Jacobi polynomials in Bernstein form

Tomas Sauer

Lehrstuhl für Numerische Mathematik, Justus–Liebig–Universität Gießen, Heinrich–Buff–Ring 44, D–35392 Gießen, Germany

Abstract

The paper describes a method to compute a basis of mutually orthogonal polynomials with respect to an arbitrary Jacobi weight on the simplex. This construction takes place entirely in terms of the coefficients with respect to the so–called Bernstein–Bézier form of a polynomial.

Key words: Jacobi polynomials, simplex, Bernstein form
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1 Introduction

This paper describes a construction of orthogonal polynomials with respect to arbitrary Jacobi weights on d–dimensional simplices, extending the approach from [1] for *Legendre polynomials*. The computations will take place entirely with respect to the so–called *Bernstein–Bézier basis* which is in several respects the most natural way to represent polynomials on the simplex, making it very popular in the area of Computer Aided Geometric Design (CAGD). Besides their appealing geometric properties, some of which we will present and exploit in this paper, the Bernstein–Bézier basis is also known to be the numerically most stable basis on simplices, cf. [2,3]. On the other hand, the condition number of any basis conversion is always worse than the gain of stability obtained by using the Bernstein–Bézier basis, so that algorithms that determine polynomials in the monomial form, say, and then convert them into the Bernstein–Bézier basis are useless from a numerical point of view. Moreover, the application of the classical Gram–Schmidt orthogonalization procedure is also not advisable in terms of the Bernstein–Bézier basis, as the computational effort of degree raising becomes overwhelming, see [3] where the same effect is pointed out in the context of polynomial interpolation.

Because of these reasons, it is reasonable to look for methods that work with polynomials entirely in terms of the Bernstein–Bézier basis, and this is what we will do here for the construction of orthogonal polynomials with arbitrary Jacobi weights. It will turn out that, in contrast to the *Rodriguez formula* which can be already found in [4] for the triangle case, this approach even determines an orthogonal basis, that is, the basis elements of degree n are not only orthogonal to all polynomials of degree $n-1$, they are even mutually orthogonal.

2 Notation

We begin by setting up some terminology. If $V \subset \mathbb{R}^d$ is a nondegenerate simplex with vertices $v_0, \ldots, v_d \in \mathbb{R}^d$, then any point $v \in V$ can be uniquely represented as the convex combination

$$v = \sum_{j=0}^{d} u_j \, v_j, \qquad u_j \geq 0, \quad \sum_{j=0}^{d} u_j = 1.$$

The vector $u = (u_0, \ldots, u_d)$ is called the *barycentric coordinates* of the point v with respect to the simplex V. Being an affine invariant "local" coordinate system with respect to a given simplex, barycentric coordinates have become an important tool in CAGD, for example in the context of multivariate spline surfaces, cf. [5], as they allow to work entirely on the *barycentric standard simplex*

$$\mathbb{S}_d := \left\{ u = (u_0, \ldots, u_d) \in \mathbb{R}^{d+1} \ : \ u_j \geq 0, 1 = |u| := \sum_{j=0}^{d} u_j \right\}. \tag{1}$$

We will also use a different way of writing the standard simplex, namely as

$$\mathbb{S}_d^* := \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \ : \ |x| \leq 1 \right\}, \tag{2}$$

and identify $x \in \mathbb{S}_d^*$ with $u \in \mathbb{S}_d$ by the straightforward relationship $u = (1 - |x|, x)$. Integration over $\mathbb{S}_d$ is now defined as

$$\begin{aligned} \int_{\mathbb{S}_d} f(u)\, du &= \int_{\mathbb{S}_d^*} f\left(1 - |x|, x\right)\, dx \\ &= \int_0^1 \int_0^{1-x_d} \cdots \int_0^{1-x_2-\cdots-x_d} f\left(1 - |x|, x_1, \ldots, x_d\right)\, dx_1 \cdots dx_d, \end{aligned}$$

yielding the normalization $\int_{\mathbb{S}_d} du = \frac{1}{d!}$. Derivatives in barycentric calculus are *directional derivatives*

$$D_v = \sum_{j=0}^{d} v_j \frac{\partial}{\partial u_j}, \qquad \sum_{j=0}^{d} v_j = 0,$$

where the side condition on the *barycentric direction* v stems from the fact that v is interpreted as the difference $u - u'$ of two points $u, u' \in \mathbb{S}_d$ and thus all its components must sum to zero.

A basis of the space Π_n of all polynomials of total degree at most n are the *Bernstein–Bézier basis* or *B–Basis* polynomials

$$B_\beta(u) := \binom{|\beta|}{\beta} u^\beta := \frac{|\beta|!}{\beta_0! \cdots \beta_d!} u_0^{\beta_0} \cdots u_d^{\beta_d},$$

according to a *homogenized multiindex* $\beta \in \Gamma_n^0 := \left\{ \gamma \in \mathbb{N}_0^{d+1} \ : \ |\gamma| = n \right\}$ of order n. Here, $|\gamma| := \gamma_0 + \cdots + \gamma_d$ is called the *length* of γ. Like with factorials $\beta!$, we also interpret all other quantities involving vectors $\mathbb{R}^{d+1}$ as products of the individual values, in particular for the Γ–function, where we set

