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d-Orthogonality of Humbert and Jacobi type polynomials

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Abstract

In this paper, we treat three questions related to the d-orthogonality of the Humbert polynomials. The first one consists to determinate the explicit expression of the d-dimensional functional vector for which the d-orthogonality holds. The second one is the investigation of the components of Humbert polynomial sequence. That allows us to introduce, as far as we know, new d-orthogonal polynomials generalizing the classical Jacobi ones. The third one consists to solve a characterization problem related to a generalized hypergeometric representation of the Humbert polynomials.

Keywords: d-Orthogonal polynomials; Generalized hypergeometric polynomials; Jacobi polynomials; Gegenbauer polynomials; Humbert polynomials; d-Symmetric polynomials

1. Introduction

The Humbert polynomials defined by the expansion [22]:

$$\left(1-(d+1)xt+t^{d+1}\right)^{-\nu}=\sum_{n\geqslant 0}h_{n,d+1}^{\nu}(x)t^{n}, \tag{1.1}$$

where $\nu>-\frac{1}{2}$, $\nu\neq 0$ and d is a positive integer, were first studied by Pincherle [30], in the limiting case $\nu=-\frac{1}{2}$, $d=2$, and later extended by Humbert [22] and Devisme [15]. Various works in the literature (see, for instance, [3, 19–21] and references therein) have focused on the analysis of some properties of these polynomials. Some of these properties are analogous to those of Gegenbauer's polynomials. But the Humbert polynomials $h_{n,d+1}^{\nu}(x)$ with $d\geqslant 2$ fulfil the following recurrence relation:

$$h_{n+1,d+1}^{\nu}(x)=\frac{(\nu+n)(d+1)}{(n+1)}xh_{n,d+1}^{\nu}(x)-\left(\frac{(d+1)(\nu-1)}{(n+1)}+1\right)h_{n-d,d+1}^{\nu}(x). \tag{1.2}$$

That is, they do not satisfy a three-term recurrence relation of the kind which is necessary for the polynomials to be orthogonal over any interval $a\leqslant x\leqslant b$ in the way that the Gegenbauer polynomials are (see, for instance, Favard the-

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orem [14]). $(d+1)$-Order recurrence relations of type (1.2) are related to the notion of d-orthogonality. That concerns a natural and canonical extension of the ordinary orthogonality. Such notion is connected with the study of vector Padé approximants, simultaneous Padé approximants, and other problems as vectorial continued fractions, polynomials solutions of higher-order differential equations. The concept of d-orthogonality appears as a particular case of the general multiple orthogonality [1,4]. It was first introduced by Van Iseghem [33] and completed by Maroni [27]. Later, some recent works were focused in the analysis of properties of d-orthogonal polynomials generalizing the classical orthogonal polynomials (see [16,17,27–29]) and many characterization theorems were derived (see [6,8–12,16,17,27, 34]).

To draw up our contribution in this paper, we recall the following notions and results which we need also throughout this work.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A polynomial sequence $\{P_n\}_{n\geqslant 0}$ is called a *polynomial set* (PS, for shorter) if and only if $\deg P_n = n$ for all non-negative integer n.

Definition 1.1. (See Van Iseghem [33] and Maroni [27].) Let d be a positive integer. A PS $\{P_n\}_{n\geqslant 0}$ is called *d-orthogonal* (d-OPS, for shorter) with respect to the d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ if it satisfies the following orthogonality relations:

$$\begin{cases} \langle \Gamma_k, P_r P_n\rangle = 0, \quad r > nd+k,\ n \in \mathbb{N} = \{0,1,2,\ldots\}, \\ \langle \Gamma_k, P_n P_{nd+k}\rangle \neq 0, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

for each integer k belonging to $\mathbb{N}_d = \{0,1,\ldots,d-1\}$.

Definition 1.2. (See Maroni [29].) The d-orthogonal PS $\{P_n\}_{n\geqslant 0}$ owns Hahn's property if the derivative sequence $\{(n+1)^{-1}\frac{d}{dx}P_{n+1}\}_{n\geqslant 1}$ is also d-orthogonal.

