Matrix Valued little q-Jacobi Polynomials Related to Matrix Valued Basic Hypergeometric Series

Noud Aldenhoven[1]

Joint work with Erik Koelink[1] and Ana M. de los Ríos[2]

[1]Radboud Universiteit Nijmegen

[2]Universidad de Sevilla

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http://www.math.ru.nl/~aldenhoven

Overview

1. Orthogonal polynomials and their connection to Lie theory.
2. Matrix valued little q-Jacobi polynomials.
3. Matrix valued basic hypergeometric series.

This talk is based on the preprint 2×2 *Matrix Valued little q-Jacobi Polynomials*, arXiv:1308.2540.

Part 1

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Orthogonal Polynomials and their connection to Lie Theory.

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Orthogonal polynomials

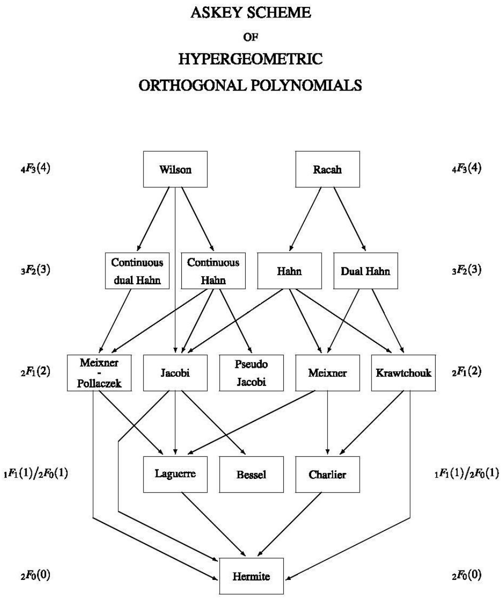
References: [KS98]

An orthogonal polynomial sequence $(p_n)_{n\geq 0}$ is a family of polynomials over $\mathbb{C}$ which are orthogonal with respect to some inner product $\langle , \rangle$, i.e.

$$\mathrm{dg}(p_n) = n, \qquad \langle p_m, p_n \rangle = C_m \delta_{m,n}.$$

We are interested in orthogonal polynomials sequences which are solutions to a second order differential or difference equation and can be represented as a hypergeometric series.

The Askey scheme



Quantum analogues of special functions

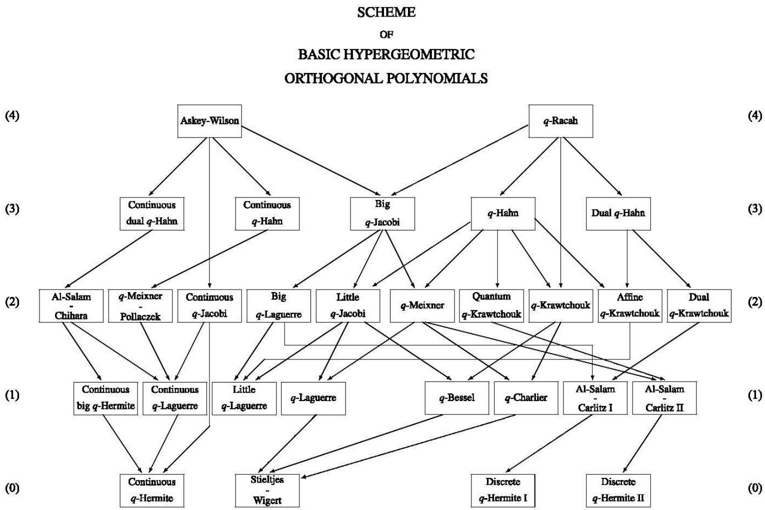
Theorem (Weierstrass, unpublished)

Let $f : \mathbb{C} \to \mathbb{C}$ *be a complex analytic function and suppose it satisfy an algebraic addition formula, i.e. there exists* $P(x, y, z) \in \mathbb{C}[x, y, z]$ *such that for all* $z_1, z_2 \in \mathbb{C}$ *we have* $P(f(x), f(y), f(x + y)) = 0$*, then*

1. f *is a rational functions in* z*,*
2. f *is a rational function in* q^z*,*
3. f *is an elliptic function.*

The Askey scheme has a quantum analogue which are all orthogonal polynomials sequences which are solutions to a second order q-difference equation and can be represented as a basic hypergeometric series.