$$\Gamma(v) = \prod_{j=0}^{d} \Gamma(v_j), \qquad v \in \mathbb{R}^{d+1}.$$

The *Bernstein–Bézier representation* or *B–Form* for short, cf. [6], of a polynomial $p \in \Pi_n$ is simply given as

$$p = \sum_{\beta \in \Gamma_n^0} p_\beta B_\beta. \tag{3}$$

Though the elements of the B–basis are indeed very special B–splines, they should not be confused with the latter as a spline is a *piecewise* polynomial while here we deal with polynomials only. A drawback of the B–basis is that *all* the basis elements of the polynomial space Π_n depend on the degree n, in contrast, for example, to the monomial basis. Hence, the problem of writing the polynomial p from (3) in terms of B_β, $\beta \in \Gamma_{n+1}^0$, is a nontrivial one. Nevertheless, it is not difficult to see, cf. [6], that

$$p = \sum_{\beta \in \Gamma_{n+1}^0} (Rp)_\beta B_\beta, \qquad (Rp)_\beta = \sum_{j=0}^{d} \frac{\beta_j}{n+1} p_{\beta - \epsilon_j}, \tag{4}$$

where the *degree raising* operator R computes the new coefficients for the degree $n+1$ representation in a very stable way as *convex combinations* of the old coefficients.

Finally, we will denote by $\Gamma_n = \left\{ \widehat{\gamma} \in \mathbb{N}_0^d \ : \ |\widehat{\gamma}| \le n \right\}$ the *non–homogeneous* multiindices of order $\le \ n$. Obviously, the two sets of multiindices are related by $\gamma = (n - |\widehat{\gamma}|, \widehat{\gamma})$.

3 Jacobi polynomials and differential operators

For $\alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{R}^{d+1}$, $\alpha_j > -1$, $j = 0, \ldots, d$, *Jacobi polynomials* are orthogonal polynomials with respect to the inner product

$$\langle f, g \rangle^\alpha := \int_{\mathbb{S}_d} f(u)\, g(u)\, u^\alpha \, du. \tag{5}$$

Jacobi polynomials on the triangle have already been studied in [4], where, for example, a Rodriguez formula has been given. Like usually in multivariate orthogonal polynomials, we define the space $\mathcal{P}_n \subset \Pi_n$ of Jacobi polynomials of degree n by

$$\langle \mathcal{P}_n, \Pi_{n-1} \rangle^\alpha = 0, \qquad \text{i.e.,} \qquad \langle p, q \rangle^\alpha = 0, \quad p \in \mathcal{P}_n,\ q \in \Pi_{n-1}.$$

Closely related to the Jacobi polynomials is the second order differential operator $\mathcal{A}^\alpha$ written as either

$$\begin{aligned} \mathcal{A}^\alpha = & \sum_{j=1}^d (1-|x|)^{-\alpha_0} \ x_j^{-\alpha_j} \frac{\partial}{\partial x_j} (1-|x|)^{\alpha_0+1} \ x_j^{\alpha_j+1} \frac{\partial}{\partial x_j} \\ & + \frac{1}{2} \sum_{j,k=1}^d x_j^{-\alpha_j} x_k^{-\alpha_k} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) x_j^{\alpha_j+1} x_k^{\alpha_k+1} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) \end{aligned} \tag{6}$$

or in the more symmetric barycentric form

$$\mathcal{A}^\alpha = \frac{1}{2} \sum_{j,k=0}^d u_j^{-\alpha_j} u_k^{-\alpha_k} \left(\frac{\partial}{\partial u_j} - \frac{\partial}{\partial u_k} \right) u_j^{\alpha_j+1} u_k^{\alpha_k+1} \left(\frac{\partial}{\partial u_j} - \frac{\partial}{\partial u_k} \right). \tag{7}$$

In [7] the differential operators have been given in the slightly different but clearly equivalent form

$$\mathcal{A}^\alpha = \frac{1}{2} u^{-\alpha} \sum_{0 \le j < k \le n} \left(\frac{\partial}{\partial u_j} - \frac{\partial}{\partial u_k} \right) u_j u_k u^\alpha \left(\frac{\partial}{\partial u_j} - \frac{\partial}{\partial u_k} \right). \tag{8}$$

For special cases of α, these operators have been given earlier, see, for example, [8,9] for $\alpha = 0$ and [10] for the degenerate case $\alpha = -\epsilon$, $\epsilon = (1, \dots, 1)$. Also note that the *second order part* of $\mathcal{A}^\alpha$ takes the form

$$\frac{1}{2}\left[\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d}\right]\begin{bmatrix} x_1(1-x_1) & -x_1x_2 & \dots & -x_1x_d \\ -x_2x_1 & x_2(1-x_2) & \dots & -x_2x_d \\ \vdots & \vdots & \ddots & \vdots \\ -x_dx_1 & -x_dx_2 & \dots & x_d(1-x_d) \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_d} \end{bmatrix} \tag{9}$$

and is *independent* of α. Moreover, this purely second order operator from (9) corresponds to the case $\alpha = -\epsilon$ and is closely related to the classical *Bernstein polynomials* as is pointed out in [10]. Some basic facts on $\mathcal{A}$ are given in the following theorem due to Braess and Schwab [7], a proof of which we briefly sketch for the sake of completeness.