In this case, we say that the sequence $\{P_n\}_{n\geqslant 0}$ is a d-orthogonal "classical" PS.

Definition 1.3. (See Douak and Maroni [16].) A PS $\{P_n\}_{n\geqslant 0}$ is called d-symmetric if it fulfils

$$P_n(w_{d+1}x) = w_{d+1}^n P_n(x), \quad n \in \mathbb{N},\ \text{where } w_{d+1} = \exp\left(\frac{2i\pi}{d+1}\right). \tag{1.4}$$

For $d = 1$, $w_2 = -1$ and a PS $\{P_n\}_{n\geqslant 0}$ satisfying (1.4) is symmetric, i.e. $P_n(-x) = (-1)^n P_n(x)$.

Lemma 1.4. *(See Maroni [27].) A PS $\{P_n\}_{n\geqslant 0}$ is a d-OPS if and only if it satisfies a $(d+1)$-order recurrence relation of type*

$$xP_n(x) = \sum_{k=0}^{d+1} \alpha_{k,d}(n) P_{n-d+k}(x), \tag{1.5}$$

where $\alpha_{d+1,d}(n)\alpha_{0,d}(n) \neq 0$, $n \geqslant d$, and by convention, $P_{-n} = 0$, $n \geqslant 1$.

Lemma 1.5. *(See Douak and Maroni [16].) A PS $\{P_n(x)\}_{n\geqslant 0}$ is d-symmetric if and only if there exists $(d+1)$ polynomial sequence $\{P_n^\mu(x)\}_{n\geqslant 0}$; $\mu \in \mathbb{N}_{d+1}$; such that*

$$P_{(d+1)n+\mu}(x) = x^\mu P_n^\mu\big(x^{d+1}\big), \quad \mu \in \mathbb{N}_{d+1},\ n \in \mathbb{N}.$$

These $(d+1)$ families are called the components of the d-symmetric PS $\{P_n(x)\}_{n\geqslant 0}$.

Lemma 1.6. *(See Douak and Maroni [16]].) Let $\{P_n(x)\}_{n\geqslant 0}$ be a d-symmetric PS and $\{u_r\}_{r\geqslant 0}$ be its dual sequence. If $\{P_n(x)\}_{n\geqslant 0}$ is d-orthogonal, then the associated components $\{P_n^\mu(x)\}_{n\geqslant 0}$; $\mu \in \mathbb{N}_{d+1}$; are also d-orthogonal and the d-dimensional functional vector $\mathcal{U}^\mu = {}^t(u_0^\mu, u_1^\mu, \ldots, u_{d-1}^\mu)$ for which $\{P_n^\mu(x)\}_{n\geqslant 0}$ are d-orthogonal is given by*

$$u_r^\mu = \sigma_{d+1}\big[x^\mu u_{\mu+r(d+1)}\big], \quad r \in \mathbb{N}_d,\ \mu \in \mathbb{N}_{d+1},$$

where σ_q designates the operator defined by

$$\sigma_q\big[f(x)\big]=f\big(x^q\big); \quad f\in\mathcal{P},$$

q being a positive integer greater or equal to two.

We return now to the Humbert polynomials. Ben Cheikh and Douak [8] showed that these polynomials are "classical," d-orthogonal and d-symmetric. This result suggested us two questions.

- The first one consists to express explicitly the d-dimensional functional vector for which the d-orthogonality of the Humbert polynomials holds. The solution, as integral representations, is known for two particular cases corresponding to Gegenbauer polynomials ($d=1$) and to Chebyshev type polynomials ($d=2$, $\nu=1$). For the second case, the solution was obtained separately by Douak and Maroni [18] and by Ben Cheikh and Ben Romdhane [6]. Two different methods were used in these two papers. The first one is based on the fact that the involved polynomials are "classical," while the second one is based on the obtention of the dual sequence of a given PS via lowering operators.
- The second question concerns the study of the components of the Humbert polynomials. For general d, Douak and Maroni [16] showed that the components are d-orthogonal (Lemma 1.6) and the first one is "classical." In the case $d=2$, for the corresponding components, Douak and Maroni [16] derived the coefficients of the third-order recurrence relation satisfied by the components, and Baker [2] obtained generalized hypergeometric representations. The case $d=1$ corresponds to the well-known relationship between Gegenbauer polynomials $C_n^{\nu}(.)$ and Jacobi polynomials $P_n^{(\alpha,\beta)}(.)$ [25, p. 39]:

