On Some Identities of $k$-Jacobsthal-Lucas Numbers

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Abstract

In this paper we present the sequence of the $k$-Jacobsthal-Lucas numbers that generalizes the Jacobsthal-Lucas sequence introduced by Horadam in 1988. For this new sequence we establish an explicit formula for the term of order $n$, the well-known Binet’s formula, Catalan’s and d’Ocagne’s Identities and a generating function.

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1 Introduction

Several recurrence sequences of positive integers have been object of study for many researchers. Examples of these are the Fibonacci, Lucas, Pell, Pell-Lucas, Modified Pell, Jacobhstal, Jacobsthal-Lucas sequences among others(see [8], [10], [12], [13]). About them there is a vast literature studying several properties, ones involving the well-known Binet’s formula, Catalan’s, Cassini’s and d’Ocagne’s identities and there is also a vast literature dedicated to the study of other properties involving each sequence (see [7] and [14]).

More recently, some of these sequences were generalized for any positive real number $k$: the study of the $k$-Fibonacci sequence, the $k$-Lucas sequence, the $k$-Pell sequence, the $k$-Pell-Lucas sequence, the Modified $k$-Pell sequence and the $k$-Jacobhstal sequence appeared (see [1], [11], [2], [4], [5], [6] and [3]).
In this paper we generalize the sequence of Jacobsthal-Lucas numbers and study by introducing the sequence of the $k$-Jacobsthal-Lucas numbers. We give an explicit formula for the term of order $n$ of this sequence, the well-known Binet’s formula, Catalan’s and d’Ocagne’s Identities and a generating function for this recurrence sequence.

2 Identities

Let us define the sequence of the $k$-Jacobsthal-Lucas numbers \( \{j_{k,n}\}_{n \in \mathbb{N}} \) as follows:

\[
j_{k,n+1} = k j_{k,n} + 2 j_{k,n-1}
\]

where the initial conditions are:

\[
\begin{align*}
\begin{cases}
  j_{k,0} &= 2 \\ 
  j_{k,1} &= k
\end{cases}
\]

for any positive real number $k$. If $k = 1$ we get the sequence of Jacobsthal-Lucas numbers defined by Horadam in [9]. The characteristic equation associated to the recurrence relation (1) is

\[
x^2 = kx + 2
\]

with roots $r_1$ and $r_2$ given by $r_1 = \frac{k + \sqrt{k^2 + 8}}{2}$ and $r_2 = \frac{k - \sqrt{k^2 + 8}}{2}$.

Note that $r_1 r_2 = -2$; $r_1 + r_2 = k$ and $r_1 - r_2 = \sqrt{k^2 + 8}$. Associated to (1) the term of order $n$ of the $k$-Jacobsthal-Lucas sequence, can be written by the following identity $j_{k,n} = c_1 r_1^n + c_2 r_2^n$ for some constants $c_1, c_2$.

Solving the system of two linear equations corresponding to the initial conditions (2),

\[
\begin{align*}
\begin{cases}
  2 &= c_1 + c_2 \\ 
  k &= c_1 r_1 + c_2 r_2
\end{cases}
\]

we obtain $c_1 = c_2 = 1$. So, we get the next Proposition:

**Proposition 2.1** (Binet’s Formula): The $n$th $k$-Jacobsthal-Lucas number $j_{k,n}$ is given by

\[
j_{k,n} = r_1^n + r_2^n
\]

where $r_1$ and $r_2$ are the roots of the characteristic equation (3) and $r_1 > r_2$. 
Proof. We use induction on \( n \). Taking into account the initial conditions (2), we note that the equation (5) is valid for \( n = 0 \) and \( n = 1 \). Now assume that (5) is true for \( 0 \leq s \leq n \), that is, \( j_{k,s} = r_1^s + r_2^s \) for every \( s \in \{0, \ldots, n\} \).

Using (1) and taking into account that

\[
\begin{align*}
\sum_{j=0}^{n-1} & k^{n-1-j} r_1^{n-j} + 2 \left( r_1^{n-1+j} + r_2^{n-1+j} \right) \\
&= r_1^{n-1} (kr_1 + 2) + r_2^{n-1} (kr_2 + 2) \\
&= r_1^{n-1} ((r_1 + r_2) r_1 + 2) + r_2^{n-1} ((r_1 + r_2) r_2 + 2) \\
&= r_1^{n-1} (r_2 + r_1 r_2 + 2) + r_2^{n-1} (r_1 r_2 + r_2^2 + 2) \\
&= r_1^{n+1} + r_2^{n+1}.
\end{align*}
\]

Consequently, the Binet’s Formula is true for any positive integer \( n \). \( \Box \)

The use of the Binet’s Formula (5) and the fact that \( r_1 r_2 = -2 \) allows us to obtain Catalan’s Identity.