Theorem 1 *The operator $\mathcal{A}^\alpha$*

(1) is elliptic on $\mathbb{S}_d$ and strictly elliptic on the interior of $\mathbb{S}_d$.
(2) is self–adjoint with respect to the inner product (22):

$$\langle \mathcal{A}^\alpha f, g\rangle^\alpha = \langle f, \mathcal{A}^\alpha g\rangle^\alpha .$$

(3) has the Jacobi polynomials as eigenfunctions whose associated eigenvalue depends only on the total degree:

$$\mathcal{A}^\alpha p = -n\,(n + |\alpha| + d)\; p, \qquad p \in \mathcal{P}_n. \tag{10}$$

PROOF. The ellipticity follows immediately from (9) as shown, for example in [11], but clearly the smallest eigenvalue becomes zero if $x_j = 0$ for some $1 \le j \le d$ and also for $|x| = 1$. Self–adjointness, on the other hand, follows by partial integration, while for (10) one first computes for $\gamma = (\gamma_1, \dots, \gamma_d) \in \Gamma_n$ that

$$\mathcal{A}x^\gamma = -|\gamma|\,(|\gamma| + |\alpha| + d)\; x^\gamma + \sum_{j=1}^{d} \gamma_j\,(\gamma_j + \alpha_j)\; x^{\gamma-\epsilon_j} \tag{11}$$

from which we conclude that $(\mathcal{A} - I)\,\Pi_n \subseteq \Pi_{n-1}$. The unique monic Jacobi polynomial $p^\alpha_{\widehat{\gamma}}$, $\widehat{\gamma} \in \Gamma_n$, of the form $p^\alpha_{\widehat{\gamma}}(x) = x^{\widehat{\gamma}} + p(x)$, $p \in \Pi_{n-1}$, satisfies

$$\left\langle \mathcal{A}\, p^\alpha_{\widehat{\gamma}}, q\right\rangle^\alpha = \left\langle p^\alpha_{\widehat{\gamma}}, \mathcal{A}\, q\right\rangle^\alpha = 0, \qquad q \in \Pi_{n-1},$$

and therefore we have that $\mathcal{A}\, p^\alpha_{\widehat{\gamma}} \in \mathcal{P}_n$. But by (11) this is again a monic polynomial with leading term $x^{\widehat{\gamma}}$ and thus is a multiple of $p^\alpha_{\widehat{\gamma}}$. Comparing leading coefficients in (11) then immediately yields (10). $\square$

4 Bernstein–Durrmeyer operators

A main tool in the construction of the Jacobi polynomials will be, like in [1], the *Bernstein–Durrmeyer operator* associated to the Jacobi weight u^α, most conveniently written as

$$V_n^\alpha f(u) = \langle f, K_n^\alpha(\cdot, u)\rangle^\alpha = \int_{\mathbb{S}_d} f(u)\, K_n^\alpha(u,v)\, v^\alpha\, dv, \tag{12}$$

with the symmetric kernel

$$K_n^\alpha(u,v) = \sum_{\beta\in\Gamma_n^0} \frac{B_\beta(u)B_\beta(v)}{\langle 1, B_\beta\rangle^\alpha}. \tag{13}$$

Note that the α–dependency of the kernel results only from the normalization coefficient for $\beta \in \Gamma_n^0$

$$\langle 1, B_\beta\rangle^\alpha = \frac{\Gamma(|\beta|+1)}{\Gamma(\beta+\epsilon)} \frac{\Gamma(\alpha+\beta+\epsilon)}{\Gamma(|\alpha+\beta+\epsilon|)} = \frac{\Gamma(\alpha+\epsilon)\,\Gamma(|\beta|+1)}{\Gamma(|\beta+\alpha+\epsilon|)} \frac{(\alpha+\epsilon)_\beta}{\beta!}, \tag{14}$$

which ensures that $V_n^\alpha 1 = 1$ for all $n \in \mathbb{N}_0$. Here $(\alpha+\epsilon)_\beta$ denotes the product Pochhammer symbols, cf. [12, p. 256]

$$(\alpha+\epsilon)_\beta = \prod_{j=0}^d (\alpha_j+1)\cdots(\alpha_j+\beta_j).$$

Next, we give some crucial facts on the Bernstein–Durrmeyer operators.

Theorem 2 *The Bernstein–Durrmeyer operators V_n^α, $n \in \mathbb{N}_0$, have the following properties:*

(1) For $\gamma \in \mathbb{N}_0^{d+1}$ the monomial $m_\gamma(u) = u^\gamma$ satisfies

$$V_n^\alpha m_\gamma = \frac{\Gamma(|\alpha+\beta+\epsilon|)}{\Gamma(|\alpha+\beta+\gamma+\epsilon|)} \sum_{\eta\le\gamma} \binom{\gamma}{\eta} \frac{n!}{(n-|\eta|)!} \frac{\Gamma(\alpha+\gamma+\epsilon)}{\Gamma(\alpha+\eta+\epsilon)} u^\eta. \tag{15}$$

In particular, $V_n^\alpha \Pi_k \subseteq \Pi_k$, $k \in \mathbb{N}_0$.