$$\begin{cases} C_{2n}^{\nu}(x)=\dfrac{(\nu)_n}{(\frac{1}{2})_n}P_n^{(\nu-\frac{1}{2},-\frac{1}{2})}\big(2x^2-1\big), & n\in\mathbb{N},\\ C_{2n+1}^{\nu}(x)=\dfrac{(\nu)_{n+1}}{(\frac{1}{2})_{n+1}}xP_n^{(\nu-\frac{1}{2},\frac{1}{2})}\big(2x^2-1\big), & n\in\mathbb{N}, \end{cases} \tag{1.6}$$

  where $(a)_p$ denotes the Pochhammer symbol given by $(a)_p=\frac{\Gamma(a+p)}{\Gamma(a)}$.

The aim of this work is to treat these two questions for general d.

Our analysis will be based on the d-orthogonality and the d-symmetry properties of the Humbert polynomials. That suggested us a characterization problem $\boldsymbol{P}$ which consists to determinate all d-OPSs in a generalized hypergeometric polynomial class containing Humbert polynomials.

The paper is structured as follows: In Section 2, by means of integral representations, we give the d-dimensional functional vector related to the Humbert polynomials. The weight functions are expressed in terms of Meijer G-functions. Some special cases corresponding to $d=1$ and $d=2$ will be discussed. In Section 3, for the components of Humbert polynomials, we derive generalized hypergeometric representations, we get the analogous of the relations (1.6) and we obtain the corresponding d-dimensional functional vector. Then we introduce a class of generalized hypergeometric polynomials containing the components of the Humbert polynomials and we derive some properties analogous to the Jacobi ones. That turns out to be a generating function, a $(d+1)$-order differential equation, a $(d+1)$-order recurrence relation of type (1.5) and the "classical" d-orthogonality. The case $d=2$ will be discussed. Finally, in Section 4, we solve the characterization problem $\boldsymbol{P}$.

2. Humbert polynomials

2.1. Dual sequence

Definition 2.1. Let $\{P_n\}_{n\geqslant 0}$ be a PS in $\mathcal{P}$. The corresponding dual sequence $\{u_n\}_{n\geqslant 0}$ in $\mathcal{P}'$ is defined by

$$\langle u_n,P_r\rangle=\delta_{n,r}; \quad n,r=0,1,2\ldots.$$

Next, we express explicitly the dual sequence associated with the Humbert polynomials.

Remark 2.2. Since $\{P_n\}_{n\geqslant 0}$ is a basis in $\mathcal{P}$, it follows from this definition that

$$\langle u_r, x^n\rangle = 0, \quad r = 1, 2, \ldots, \ r > n. \tag{2.1}$$

Theorem 2.3. *The moments of the dual sequence $\{u_r\}_{r\geqslant 0}$ associated with the Humbert PS $\{h^{\nu}_{n,d+1}(x)\}_{n\geqslant 0}$ defined by* (1.1) *are given by*

$$\langle u_r, x^n\rangle = 0 \quad \text{if } n < r, \tag{2.2}$$

and, if $n \geqslant r$,

$$\langle u_r, x^n\rangle = \delta_{i',i} \int_0^{d^{-\frac{d}{d+1}}} \xi^n \varphi_{r,d}(\xi)\, d\xi, \tag{2.3}$$

where

$$\begin{cases} n = i + (d+1)k, & k \in \mathbb{N},\ i \in \mathbb{N}_{d+1}, \\ r = i' + (d+1)k', & k' \in \mathbb{N},\ i' \in \mathbb{N}_{d+1}, \end{cases} \tag{2.4}$$