The q-Askey Scheme



Non-commutative orthogonal polynomials

Now suppose we take orthogonal polynomial sequences over some non-commutative algebra - for example $\mathrm{Mat}_2(\mathbb{C})$.

Question (The big question)

Does there exist a non-commutative Askey Scheme?

Problem (The big problem)

I have no clue at all how to find a non-commutative Askey Scheme and if it would exist, it will be too big and contain too many non-interesting cases.

Problem (A smaller problem)

Do there exist interesting examples of non-commutative orthogonal polynomials?

Approach

Use Lie theory to find interesting non-commutative examples.

Spherical functions

Fix pair (G, K) where G is a compact Lie group and K a compact subgroup of G.

Definition

Let $t \in \hat{K}$. A spherical function $\phi : G \to \mathrm{End}(V)$ on G of type t is a continuous function such that

1. $\phi(e) = I$,
2. $\phi(k_1 a k_2) = t(k_1)\phi(a)t(k_2)$ for all $k_1, k_2 \in K$ and $a \in G$.
3. $\phi(x)\phi(y) = \int_K \xi_t(k^{-1})\phi(xky)dk$ for all $x, y \in G$,

where ξ_t is the character times the degree of t.

Examples

Suppose we take the trivial representation $t = 1$. If we have a KAK-decomposition, $G = KAK$, then by property 2 ($\phi(k_1 a k_2) = t(k_1)\phi(a)t(k_2)$) a spherical function is defined on A and $\phi : A \to \mathbb{C}$.

Example

- On $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$, $\phi(a)$ is a Chebychev polynomial.
- On $(\mathrm{SU}(3), \mathrm{U}(2))$, $\phi(a)$ is a Jacobi polynomial.

Examples

References: [GPT03], [KvPR12]

Now suppose that t is not the trivial representation. The spherical function $\phi : G \to \mathrm{End}(V)$ can take values in a non-commutative matrix ring of $\mathrm{End}(V)$.

Example

- On $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$, $\phi(a)$ is a matrix-valued analogue of Chebychev polynomials.
- On $(\mathrm{SU}(3), \mathrm{U}(2))$, $\phi(a)$ is a matrix-valued analogue of Jacobi polynomials.

Quantum spherical functions

References: [Let04], Dijkhuizen, Sugitani, Koornwinder, Noumi, Kolb

Let G be a compact Lie group and $\mathfrak{g}$ its semi-simple Lie algebra. $\mathcal{U}_q(\mathfrak{g})$ is the quantised universal algebra, $\mathcal{A}_q(G)$ the quantised function algebra.

Theorem

For every Gel'fand pair (G, K) there exists a q-analogue $(\mathcal{U}_q(\mathfrak{g}), \mathcal{B})$, where $\mathcal{B}$ is a right coideal of $\mathcal{U}_q(\mathfrak{g})$, i.e. $\Delta(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{U}_q(\mathfrak{g})$.

Definition

$\phi \in \mathcal{A}_q(G)$ is called a scalar quantum spherical function if $\mathcal{B}.\phi = 0 = \phi.\mathcal{B}$.

Examples

For these quantum Gel'fand pairs $(\mathcal{U}_q(\mathfrak{g}), \mathcal{B})$ there exists a quantum Iwasawa decomposition such that $\mathcal{U}_q(\mathfrak{g}) \simeq \mathcal{B} \otimes \mathcal{A} \otimes N$.

Examples

If we restrict the quantum zonal spherical to $\mathcal{A}$ we get:

- For $(\mathcal{U}_q(\mathfrak{sl}(2)) \otimes \mathcal{U}_q(\mathfrak{sl}(2)), \mathcal{B})$, $\phi(a)$ is a (quantum) Chebychev polynomial.
- For $(\mathcal{U}_q(\mathfrak{sl}(3)), \mathcal{B}')$, $\phi(a)$ is a little q-Jacobi polynomial.

Question

What is the matrix valued picture of the quantum spherical functions?

Question

What are matrix valued little q-Jacobi polynomials?

Part 2

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Matrix valued little q-Jacobi polynomials.