(2) They commute with the differential operator $\mathcal{A}^\alpha$*:*

$$V_n^\alpha \mathcal{A}^\alpha = \mathcal{A}^\alpha V_n^\alpha. \tag{16}$$

(3) The Jacobi polynomials are eigenfunctions: for any $p \in \mathcal{P}_m$*,* $m \leq n$*, we have that*

$$V_n^\alpha\, p = \frac{n!}{(n-m)!}\,\frac{\Gamma\left(|\alpha+\epsilon|+m\right)}{\Gamma\left(|\alpha+\epsilon|+m+n\right)}\, p =: \lambda_{n,m}\, p. \tag{17}$$

PROOF. To prove (15), we assume that $|\gamma| \leq n$, as otherwise the statement is trivially true, and first note that for $\beta \in \Gamma_n^0$ we have

$$\frac{\langle m_\gamma, B_\beta\rangle^\alpha}{\langle 1, B_\beta\rangle^\alpha} = \frac{\Gamma\left(|\alpha+\beta+\epsilon|\right)}{\Gamma\left(|\alpha+\beta+\gamma+\epsilon|\right)}\left(\alpha+\beta+\epsilon\right)_\gamma,$$

hence,

$$V_n^\alpha\, m_\gamma = \frac{\Gamma\left(|\alpha+\beta+\epsilon|\right)}{\Gamma\left(|\alpha+\beta+\gamma+\epsilon|\right)} \sum_{\beta\in\Gamma_n^0} \left(\alpha+\beta+\epsilon\right)_\gamma\, B_\beta. \tag{18}$$

On the other hand, the (formal) identity

$$\begin{aligned} u^{-\alpha}\frac{\partial^{|\gamma|}}{\partial u^\gamma}\left(u^{\gamma+\alpha}\left(u_0+\cdots+u_d\right)^n\right) &= u^{-\alpha}\frac{\partial^{|\gamma|}}{\partial u^\gamma}\sum_{\beta\in\Gamma_n^0} u^{\gamma+\alpha}B_\beta(u) \\ = u^{-\alpha}\sum_{\beta\in\Gamma_n^0}\binom{n}{\beta}\frac{\partial^{|\gamma|}}{\partial u^\gamma}u^{\alpha+\beta+\gamma} &= \sum_{\beta\in\Gamma_n^0}\left(\alpha+\beta+\epsilon\right)_\gamma\, B_\beta \end{aligned}$$

together with the (formal) Leibniz expansion

$$u^{-\alpha}\frac{\partial^{|\gamma|}}{\partial u^\gamma}\left(u^{\gamma+\alpha}\left(u_0+\cdots+u_d\right)^n\right) = \sum_{\eta\leq\gamma}\binom{\gamma}{\eta}\frac{n!}{(n-|\eta|)!}\,\frac{\Gamma\left(\alpha+\gamma+\epsilon\right)}{\Gamma\left(\alpha+\eta+\epsilon\right)}u^\eta$$

immediately yields (15). Now, setting $\gamma_0 = 0$, then also $\eta_0 = 0$ for all $\eta \leq \gamma$ and thus the monomial $x^{\widehat{\gamma}}$ is mapped to a linear combination of the monomials $x^{\widehat{\eta}}$, $\eta \leq \gamma$.

Equation (16) is a consequence of the symmetry of the kernel K_n^α and the self adjointness of $\mathcal{A}^\alpha$.

To prove (17), we again use the monic Jacobi polynomials $p_{\widehat{\gamma}}^{\alpha}$, $|\widehat{\gamma}| = m$, and conclude from (15) that

$$V_n^\alpha \, p_{\widehat{\gamma}}^{\alpha} = \frac{n!}{(n-m)!} \, \frac{\Gamma\left(|\alpha+\epsilon|+m\right)}{\Gamma\left(|\alpha+\epsilon|+m+n\right)} \, p_{\widehat{\gamma}}^{\alpha} + q \tag{19}$$

for some $q \in \Pi_{n-1}$. Expanding q in terms of $p_{\widehat{\eta}}^{\alpha}$, $|\widehat{\eta}| < n$, and using the commuting property (16) together with (19) to compare the expansion coefficients in $\mathcal{A}^\alpha V_n^\alpha p_{\widehat{\gamma}}^{\alpha} = V_n^\alpha \mathcal{A}^\alpha p_{\widehat{\gamma}}^{\alpha}$, a simple inductive argument shows that q must be zero, providing us with (17). $\square$

Based on the properties listed in Theorem 2, one could also study approximation properties of the de la Vallée–Poussin like summation procedure given by the positive linear operator V_n^α, just like in [9,10]. Since we are interested in the construction of orthogonal polynomials, however, we will not pursue this line any further here.