and $\varphi_{r,d}(\xi)$ is the function defined by

$$\varphi_{r,d}(\xi) = \frac{r!}{(d+1)^{r-1}(\nu)_r} \frac{\prod_{j=1}^{d} \Gamma(\frac{\nu+r+j}{d})}{\prod_{j=1}^{d+1} \Gamma(\frac{r+j}{d+1})} \xi^{-(r+1)} G^{d+1,0}_{d+1,d+1}\left(d^d \xi^{d+1} \,\middle|\, \begin{matrix} \frac{\nu+r+1}{d}, \ldots, \frac{\nu+r+d}{d}, 1 \\ \frac{r+1}{d+1}, \ldots, \frac{r+(d+1)}{d+1} \end{matrix}\right). \tag{2.5}$$

$G^{m,n}_{p,q}$ designates the Meijer's G-function defined by [26, p. 143]

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right) = (2\pi i)^{-1} \int_L z^{\tau} \frac{\prod_{j=1}^{m} \Gamma(b_j - \tau) \prod_{j=1}^{n} \Gamma(1 - a_j + \tau)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \tau) \prod_{j=n+1}^{p} \Gamma(a_j - \tau)}\, d\tau,$$

where (a_p) abbreviates the set $\{a_1, a_2, \ldots, a_p\}$. We refer to [26, p. 144] *for the details regarding the type of the contour L.*

Proof. (2.2) follows from (2.1). In order to prove (2.3) we recall that the inversion formula related to the Humbert polynomial set $\{h^{\nu}_{n,d+1}(x)\}_{n\geqslant 0}$ is given by [7]

$$x^n = \sum_{j=0}^{[\frac{n}{d+1}]} \frac{(\nu + n - (d+1)j)}{(\nu)_{n+1-j}} \frac{n!}{(d+1)^n j!} h^{\nu}_{n-(d+1)j,d+1}(x). \tag{2.6}$$

On the other hand, it is easy to verify that $\delta_{r,n-(d+1)j} = \delta_{i,i'}\delta_{j,k-k'}$, where $j = 0, \ldots, [\frac{n}{d+1}]$ and i, i' are two integers defined by (2.4).

Then, according to (2.4) and (2.6), we have

$$\begin{aligned} \langle u_r, x^n\rangle &= \sum_{j=0}^{[\frac{n}{d+1}]} \frac{(\nu + n - (d+1)j)}{(\nu)_{n+1-j}} \frac{n!}{(d+1)^n j!} \langle u_r, h^{\nu}_{n-(d+1)j,d+1}(x)\rangle \\ &= \delta_{i',i} \sum_{j=0}^{[\frac{n}{d+1}]} \frac{(\nu + n - (d+1)j)}{(\nu)_{n+1-j}} \frac{n!}{(d+1)^n j!} \delta_{j,k-k'} \\ &= \delta_{i',i} \frac{(\nu + r)}{(d+1)^{r+(d+1)(k-k')}} \frac{(1)_{r+(d+1)(k-k')}}{(k-k')!(\nu)_{r+1+d(k-k')}}. \end{aligned}$$

Putting $k'' = k - k'$, we obtain

$$\langle u_r, x^n\rangle = \delta_{i',i} \frac{(\nu + r)}{(d+1)^{r+(d+1)k''}} \frac{(1)_{r+(d+1)k''}}{k''!(\nu)_{r+1+dk''}}. \tag{2.7}$$