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little q-Jacobi Polynomials

Fix $0 < q < 1$ and take $0 < a < q^{-1}$ and $b < q^{-1}$. The little q-Jacobi polynomials $(p_n(x))_{n\geq 0}$ are defined by

$$p_n(x; a, b|q) = {}_2\phi_1\left[\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right].$$

little q-Jacobi polynomials

The little q-Jacobi polynomials satisfy the q-Difference equation.

$$\lambda_n p_n(x) = B(x)p_n(qx) - (B(x) + D(x))p_n(x) + D(x)p_n(q^{-1}x),$$

where

$$\begin{aligned}\lambda_n &= q^{-n}(1 - q^n)(1 - abq^{n+1}),\\ B(x) &= a(bq - x^{-1}),\\ D(x) &= 1 - x^{-1}.\end{aligned}$$

Every polynomials solution is unique up to a constant.

little q-Jacobi polynomials

The little q-Jacobi polynomials are orthogonal with respect to the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} (aq)^n \frac{(bq;q)_n}{(q;q)_n} f(q^n)\overline{g(q^n)}$$

i.e.

$$\langle p_m, p_n \rangle = C_n \delta_{m,n}.$$

where $C_n > 0$.

Matrix valued polynomials

Let $N \geq 1$ and $P \in \mathrm{Mat}_N(\mathbb{C})[x]$. Then

$$P(x) = R_m x^m + R_{m-1} x^{m-1} + \ldots + R_1 x + R_0,$$

where $R_i \in \mathrm{Mat}_N(\mathbb{C})$.

Matrix valued *orthogonal* polynomials

References: [DPS08]

Let $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$ and define an inner product by

$$\langle P, Q \rangle = \sum_{n=0}^{\infty} q^n P(q^n) W(q^n) (Q(q^n))^* \in \mathrm{Mat}_N(\mathbb{C}),$$

where $W : \{q^n : n \geq 0\} \to \mathrm{Mat}_N(\mathbb{C})$, such that

- $W(q^n)$ is positive definite for all n.
- W is Hermitian, i.e. $(W(q^n))^* = W(q^n)$.
- $\langle x^n I, I \rangle \in \mathrm{Mat}_N(\mathbb{C})$ for all $n \geq 0$.
- If $P \in \mathrm{Mat}_N(\mathbb{C})[x]$ has a non-singular leading coefficient, then $\langle P, P \rangle$ is non-singular.

Matrix valued orthogonal polynomials

A sequence $(P_m)_{m\geq 0}$ of matrix-valued polynomials is orthogonal if for all $m, n \geq 0$:

- $\deg(P_m) = m$.
- P_m has non-singular leading coefficient.
- $\langle P_m, P_n\rangle = \Lambda_m \delta_{m,n}$, where Λ_m is a positive definite matrix.

Trivial matrix valued orthogonal polynomials

A weight matrix $W(x)$ is called trivial if there exists a unitary matrix K, independent of x, and a diagonal matrix $D(x)$ such that for all $n \geq 0$

$$W(q^n) = K\, D(q^n)\, K^*.$$

In this case the orthogonal polynomials are of the form

$$P_n(x) = K \begin{pmatrix} p_{n,1}(x) & 0 & \cdots & 0 \\ 0 & p_{n,2}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{n,N}(x) \end{pmatrix}$$

Symmetric q-difference operators

References: [GT07]

Let D be a q-difference operator defined by

$$DP(x) = P(qx)F_1(x) + P(x)F_0(x) + P(q^{-1}x)F_{-1}(x),$$

where $P : \mathbb{C} \to \mathrm{Mat}_N(\mathbb{C})$ and $F_1, F_0, F_{-1} \in \mathrm{Mat}_N(\mathbb{C})[x^{-1}]$.

Definition

A q-difference operator D is called symmetric if $\langle DP, Q \rangle = \langle P, DQ \rangle$ for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$.

Theorem

If D is symmetric and preserves polynomials then there exists a MVOPS $(P_n)_{n\geq 0}$ which are eigenvector of D, i.e. $DP_n = \Lambda_n P_n$.

Symmetric q-difference operators

Theorem

Let

$$DP(x) = P(qx)F_1(x) + P(x)F_0(x) + P(q^{-1}x)F_{-1}(x),$$

$$\langle P, Q\rangle = \sum_{n=0}^{\infty} q^n P(q^n)W(q^n)(Q(q^n))^*.$$

If symmetry equations

$$F_0(q^n)W(q^n) = W(q^n)(F_0(q^n))^*, \qquad n \in \mathbb{N},$$

$$qF_1(q^{n-1})W(q^{n-1}) = W(q^n)(F_{-1}(q^n))^*, \qquad n \in \mathbb{N}\backslash\{0\},$$

and boundary conditions

$$W(1)(F_{-1}(1))^* = 0, \quad F_1(q^n)W(q^n) = o(q^{-n}), \quad \text{as } n \to \infty,$$

hold, then D is symmetric.