5 The discrete inner product and a characterization of orthogonal polynomials

For a given degree $n \in \mathbb{N}_0$ and polynomials $p, q \in \Pi_n$ with respective Bézier coefficients p_β and q_β, $\beta \in \Gamma_n^0$, we define the *discrete inner product* of order n

$$\langle p, q \rangle_n^\alpha := \sum_{\beta \in \Gamma_n^0} p_\beta \, q_\beta \, w^\alpha(\beta), \qquad w^\alpha(\beta) := \frac{(\alpha+\epsilon)_\beta}{\beta!} \tag{20}$$

and observe that for $\alpha = 0$ the identity $(\epsilon)_\beta = \beta!$ yields $w^\alpha \equiv 1$ and thus the inner product becomes the one from [1] in this situation. If $p \in \Pi_m$ and $q \in \Pi_n$, $m \leq n$, then we use the degree raising operator to extend the product as

$$\langle p, q \rangle_n^\alpha = \left\langle R^{n-m} p, q \right\rangle_n^\alpha ;$$

therefore, the inner product depends only on the polynomials, not on their specific representation of a certain degree. With respect to the inner product $\langle \cdot, \cdot \rangle_n^\alpha$, the degree raising operator R has an adjoint which can be computed to be

$$\left\langle R^T p, q \right\rangle_n^\alpha = \langle p, Rq \rangle_n^\alpha = \sum_{\beta \in \Gamma_n^0} p_\beta \, (Rq)_\beta \, w^\alpha(\beta) = \sum_{j=0}^{d} \sum_{\beta \in \Gamma_n^0} p_\beta \, \frac{\beta_j}{n} \, q_{\beta-\epsilon_j} \, w^\alpha(\beta)$$

$$
\begin{aligned}
&= \sum_{j=0}^{d} \sum_{\beta \in \Gamma_{n-1}^0} p_{\beta+\epsilon_j} \frac{\beta_j + 1}{n} q_\beta \, w^\alpha (\beta + \epsilon_j) \\
&= \sum_{\beta \in \Gamma_{n-1}^0} \left(\sum_{j=0}^{d} \frac{\beta_j + 1}{n} \frac{\alpha_j + \beta_j + 1}{\beta_j + 1} p_{\beta+\epsilon_j} \right) q_\beta \, w^\alpha(\beta),
\end{aligned}
$$

yielding the explicit formula for the degree reduction operator R^T as

$$
\left(R^T p\right)_\beta = \sum_{j=0}^{d} \frac{\alpha_j + \beta_j + 1}{n} p_{\beta+\epsilon_j}, \qquad \beta \in \Gamma_{n-1}^0. \tag{21}
$$

Though this operator is based on convex combinations if and only if $|\alpha + \epsilon| = 1$, for example when $\alpha = -\frac{d}{d+1}\epsilon$, we still note that all coefficients appearing in (21) are at least *positive* since $\alpha_j > -1$, $j = 0, \dots, d$. Now we are in position to characterize Jacobi polynomials in terms of the adjoint degree raising operator.

Theorem 3 *A polynomial $p \in \Pi_n$ belongs to $\mathcal{P}_n$ if and only if $R^T p = 0$.*

PROOF. Theorem 3 is direct consequence of the case $m = n$ in Proposition 4 that follows immediately. Indeed, if $p \in \mathcal{P}_n$ and $q \in \Pi_{n-1}$, then (22) implies that

$$
0 = \langle p, Rq \rangle_n^\alpha = \left\langle R^T p, q \right\rangle_n^\alpha,
$$

and this holds true for *any* $q \in \Pi_{n-1}$ if and only if $R^T p = 0$. $\Box$

The next proposition is not only a useful tool for the proof of Theorem 3, it also shows how the inner product between a polynomial from $\mathcal{P}_m$, $m \neq n$, and an arbitrary polynomial of degree n can be computed in terms of the discrete inner product.

Proposition 4 *For $m \leq n$ let $p \in \mathcal{P}_m$ and $q \in \Pi_n$. Then*

$$
\langle p, q \rangle^\alpha = \lambda_{n,m} \frac{\Gamma\left(n + |\alpha + \epsilon|\right)}{\Gamma\left(\alpha + \epsilon\right) n!} \langle p, q \rangle_n^\alpha \tag{22}
$$

PROOF. We first notice that

$$
\sum_{\beta \in \Gamma_n^0} \left(R^{m-n} p\right)_\beta B_\beta = p = \lambda_{n,m}^{-1} V_n^\alpha p = \lambda_{n,m}^{-1} \sum_{\beta \in \Gamma_n^0} \frac{\langle p, B_\beta \rangle^\alpha}{\langle 1, B_\beta \rangle^\alpha} B_\beta
$$

so that comparison of coefficients and (14) give

$$\begin{aligned}\langle p, B_\beta\rangle^\alpha &= \lambda_{n,m} \left(R^{m-n} p\right)_\beta \langle 1, B_\beta\rangle^\alpha \\ &= \lambda_{n,m} \left(R^{m-n} p\right)_\beta w^\alpha(\beta) \frac{\Gamma(|\beta+\alpha+\epsilon|)}{\Gamma(\alpha+\epsilon)\,\Gamma(|\beta|+1)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\langle p, q\rangle^\alpha &= \sum_{\beta\in\Gamma_n^0} q_\beta \langle p, B_\beta\rangle^\alpha \\ &= \lambda_{n,m} \frac{\Gamma(n+|\alpha+\epsilon|)}{\Gamma(\alpha+\epsilon)\; n!} \sum_{\beta\in\Gamma_n^0} \left(R^{n-m} p\right)_\beta q_\beta\, w^\alpha(\beta)\end{aligned}$$

which is (22). □

We next give a slightly uncommon way to parameterize the Jacobi polynomials, namely by their restriction on a face of $\mathbb{S}_d$. The that end, we denote by

$$\partial_j \mathbb{S}_d := \{u \in \mathbb{S}_d \ : \ u_j = 0\}$$

the j*–face* of $\mathbb{S}_d$. Any such face is isomorphic to $\mathbb{S}_{d-1}$ and their union forms the boundary $\partial\mathbb{S}_d$ of $\mathbb{S}_d$. Analogously, we also define

$$\partial_j \Gamma_n^0 = \left\{\beta \in \Gamma_n^0 \ : \ \beta_j = 0\right\}.$$