The use of the identity (2.7) and the following transformations (see [26], for instance):

$$(a)_{i+j} = (a)_i (a+i)_j, \quad i, j \in \mathbb{N}, \tag{2.8}$$

$$(a)_{ms} = m^{ms} \prod_{j=0}^{m-1} \left(\frac{a+j}{m}\right)_s, \quad s \in \mathbb{N}, \tag{2.9}$$

leads to

$$\begin{aligned}\langle u_r, x^n \rangle &= \delta_{i',i} \frac{r!}{(d+1)^r (\nu)_r} \frac{1}{d^{dk''}} \frac{\prod_{j=1}^{d+1} (\frac{r+j}{d+1})_{k''}}{k''! \prod_{j=1}^{d} (\frac{\nu+r+j}{d})_{k''}} \\ &= \frac{r! \delta_{i',i}}{(d+1)^r (\nu)_r} \frac{\prod_{j=1}^{d} (\frac{\nu+r+j}{d})}{\prod_{j=1}^{d+1} (\frac{r+j}{d+1})} \underbrace{\frac{\prod_{j=1}^{d+1} \Gamma(\frac{r+j}{d+1} + k'')}{d^{dk''} \Gamma(1+k'') \prod_{j=1}^{d} \Gamma(\frac{\nu+r+j}{d} + k'')}}_{A_{k'',r}(d)}.\end{aligned} \tag{2.10}$$

Setting

$$\begin{cases} \alpha_j = \dfrac{r+j}{d+1} + k'' - 1, & 1 \leqslant j \leqslant d+1, \\ \beta_j = \dfrac{\nu(d+1)+r+j}{d(d+1)}, & 1 \leqslant j \leqslant d, \\ \beta_{d+1} = -\dfrac{r}{d+1}, \end{cases}$$

we obtain

$$A_{k'',r}(d) = \frac{1}{d^{dk''}} \prod_{j=1}^{d+1} \left[\frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+1+\beta_j)}\right]. \tag{2.11}$$

On the other hand, Ben Cheikh and Douak [9] showed that

$${}_pF_q\left(\begin{matrix}(a_p) \\ (\alpha_q+\beta_q+1)\end{matrix}; x\right) = \prod_{i=1}^{q} \left(\frac{\Gamma(\alpha_i+1+\beta_i)}{\Gamma(\alpha_i+1)}\right) \int_0^1 G_{q,q}^{q,0}\left(t \,\middle|\, \begin{matrix}(\alpha_q+\beta_q) \\ (\alpha_q)\end{matrix}\right) {}_pF_q\left(\begin{matrix}(a_p) \\ (\alpha_q+1)\end{matrix}; xt\right) dt, \tag{2.12}$$

where $\sum_{j=1}^{d+1} \beta_j > 0$. ${}_pF_q(z)$ designates the generalized hypergeometric function with p numerator and q denominator parameters [26]

$${}_pF_q\left(\begin{matrix}(a_p) \\ (b_q)\end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{[a_p]_k}{[b_q]_k} \frac{z^k}{k!}, \tag{2.13}$$

where $[a_r]_p = \prod_{i=1}^{r} (a_i)_p$.

The identity (2.12), for $x = 0$ and $q = d+1$, is reduced to

$$\prod_{j=1}^{d+1} \left[\frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+1+\beta_j)}\right] = \int_0^1 G_{d+1,d+1}^{d+1,0}\left(t \,\middle|\, \begin{matrix}(\alpha_{d+1}+\beta_{d+1}) \\ (\alpha_{d+1})\end{matrix}\right) dt. \tag{2.14}$$

Thus, for $\nu > \frac{-1}{2}$, the identity (2.11) can be rewritten under the form

$$A_{k'',r}(d) = \frac{1}{d^{dk''}} \int_0^1 G_{d+1,d+1}^{d+1,0}\left(t \,\middle|\, \begin{matrix}\frac{\nu+r+1}{d} - 1 + k'', \ldots, \frac{\nu+r+d}{d} - 1 + k'', k'' \\ \frac{r+1}{d+1} - 1 + k'', \ldots, \frac{r+(d+1)}{d+1} - 1 + k''\end{matrix}\right) dt. \tag{2.15}$$