Symmetric q-difference operators

Idea of the proof: Take the truncated inner product

$$\langle P, Q\rangle_M = \sum_{n=0}^{M} q^n P(q^n) W(q^n) Q^*(q^n).$$

Write $P(q^n) = A_0 + \mathcal{O}(q^n)$ and $Q(q^n) = B_0 + \mathcal{O}(q^n)$. Use symmetry equations and boundary conditions to calculate

$$\begin{aligned}\langle DP, Q\rangle_M - \langle P, DQ\rangle_M \\ &= q^M A_0 \left(F_1(q^M) W(q^M) - W(q^M)(F_1(q^M))^*\right) B_0 + \mathcal{O}(q^M).\end{aligned}$$

Then by the last boundary condition

$$\lim_{M\to\infty} (\langle DP, Q\rangle_M - \langle P, DQ\rangle_M) = 0.$$

A construction

If

$$F_1(q^{n-1})F_{-1}(q^n) = |s(q^n)|^2 I, \qquad n \in \mathbb{N}\backslash\{0\}.$$

where $s(q^n) \in \mathbb{C}\backslash\{0\}$. Let $T(1) = I$ and

$$T(q^n) = q^{\frac{1}{2}} \frac{s(q^n)}{|s(q^n)|^2} F_1(q^{n-1})T(q^{n-1}), \qquad n \in \mathbb{N}\backslash\{0\}.$$

The inner product

$$\langle P, Q \rangle = \sum_{n=0}^{\infty} P(q^n)T(q^n)T^*(q^n)Q^*(q^n) = \sum_{n=0}^{\infty} q^n P(q^n)W(q^n)Q^*(q^n)$$

satisfies the symmetry equation

$$qF_1(q^{n-1})W(q^{n-1}) = W(q^n)F_{-1}^*(q^n), \qquad n \in \mathbb{N}\backslash\{0\}.$$

2×2 Matrix valued little q-Jacobi polynomials

Consider

$$DP(x) = P(qx)F_1(x) + P(x)F_0(x) + P(q^{-1}x)F_{-1}(x),$$

where

$$F_{-1}(x) = (x^{-1} - 1)A^{-1}, \qquad F_1(x) = (ax^{-1} - abq)A,$$

and

$$A = \begin{pmatrix} q & v(q-1) \\ 0 & 1 \end{pmatrix},$$

where $v \in \mathbb{C}$. Then

$$T(q^n)T^*(q^n) = (aq)^n \frac{(bq;q)_n}{(q;q)_n} A^n (A^*)^n$$

2×2 Matrix valued little q-Jacobi polynomials

Take

$$F_0(x) = K - x^{-1}(A^{-1} + aA), \quad K = \begin{pmatrix} 0 & v(1-q)(q^{-1}-a) \\ 0 & 0 \end{pmatrix}.$$

Theorem

The sequence of matrix-valued orthogonal polynomials $(P_n)_{n\geq 0}$ with respect to the inner product

$$\langle P, Q \rangle = \sum_{n=0}^{\infty} q^n P(q^n)\, a^n \frac{(bq;q)_n}{(q;q)_n} A^n (A^*)^n\, Q^*(q^n)$$

are eigenvectors of

$$DP(x) = P(qx)F_1(x) + P(x)F_0(x) + P(q^{-1}x)F_{-1}(x),$$

2×2 Matrix valued little q-Jacobi polynomials

$$A = \begin{pmatrix} q & v(q-1) \\ 0 & 1 \end{pmatrix}$$

If $v = 0$ then

$$W(q^n) = a^n \frac{(bq;q)_n}{(q;q)_n} A^n (A^*)^n = \begin{pmatrix} w_1(q^n) & 0 \\ 0 & w_2(q^n), \end{pmatrix}.$$

where w_1 and w_2 are little q-Jacobi weights. However in general $W(q^m)W(q^n) \neq W(q^n)W(q^m)$, hence W is not trivial.

Part 3

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Matrix valued basic hypergeometric series.