One appealing geometric property of representing polynomials in B–form is that the restriction of a polynomial $p \in \Pi_n$ to $\partial_j\mathbb{S}_d$ depends only on the coefficients p_β, $\beta \in \partial_j\Gamma_n^0$, and therefore $p, q \in \Pi_n$ coincide on $\partial_j\mathbb{S}_d$ if and only if $p_\beta = q_\beta$, $\beta \in \partial_j\Gamma_n^0$.

Theorem 5 *For given $j \in \{0,\dots,d\}$ and $q \in \Pi_n$ there exists a unique polynomial $p \in \mathcal{P}_n$ such that*

$$p|_{\partial_j\mathbb{S}_d} = q|_{\partial_j\mathbb{S}_d}, \qquad \textit{i.e.,} \qquad p(u) = q(u), \quad u \in \partial_j\mathbb{S}_d.$$

PROOF. Writing q in its B–form of degree n, we get coefficients q_β, $\beta \in \Gamma_n^0$. Setting $p_\beta = q_\beta$ for $\beta \in \partial_j\Gamma_n^0$, we immediately obtain coincidence of the two polynomials on $\partial_j\mathbb{S}_d$. For any $\beta \in \partial_j\Gamma_{n-1}^0$, the requirement $\left(R^T p\right)_\beta = 0$ can then be rewritten as

$$p_{\beta+\epsilon_j} = -\sum_{k\neq j} \frac{\alpha_k+\beta_k+1}{\alpha_j+\beta_j+1}\, p_{\beta+\epsilon_k}, \qquad \beta+\epsilon_k \in \partial_j\Gamma_n^0,\ k\neq j, \tag{23}$$

which defines the coefficients p_β, $\beta \in \Gamma_n^0$, $\beta_j = 1$. Repeating this argument, we also define the coefficients p_β, $\beta_j = 2, 3, \dots, n$, and end up with the coefficients for a polynomial p such that $R^T p = 0$. By Theorem 3, p belongs to $\mathcal{P}_n$. $\square$

Theorem 5 tells us that we can normalize the Jacobi polynomials in $\mathcal{P}_n$ by prescribing polynomials of degree at most n on one fixed face of $\mathbb{S}_d$. This is in full coincidence with the univariate case where usually the orthogonal polynomials are fixed by their behavior on some zero dimensional face of the interval, i.e., by their behavior in one of its endpoints.

6 Construction and orthogonality

The basic idea of the construction of a basis for $\mathcal{P}_n$, $n \ge 0$, is suggested by Theorem 5: we fix j, take a basis of orthogonal polynomials of degree $\le n$ in $d-1$ variables, obtained by a recursive application of the method, and extend any of these polynomials to an element of $\mathcal{P}_n$ by means of (23). This way, we obtain a basis

$$\left\{ p^\alpha[\widehat{\beta}] \in \mathcal{P}_{\left|\widehat{\beta}\right|} \ : \ \widehat{\beta} \in \Gamma_n \right\} \subset \Pi_n, \qquad n \in \mathbb{N}_0,$$

of orthogonal polynomials. Using $\widehat{\beta}' := (\beta_1, \dots, \beta_{d-1})$ for the "truncated" multiindex, the Bézier coefficients of $p = p^\alpha[\widehat{\beta}]$ are defined recursively by

$$p_\gamma = \left(R^{n-\left|\widehat{\beta}'\right|} p^\alpha[\widehat{\beta}'] \right)_{\gamma^j}, \qquad \gamma \in \partial_j \Gamma_0^n, \tag{24}$$

$$p_{\gamma+\epsilon_j} = -\sum_{k \ne j} \frac{\alpha_k + \gamma_k + 1}{\alpha_j + \gamma_j + 1} p_{\gamma+\epsilon_k}, \qquad \gamma_j = \ell,\ \ell = 0, \dots, n-1. \tag{25}$$

However, this construction even gives us more: the orthogonal polynomials are not only orthogonal to those of lower degree, they are even *mutually* orthogonal. In other words, we have the following result.