Then, according to the transformation [32, p. 46]

$$z^{\sigma} G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix}\right) = G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} \alpha_1+\sigma, \dots, \alpha_p+\sigma \\ \beta_1+\sigma, \dots, \beta_q+\sigma \end{matrix}\right), \tag{2.16}$$

we get

$$A_{k'',r}(d) = \int_0^1 \left(\frac{t}{d^d}\right)^{k''} G^{d+1,0}_{d+1,d+1}\left(t \,\middle|\, \begin{matrix} \frac{\nu+r+1}{d}-1, \dots, \frac{\nu+r+d}{d}-1, 0 \\ \frac{r+1}{d+1}-1, \dots, \frac{r+(d+1)}{d+1}-1 \end{matrix}\right) dt.$$

That, upon the change of variables $t = d^d \xi^{(d+1)}$, leads to

$$A_{k'',r}(d) = \int_0^{d^{-\frac{d}{d+1}}} \xi^{k''(d+1)} G^{d+1,0}_{d+1,d+1}\left(d^d \xi^{d+1} \,\middle|\, \begin{matrix} \frac{\nu+r+1}{d}-1, \dots, \frac{\nu+r+d}{d}-1, 0 \\ \frac{r+1}{d+1}-1, \dots, \frac{r+(d+1)}{d+1}-1 \end{matrix}\right) (d+1)\, d^d \xi^d \, d\xi. \tag{2.17}$$

Substituting (2.17) in (2.10), we obtain

$$\begin{aligned} \langle u_r, x^n\rangle &= \delta_{i',i} \frac{r!}{(d+1)^{r-1}(\nu)_r} \frac{\prod_{j=1}^{d} \Gamma(\frac{\nu+r+j}{d})}{\prod_{j=1}^{d+1} \Gamma(\frac{r+j}{d+1})} \int_0^{d^{-\frac{d}{d+1}}} \xi^{r+k''(d+1)} \xi^{-(r+1)} \\ &\quad \times \left(d^d \xi^{d+1}\right) G^{d+1,0}_{d+1,d+1}\left(d^d \xi^{d+1} \,\middle|\, \begin{matrix} \frac{\nu+r+1}{d}-1, \dots, \frac{\nu+r+d}{d}-1, 0 \\ \frac{r+1}{d+1}-1, \dots, \frac{r+(d+1)}{d+1}-1 \end{matrix}\right) d\xi. \end{aligned} \tag{2.18}$$

Replacing in the identity (2.18) k'' by $k-k'$, and using the identity (2.16), we deduce

$$\begin{aligned} \langle u_r, x^n\rangle &= \delta_{i',i} \frac{r!}{(d+1)^{r-1}(\nu)_r} \frac{\prod_{j=1}^{d} \Gamma(\frac{\nu+r+j}{d})}{\prod_{j=1}^{d+1} \Gamma(\frac{r+j}{d+1})} \int_0^{d^{-\frac{d}{d+1}}} \xi^{r+(k-k')(d+1)} \\ &\quad \times \xi^{-(r+1)} G^{d+1,0}_{d+1,d+1}\left(d^d \xi^{d+1} \,\middle|\, \begin{matrix} \frac{\nu+r+1}{d}-1, \dots, \frac{\nu+r+d}{d}-1, 0 \\ \frac{r+1}{d+1}-1, \dots, \frac{r+(d+1)}{d+1}-1 \end{matrix}\right) d\xi. \end{aligned} \tag{2.19}$$

On the other hand, from (2.4) with $i = i'$, it can be readily shown that $n = r + (k-k')(d+1)$.

That, by virtue of (2.19), leads to (2.3). □

2.2. d-Dimensional functional vector

Our interest here is to determinate the d-dimensional functional vector for which we have the d-orthogonality of the Humbert polynomials.

As proved by Maroni [28], a PS $\{P_n\}_{n\geqslant 0}$ is d-orthogonal with respect to a d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \dots, \Gamma_{d-1})$ if and only if it is also d-orthogonal with respect to the vector $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$, where the functionals $u_0, u_1, \dots, u_{d-1}$ are the d first elements of the dual sequence $\{u_n\}_{n\geqslant 0}$ associated to the PS $\{P_n\}_{n\geqslant 0}$. Consequently, for the Humbert polynomials, we consider the d first elements of the corresponding dual sequence as the d-dimensional functional vector ensuring the d-orthogonality of these polynomials. That leads to the following.