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Matrix valued basic hypergeometric series

References: [Tir03], [CS10]

Let $(P_n)_{n\geq 0}$ the monic polynomials. Then $DP_n = \Lambda_n P_n$. If

$$E_n = \begin{pmatrix} 1 & \frac{-\mu_n}{1-abq^{2n+2}} v \\ 0 & 1 \end{pmatrix}, \quad \text{where } \mu_n = 1 - q^n + aq^{n+1} - abq^{2n+2},$$

then

$$\tilde{\Lambda}_n = E_n^{-1}\Lambda_n E_n = \mathrm{diag}(-q^{-n} - abq^{n+2}, -q^{-n} - abq^{n+1}).$$

Let $\tilde{P}_n = E_n P_n$, then

$$D\tilde{P}_n = E_n DP_n = E_n \Lambda_n P_n = E_n \Lambda_n E_n^{-1} E_n P_n = \tilde{\Lambda}_n \tilde{P}_n.$$

Matrix valued basic hypergeometric series

We can write

$$\begin{aligned}\tilde{\Lambda}_n\tilde{P}_n(x) &= D\tilde{P}_n(x)\\ &= \tilde{P}_n(q^{-1}x)(x^{-1}-1)A^{-1} + \tilde{P}_n(x)(K - x^{-1}(A^{-1}-aA))\\ &\quad + \tilde{P}_n(qx)(ax^{-1}-abq)A.\end{aligned}$$

Let $\tilde{P}_{i,n}$ be the i-th row of $\tilde{P}_n$, $i = 1, 2$. Multiply from the right by xA such that

$$\begin{aligned}\tilde{P}_{i,n}(q^{-1}x)(1-x) + \tilde{P}_{i,n}(x)(-I - aA^2 + (KA - \tilde{\Lambda}_{i,n}I)x)\\ + \tilde{P}_{i,n}(qx)(aA^2 - abqA^2x) = 0.\end{aligned}$$

Therefore rewrite

$$\tilde{P}_{i,n}(q^{-1}x)(1-x) + \tilde{P}_{i,n}(x)(-I - C + Ax) + \tilde{P}_{i,n}(qx)(C + Bx) = 0.$$

Matrix valued basic hypergeometric series

Suppose we want to solve

$$F(q^{-1}x)(1-x) + F(x)(-I - C + Ax) + F(qx)(C + Bx) = 0,$$

where $F : q^{\mathbb{N}} \to \mathbb{C}^N$ (row-valued!). Use the Frobenius method. Suppose $F(x) = \sum_{k=0}^{\infty} F^k x^k$, $F^k \in \mathbb{C}^N$ (rows). Then

$$F^k(C - q^k I) = \frac{q}{1-q^k} F^{k-1}(I - q^{k-1}A - q^{2k-2}B), \quad k \geq 1.$$

If $C - q^k I$ non-singular, then

$$F^k = \frac{q}{1-q^k} F^{k-1}(I - q^{k-1}A - q^{2k-2}B)(C - q^k I)^{-1}, \quad k \geq 1.$$

Matrix valued basic hypergeometric series

Let $A, B, C \in \mathrm{Mat}_N(\mathbb{C})$. Suppose that the eigenvalues of C are not in $q^{-\mathbb{N}\setminus\{0\}}$. Define

$$(A, B; C; q)_0 = I,$$
$$(A, B; C; q)_k = (A, B; C; q)_{k-1}(I - q^{k-1}A - q^{2k-2}B)(I - q^k C)^{-1},$$

and

$${}_2\eta_1\left[\begin{matrix} A, B \\ C \end{matrix}; q, x\right] = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}(A, B; C; q)_k.$$

Matrix valued basic hypergeometric series

Theorem

$$F(x) = F_0 \; {}_2\eta_1\left[\begin{matrix} A, B \\ C \end{matrix}; q, qx\right] = F_0 \; \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} x^k (A, B; C; q)_k,$$

where $F_0 \in \mathbb{C}^N$*, is a solution of*

$$F(q^{-1}x)(1-x) + F(x)(-I - C + Ax) + F(qx)(C + Bx) = 0.$$

Matrix valued basic hypergeometric series

We had

$$\tilde{P}_{i,n}(q^{-1}x)(1-x) + \tilde{P}_{i,n}(x)(-I - aA^2 + (KA - \tilde{\Lambda}_{i,n}I)x) \\ + \tilde{P}_{i,n}(qx)(aA^2 - abqA^2x) = 0.$$

and therefore by the previous theorem

$$\tilde{P}_{i,n}(x) = \tilde{P}_{i,n}(0)\ {}_2\eta_1\left[\begin{matrix} KA - \tilde{\Lambda}_{i,n}I, -abqA^2 \\ aA^2 \end{matrix}; q, qx\right].$$

Gracias!

The Netherlands

Argentina

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