**Proposition 2.2 (Catalan’s Identity):**

\[
j_{k,n-r} j_{k,n+r} - j_{k,n}^2 = (-2)^{n-r} (j_{k,r}^2 - (-2)^{r+2}). \tag{6}
\]

Proof. We have

\[
j_{k,n-r} j_{k,n+r} - j_{k,n}^2 = \left( r_1^{n-r} + r_2^{n-r} \right) \left( r_1^{n+r} + r_2^{n+r} \right) - \left( r_1^n + r_2^n \right)^2
\]

\[
= (-2)^n \left( \frac{r_1^2}{r_2^r} + \frac{r_2^r}{r_1^r} - 2 \right)
\]

\[
= (-2)^n \left( \frac{r_2^r + r_1^r - 2(r_1 r_2)^r}{(r_1 r_2)^r} \right)
\]

\[
= (-2)^n \left( \frac{r_2^r + r_1^r - 2(r_1 r_2)^r}{(-2)^r} \right)
\]

\[
= (-2)^{n-r} (r_2^{2r} + r_1^{2r} - 2(r_1 r_2)^r)
\]

\[
= (-2)^{n-r} (r_1^r + r_2^r)^2 - 4(r_1 r_2)^r)
\]

\[
= (-2)^{n-r} \left( j_{k,r}^2 - 4(-2)^r \right),
\]

as required. \( \Box \)

Substituting \( r = 1 \) in Catalan’s Identity (6), yields

\[
j_{k,n-1} j_{k,n+1} - j_{k,n}^2 = (-2)^{n-1} \left( j_{k,1}^2 - 4(-2) \right)
\]

and using the initial condition \( j_{k,1} = k \), we obtain the Cassini’s identity for \( k \)-Jacobsthal-Lucas sequence.
Proposition 2.3 (Cassini’s Identity):
\[ j_{k,n-1} j_{k,n+1} - j_{k,n}^2 = (-2)^{n-1} (k^2 + 8). \] (7)

The d’Ocagne’s identity can also be obtained from the Binet’s Formula (5) and the fact that \( r_1 r_2 = -2 \) and \( m > n \).

Proposition 2.4 (d’Ocagne’s Identity): For \( m > n \),
\[ j_{k,m} j_{k,n+1} - j_{k,m+1} j_{k,n} = (-2)^n \sqrt{k^2 + 8} \left( j_{k,m-n} - 2^{n-m+1} \left( k + \sqrt{k^2 + 8} \right)^{m-n} \right). \]

Proof. For \( m > n \), we have
\[
j_{k,m} j_{k,n+1} - j_{k,m+1} j_{k,n} = (r_1^m + r_2^m) (r_1^{n+1} + r_2^{n+1}) - (r_1^{m+1} + r_2^{m+1}) (r_1^n + r_2^n)
= (-2)^n (r_1^{m-n} r_2 + r_1 r_2^{m-n} - r_1^{m-n} r_1 - r_2^{m-n} r_2)
= (-2)^n (r_1 (r_2 - r_1) + r_2 (r_1 - r_2))
= (-2)^n (r_1 - r_2) (r_2^{m-n} - r_1^{m-n})
= (-2)^n \sqrt{k^2 + 8} (r_1^{m-n} + r_2^{m-n} - 2 r_1^{m-n})
= (-2)^n \sqrt{k^2 + 8} (j_{k,m-n} - 2 (k + \sqrt{k^2 + 8})^{m-n})
= (-2)^n \sqrt{k^2 + 8} (j_{k,m-n} - 2^{n-m+1} (k + \sqrt{k^2 + 8})^{-m-n}).
\]
as required. \(\square\)

The limit property stated in the following Proposition is also deduced using Binet’s Formula (5).

Proposition 2.5 For \( m > n \),
\[ \lim_{n \to \infty} \frac{j_{k,n}}{j_{k,n-1}} = r_1. \] (8)

Proof. We have
\[ \lim_{n \to \infty} \frac{j_{k,n}}{j_{k,n-1}} = \lim_{n \to \infty} \frac{r_1^n + r_2^n}{r_1^{n-1} + r_2^{n-1}}. \]

Since \( \left| \frac{r_2}{r_1} \right| < 1 \), then \( \lim_{n \to \infty} \left( \frac{r_2}{r_1} \right)^n = 0 \) and therefore
\[ \lim_{n \to \infty} \frac{j_{k,n}}{j_{k,n-1}} = \lim_{n \to \infty} \frac{1 + \left( \frac{r_2}{r_1} \right)^n}{ \frac{1}{r_1} + \left( \frac{r_2}{r_1} \right)^{n-1} \frac{1}{r_2}} = \frac{1}{r_1}, \]
and the result follows. \(\square\)
3 Generating Function

In the next Proposition we present a generating function for the sequence of the $k$-Jacobsthal-Lucas numbers.

**Proposition 3.1** (*Generating function of the $k$-Jacobsthal-Lucas numbers*)

\[
j_k(x) = \frac{2 - kx}{1 - kx - 2x^2}
\]

**Proof.** Let us suppose that the $k$-Jacobsthal-Lucas numbers are the coefficients of a power series centered at the origin, that is convergent in \( \left[-\frac{1}{r_1}, \frac{1}{r_1}\right] \), taking in account the Proposition (2.5). To the sum of this power series, \( j_k(x) \), we call generating function of the $k$-Jacobsthal-Lucas numbers. So we have

\[
j_k(x) = j_{k,0} + j_{k,1}x + j_{k,2}x^2 + \cdots + j_{k,n}x^n + \cdots
\]

and then,

\[
kxj_k(x) = kj_{k,0}x + kj_{k,1}x^2 + kj_{k,2}x^3 + \cdots + kj_{k,n}x^{n+1} + \cdots
\]

\[
2x^2j_k(x) = 2j_{k,0}x^2 + 2j_{k,1}x^3 + 2j_{k,2}x^4 + \cdots + 2j_{k,n}x^{n+2} + \cdots.
\]

Since (1) e (2) we obtain

\[
j_k(x) - kxj_k(x) - 2x^2j_k(x) = 2 - kx
\]

and then we conclude that

\[
j_k(x) = \frac{2 - kx}{1 - kx - 2x^2}
\]

\[
\Box
\]

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