Theorem 2.4. *The Humbert PS $\{h^{\nu}_{n,d+1}(x)\}_{n\geqslant 0}$ defined by* (1.1) *is a d-OPS with respect to the d-dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$ given by their moments*:

$$\langle u_r, x^n\rangle = 0 \quad \text{if } n < r, \tag{2.20}$$

and, if $n \geqslant r$,

$$\langle u_r, x^n\rangle = \delta_{r,i} \int_0^{d^{-\frac{d}{d+1}}} \xi^n \varphi_{r,d}(\xi)\, d\xi, \tag{2.21}$$

where $n = i + (d+1)k$, $k \in \mathbb{N}$, $i \in \mathbb{N}_{d+1}$, $r \in \mathbb{N}_d$ *and*

$$\varphi_{r,d}(\xi) = \frac{r!}{(d+1)^{r-1}(\nu)_r} \frac{\prod_{j=1}^{d} \Gamma(\frac{\nu+r+j}{d})}{\prod_{j=1}^{d+1} \Gamma(\frac{r+j}{d+1})} \xi^{-(r+1)} G_{d+1,d+1}^{d+1,0}\left(d^d \xi^{d+1} \,\middle|\, \begin{matrix} \frac{\nu+r+1}{d}, \ldots, \frac{\nu+r+d}{d}, 1 \\ \frac{r+1}{d+1}, \ldots, \frac{r+(d+1)}{d+1} \end{matrix}\right). \tag{2.22}$$

Proof. From Theorem 2.3, it follows that the moments of the d first elements of the dual sequence $\{u_n\}_{n\geqslant 0}$ associated to Humbert PS are given by

$$\langle u_r, x^n \rangle = 0, \quad r > n,$$

$$\langle u_r, x^n \rangle = \delta_{i',i} \int_0^{d^{-\frac{d}{d+1}}} \xi^n \varphi_{r,d}(\xi)\, d\xi, \quad r \leqslant n, \tag{2.23}$$

where i, i' are two integers given by (2.4), and $\varphi_{r,d}(\xi)$ is the weight function defined by (2.5).

Taking into account the fact that $r \in \mathbb{N}_d$, we deduce $k' = 0$ and $r = i'$. That, by virtue of (2.23), leads to the desired result. □

2.3. Special cases

In this subsection, we consider some particular cases of Humbert polynomials by specializing the parameters d or ν to show that known and new results concerning the corresponding d-dimensional functional vectors may be deduced from Theorem 2.4.

2.3.1. Gegenbauer polynomials ($d = 1$)

By letting $d = 1$ in (1.1), we meet the generating function of Gegenbauer polynomials $\{C_n^{\nu}(x)\}_{n\geqslant 0}$. These polynomials are orthogonal with respect to the well-known weight function [25]

$$\varphi_{0,1}(\xi) = \frac{\nu(\Gamma(\nu))^2}{\pi\Gamma(2\nu)2^{1-2\nu}} \left(1-\xi^2\right)^{\nu-\frac{1}{2}}, \quad -1 \leqslant \xi \leqslant 1. \tag{2.24}$$

Next, we show that this result may be also deduced from Theorem 2.4. Indeed, from (2.21) with $d = 1$ and the transformation [32, p. 46]

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} \alpha_1, \ldots, \alpha_{p-1}, \beta_1 \\ \beta_1, \ldots, \beta_q \end{matrix}\right) = G_{p-1,q-1}^{m-1,n}\left(z \,\middle|\, \begin{matrix} \alpha_1, \ldots, \alpha_{p-1} \\ \beta_2, \ldots, \beta_q \end{matrix}\right); \quad m, p, q \geqslant 1; \tag{2.25}$$

we have

$$\varphi_{0,1}(\xi) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}} \xi G_{1,1}^{1,0}\left(\xi^2 \,\middle|\, \begin{matrix} \nu \\ -\frac{1}{2} \end{matrix}\right).$$