Theorem 6 *The polynomials $p^\alpha[\widehat{\beta}]$, $\widehat{\beta} \in \Gamma_n$, $n \in \mathbb{N}_0$, are an orthogonal basis of the space Π of all polynomials.*

PROOF. For $\widehat{\beta}, \widehat{\gamma} \in \Gamma_n$ we have to show that

$$0 = \left\langle p^\alpha[\widehat{\beta}], p^\alpha[\widehat{\gamma}] \right\rangle^\alpha =: \langle p, q \rangle^\alpha, \qquad \widehat{\beta} \ne \widehat{\gamma}, \tag{26}$$

where, for brevity, we set $p = p^\alpha_{\widehat{\beta}}$ and $q = p^\alpha_{\widehat{\gamma}}$. The identity (26) follows directly from Theorem 3 if $\left|\widehat{\beta}\right| \neq |\widehat{\gamma}|$. Otherwise, we use a different interpretation of (25) in terms of the coefficient vectors

$$p^k[\widehat{\beta}] := \left[p^k_\eta \;:\; \eta \in \partial_j \Gamma^0_{n-k}\right] := \left[p^\alpha[\widehat{\beta}]_\gamma \;:\; \gamma_j = k\right], \qquad k = 0, \dots, n,$$

which we consider to be coefficient vectors associated to polynomials of degree $n-k$ defined on $\partial_j \mathbb{S}_d$. Then the recursion (24) and (25) can then be rewritten in vector form as

$$p^0[\widehat{\beta}] = R^{n-\left|\widehat{\beta}'\right|} p^\alpha[\widehat{\beta}'], \qquad p^{k+1}[\widehat{\beta}] = -\frac{n-k}{\alpha_j + k + 1} R^T p^k[\widehat{\beta}], \tag{27}$$

i.e.,

$$p^k[\widehat{\beta}] = \frac{n!}{(n-k)!\,(\alpha_j+1)_k} \left(R^T\right)^k R^{n-\left|\widehat{\beta}'\right|} p^\alpha[\widehat{\beta}'], \tag{28}$$

where the operators R^T and R have to be understood as acting in $d-1$ variables. By Lemma 8, which will be stated and proven next, this implies for $k \leq n - \left|\widehat{\beta}'\right|$ that

$$p^k[\widehat{\beta}] = \frac{n!}{(n-k)!\,(\alpha_j+1)_k} \mu_{\left|\widehat{\beta}'\right|, n-\left|\widehat{\beta}'\right|, k} R^{n-\left|\widehat{\beta}'\right|-k} p^\alpha[\widehat{\beta}^j] \tag{29}$$

and since $p^{n-\left|\widehat{\beta}'\right|}[\widehat{\beta}]$ is a multiple of the coefficient vector of the orthogonal polynomial $p^\alpha[\widehat{\beta}']$ it follows that

$$p^k[\widehat{\beta}] = 0, \qquad k > \widehat{\beta}_j. \tag{30}$$

Now we are in position to complete the proof of (26). Whenever $\widehat{\beta} \neq \widehat{\gamma}$ and $\left|\widehat{\beta}\right| = |\widehat{\gamma}| =: n$ it follows that $\widehat{\beta}' \neq \widehat{\gamma}'$ and without loss of generality we can also assume that $\left|\widehat{\beta}'\right| \geq |\widehat{\gamma}'|$. Using (29) and (28) we then get that

$$\begin{aligned}
&\lambda^{-1}_{n,n} \frac{\Gamma(\alpha+\epsilon)\, n!}{\Gamma(n + |\alpha+\epsilon|)} \left\langle p^\alpha[\widehat{\beta}], p^\alpha[\widehat{\gamma}]\right\rangle^\alpha = \left\langle p^\alpha[\widehat{\beta}], p^\alpha[\widehat{\gamma}]\right\rangle^\alpha_n \\
&= \sum_{k=0}^{n} \left\langle p^k[\widehat{\beta}], p^k[\widehat{\gamma}]\right\rangle^\alpha_{n-k} \frac{(\alpha_j+1)_k}{k!} \\
&= \sum_{k=0}^{n-\left|\widehat{\beta}'\right|} \left(\frac{n!}{(n-k)!\,(\alpha_j+1)_k}\right)^2 \mu_{\left|\widehat{\beta}'\right|, k, n-\left|\widehat{\beta}'\right|} \mu_{\left|\widehat{\gamma}'\right|, k, n-\left|\widehat{\gamma}'\right|} \frac{(\alpha_j+1)_k}{k!} \times
\end{aligned}$$

$$\times \left\langle R^{n-|\widehat{\beta}'|-k}\, p^\alpha[\widehat{\beta}^j], R^{n-|\widehat{\beta}'|-k}\, p^\alpha[\widehat{\gamma}^j] \right\rangle_{n-k}^{\alpha} = 0,$$

due to Proposition 4. This finally verifies (26).

Remark 7 *The vector recursion (27) fails to define p^1 if $\alpha = -\epsilon$, reflecting the fact that in this case the behavior of the orthogonal polynomials on the boundary and inside the simplex are completely decoupled due to the singularity of the weight function, see [10].*

To complete the proof of Theorem 6, we finally need the following technical result that says that R^T is essentially a left inverse of R.

