4)}{155(\omega_1-\omega_2-\omega_3-\omega_4)}\omega_1^n \\ &+ \frac{3(14+25\omega_1-24\omega_2-3\omega_3+12\omega_4)}{155(\omega_1-\omega_2-\omega_3-\omega_4)}\omega_2^n + \frac{3(14-12\omega_1-3\omega_2-24\omega_3+25\omega_4)}{155(\omega_1-\omega_2-\omega_3-\omega_4)}\omega_3^n \\ &- \frac{3(14-3\omega_1+25\omega_2-12\omega_3-24\omega_4)}{155(\omega_1-\omega_2-\omega_3-\omega_4)}\omega_4^n, \end{aligned}$$

Simson's identity

$$\begin{vmatrix} J_{n+4}^{(5)} & J_{n+3}^{(5)} & J_{n+2}^{(5)} & J_{n+1}^{(5)} & J_{n}^{(5)} \\ J_{n+3}^{(5)} & J_{n+2}^{(5)} & J_{n+1}^{(5)} & J_{n}^{(5)} & J_{n-1}^{(5)} \\ J_{n+2}^{(5)} & J_{n+1}^{(5)} & J_{n}^{(5)} & J_{n-1}^{(5)} & J_{n-2}^{(5)} \\ J_{n+1}^{(5)} & J_{n}^{(5)} & J_{n-1}^{(5)} & J_{n-2}^{(5)} & J_{n-3}^{(5)} \\ J_{n}^{(5)} & J_{n-1}^{(5)} & J_{n-2}^{(5)} & J_{n-3}^{(5)} & J_{n-4}^{(5)} \end{vmatrix} = 2^{n-2} \cdot 11.$$

$$\begin{vmatrix} j_{n+4}^{(5)} & j_{n+3}^{(5)} & j_{n+2}^{(5)} & j_{n+1}^{(5)} & j_{n}^{(5)} \\ j_{n+3}^{(5)} & j_{n+2}^{(5)} & j_{n+1}^{(5)} & j_{n}^{(5)} & j_{n-1}^{(5)} \\ j_{n+2}^{(5)} & j_{n+1}^{(5)} & j_{n}^{(5)} & j_{n-1}^{(5)} & j_{n-2}^{(5)} \\ j_{n+1}^{(5)} & j_{n}^{(5)} & j_{n-1}^{(5)} & j_{n-2}^{(5)} & j_{n-3}^{(5)} \\ j_{n}^{(5)} & j_{n-1}^{(5)} & j_{n-2}^{(5)} & j_{n-3}^{(5)} & j_{n-4}^{(5)} \end{vmatrix} = 2^{n-3} \cdot 3^4 \cdot 19.$$

Summation formulas

$$\sum_{k=0}^{n} J_k^{(5)} = \begin{cases} J_{n+1}^{(5)} & \text{if } n \equiv \pm 1 \bmod 5 \\ J_{n+1}^{(5)} - 1 & \text{if } n \equiv 0 \bmod 5 \\ J_{n+1}^{(5)} + 1 & \text{if } n \equiv 2 \bmod 5 \\ J_{n+1}^{(5)} + 2 & \text{if } n \equiv 3 \bmod 5 \end{cases}, \sum_{k=0}^{n} j_k^{(5)} = \begin{cases} j_{n+1}^{(5)} - 2 & \text{if } n \not\equiv 0 \bmod 5 \\ j_{n+1}^{(5)} + 1 & \text{if } n \equiv 0 \bmod 5 \end{cases}.$$

Miscellaneous fifth order identities

$$j_n^{(5)} + 6J_n^{(5)} = \begin{cases} 2^{n+1} & \text{if } n \equiv 0 \bmod 5 \\ 2^{n+1} + 3 & \text{if } n \equiv 1 \bmod 5 \\ 2^{n+1} + 3 & \text{if } n \equiv 2 \bmod 5 \\ 2^{n+1} & \text{if } n \equiv 3 \bmod 5 \\ 2^{n+1} - 6 & \text{if } n \equiv 4 \bmod 5 \end{cases}. \tag{5.2}$$

$$j_n^{(5)} - 6J_n^{(5)} = \begin{cases} 2^{n-1} - 3(J_{n-3}^{(5)} - 1) & \text{if } n \equiv 0 \bmod 5 \\ 2^{n-1} - 3(J_{n-3}^{(5)} + 2) & \text{if } n \equiv 1 \bmod 5 \\ 2^{n-1} - 3(J_{n-3}^{(5)} + 2) & \text{if } n \equiv 2 \bmod 5 \\ 2^{n-1} - 3J_{n-3}^{(5)} & \text{if } n \equiv 3 \bmod 5 \\ 2^{n-1} - 3(J_{n-3}^{(5)} - 3) & \text{if } n \equiv 4 \bmod 5 \end{cases}. \tag{5.3}$$

If we let the right hand side of (5.2) be M and that of (5.3) N, then the following are noted

$$j_n^{(5)} = \frac{M+N}{2}, J_n^{(5)} = \frac{M-N}{12}.$$

$$j_n^{(5)} + J_n^{(5)} = \frac{7M+5N}{12}, j_n^{(5)} - J_n^{(5)} = \frac{5M+7N}{12}, J_n^{(5)} j_n^{(5)} = \frac{M^2-N^2}{24}.$$

and finally

$$(j_n^{(5)})^2 - 36(J_n^{(5)})^2 = MN \text{ and } (j_n^{(5)})^2 + 36(J_n^{(5)})^2 = \frac{M^2+N^2}{2}.$$

6. Concluding comments

The authors believe that most of these results are new but unfortunately, many of them do not seem to fall into a convenient pattern for generalization to an n^{th} order case. While investigating the Simson (Cassini/Catalan) identity for higher order Jacobsthal numbers a general Simson identity for an arbitrary n^{th} order recursive relation was discovered and proved. This generalized Simson identity has resulted in a short paper that will be submitted to the Fibonacci Quarterly. Certainly many more identities could be generated from those obtained here and by investigating Jacobsthal and Jacobsthal-Lucas polynomials. For example, using the methods presented in [1, 2, 6, 13, 16] a plethora of identities generated from ordinary generating functions should be possible; and similarly using [2, 5, 12, 14], identities obtained from the exponential generating functions should arise. Further investigations for these and other methods useful in discovering identities for the higher order Jacobsthal and Jacobsthal-Lucas numbers will be addressed in a future paper.

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