Lemma 8 *For $p \in \mathcal{P}_n$ we have that*

$$\left(R^T\right)^j \, R^k p = \mu_{n,j,k}\, R^{k-j} p, \qquad 0 \le j \le k, \tag{31}$$

where

$$\mu_{n,j,k} = \frac{(n+k)!}{(n+k-j)!} \frac{\Gamma\left(n+k-j+|\alpha+\epsilon|\right)}{\Gamma\left(n+k+|\alpha+\epsilon|\right)} \frac{\lambda_{n+k-j,n}}{\lambda_{n+k,n}}.$$

PROOF. For any $q \in \Pi_m$, $n \le m \le n+k-j$, we note that Proposition 4 gives

$$\begin{aligned}
&\left\langle R^{n+k-j-m} q, \left(R^T\right)^j \, R^k p \right\rangle_{n+k-j}^{\alpha} = \left\langle R^{n+k-m} q, R^k p \right\rangle_{n+k}^{\alpha} \\
&= \lambda_{n+k,n}^{-1} \frac{\Gamma\left(\alpha+\epsilon\right)\Gamma\left(n+k+1\right)}{\Gamma\left(n+k+|\alpha+\epsilon|\right)} \left\langle q, p \right\rangle^{\alpha}
\end{aligned}$$

as well as

$$\left\langle R^{n+k-j-m} q, R^{k-j} p \right\rangle_{n+k-j}^{\alpha} = \lambda_{n+k-j,n}^{-1} \frac{\Gamma\left(\alpha+\epsilon\right)\Gamma\left(n+k-j+1\right)}{\Gamma\left(n+k-j+|\alpha+\epsilon|\right)} \left\langle q, p \right\rangle^{\alpha}$$

yielding that

$$\left\langle R^{n+k-j-m} q, \left(R^T\right)^j \, R^k p - \mu_{n,j,k}\, R^{k-j} p \right\rangle_{n+k-j}^{\alpha} = 0,$$

and varying q over Π_{n+k-j} allows us conclude that (31) holds true. □

7 Examples

In this section, we give finally give some examples what the results of the construction methods above look like.

Let us begin with the univariate case and the Jacobi polynomials $p^\alpha[k]$, $k \in \mathbb{N}_0$, $\alpha = (\alpha_0, \alpha_1)$ normalized by $p^\alpha[k](1) = 1$. This implies that $p_{(0,n)} = 1$ and by means of (25) we are lead to the explicit B–form

$$\begin{aligned} p^\alpha[n] &= \sum_{k=0}^{n} (-1)^{n+k} \frac{(\alpha_1 + n - k + 1)_k}{(\alpha_0 + 1)_k} B_{(k,n-k)} \\ &= \sum_{k=0}^{n} (-1)^{n+k} \frac{(\alpha_1 + 1)_n}{(\alpha_0 + 1)_k \, (\alpha_1 + 1)_{n-k}} B_{(k,n-k)}; \end{aligned} \tag{32}$$

this expression, that can also be easily derived from the Rodriguez formula, has been given in [1] for the case $\alpha_0 = \alpha_1 = 0$.

In two variables, this already becomes more complicated. The two extremal cases of degree n occur when the restriction to $\partial_j \mathbb{S}_2$, where for convenience we set $j = 0$ from now on, is either the constant polynomial $p^\alpha[0] \equiv 1$ or the univariate Jacobi polynomial of degree n, that is, $p^\alpha[n]$. We first observe in the case of the boundary polynomial $p^\alpha[0]$ that for any $k \in \mathbb{N}_0$ the vector $R^k p^\alpha[0]$ has all coefficients equal to 1 and so (29) yields that

$$p^\alpha[(0,n)]_\gamma = \frac{n!}{(n-\gamma_0)! \, (\alpha_0 + 1)_{\gamma_0}} \, \mu_{0,\gamma_0,n}, \qquad \gamma \in \Gamma_n^0,$$

leading to a polynomial that depends only on u_0:

$$p^\alpha[(0,n)](u) = \sum_{k=0}^{n} \frac{n!}{(n-\gamma_0)! \, (\alpha_0 + 1)_{\gamma_0}} \, \mu_{0,\gamma_0,n} \, B_{(n-k,k)} \, (u_0) \, .$$

The opposite extreme is the case of $p^\alpha[(n,0)]$ that restricts to $p^\alpha[n]$ on the boundary. Since $R^T p^\alpha[n] = 0$, we thus get that

$$p^\alpha[(n,0)]_\gamma = \begin{cases} (-1)^{n+\gamma_1} \dfrac{(\alpha_2 + 1)_n}{(\alpha_1 + 1)_{\gamma_1} \, (\alpha_2 + 1)_{\gamma_2}}, & \gamma_0 = 0, \\ 0, & \gamma_0 \neq 0, \end{cases}$$

from which it follows that

$$p^{\alpha}\left[(n,0)\right](u)$$
$$= (1-u_0)^n \sum_{\gamma\in\partial_0\Gamma_n^0} (-1)^{n+\gamma_1} \frac{(\alpha_2+1)_n}{(\alpha_1+1)_{\gamma_1}\,(\alpha_2+1)_{\gamma_2}} B_{(\gamma_1,\gamma_2)}\left(\frac{u_2}{1-u_0}\right).$$

This formula admits a geometric interpretation on the extension of "boundary polynomials" to the simplex: a point $u \in \mathbb{S}_d$ is connected to the vertex ϵ_0 and the value of $p^{\alpha}\left[(n,0)\right](u)$ is determined by the value of the intersection of this line with the face $\partial_0\mathbb{S}_d$ multiplied by the "distance" term $(1-u_0)^n$. It also shows that for increasing n these polynomials "live" essentially on the boundary.

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