Taking into account the following identity [9]:

$$G_{1,1}^{1,0}\left(x \,\middle|\, \begin{matrix} \alpha+\beta \\ \alpha \end{matrix}\right) = \frac{1}{\Gamma(\beta)} (1-x)^{\beta-1} x^{\alpha}, \tag{2.26}$$

we obtain

$$\varphi_{0,1}(\xi) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \left(1-\xi^2\right)^{\nu-\frac{1}{2}}. \tag{2.27}$$

Under the Gauss's multiplication theorem [32, p. 23]

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots. \tag{2.28}$$

The identity (2.27) becomes

$$\varphi_{0,1}(\xi)=\frac{\nu(\Gamma(\nu))^2}{\pi\Gamma(2\nu)2^{-2\nu}}\left(1-\xi^2\right)^{\nu-\frac{1}{2}}.$$

Consequently, the Gegenbauer functional is given by its moments

$$\langle u_0,x^n\rangle=\delta_{0,i}\int_0^1\xi^{2k}\left(1-\xi^2\right)^{\nu-\frac{1}{2}}\frac{\nu(\Gamma(\nu))^2}{\pi\Gamma(2\nu)2^{-2\nu}}\,d\xi,$$

where $n=i+2k,\ k\in\mathbb{N}$ and $i=0,1$.

Or equivalently

$$\langle u_0,x^n\rangle=\delta_{0,i}\int_{-1}^1\xi^{n}\left(1-\xi^2\right)^{\nu-\frac{1}{2}}\frac{\nu(\Gamma(\nu))^2}{\pi\Gamma(2\nu)2^{1-2\nu}}\,d\xi. \tag{2.29}$$

2.3.2. Pincherle type polynomials ($d=2$)

Recall that the Pincherle type polynomials are defined by [22]:

$$\left(1-3xt+t^3\right)^{-\nu}=\sum_{n\geqslant 0}P_n^\nu(x)t^n, \tag{2.30}$$

which for $\nu=-\frac{1}{2}$ reduce to the Pincherle polynomials [30].

By letting $d=2$ in the identity (1.1), we deduce that $P_n^\nu(x)=h_{n,3}^\nu(x)$. In this case Theorem 2.4 provides the following.

Corollary 2.5. *The Pincherle type polynomials $\{P_n^\nu(x)\}_{n\geqslant 0}$ are 2-orthogonal with respect to the 2-dimensional functional vector $\mathcal{U}={}^t(u_0,u_1)$ given by their moments*:

$$\langle u_r,x^n\rangle=\delta_{r,i}\int_0^{2^{\frac{-2}{3}}}\xi^n\varphi_{r,2}(\xi)\,d\xi,\quad r=0,1,\ n=i+3k,\ k\in\mathbb{N}, \tag{2.31}$$

where the weight functions $\varphi_{r,2}(\xi)$; $r=0,1$; are given by

$$\varphi_{0,2}(\xi)=\frac{3\sqrt{3}\,2^{\frac{1}{3}-\nu}\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\xi\left(1-4\xi^3\right)^{\nu-\frac{1}{2}}{}_2F_1\left(\begin{matrix}\frac{\nu}{2}+\frac{2}{3},\frac{\nu}{2}+\frac{1}{6}\\ \nu+\frac{1}{2}\end{matrix};1-4\xi^3\right),$$

$$\varphi_{1,2}(\xi)=\frac{3\sqrt{3}\,2^{\frac{2}{3}-\nu}\Gamma(\nu+2)}{\nu\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\xi^2\left(1-4\xi^3\right)^{\nu-\frac{1}{2}}{}_2F_1\left(\begin{matrix}\frac{\nu}{2}+\frac{5}{6},\frac{\nu}{2}+\frac{1}{3}\\ \nu+\frac{1}{2}\end{matrix};1-4\xi^3\right).$$

Proof. From the identity (2.25) and Theorem 2.4, we deduce

$$\langle u_r,x^n\rangle=\delta_{r,i}\int_0^{2^{-\frac{2}{3}}}\xi^n\varphi_{r,2}(\xi)\,d\xi,\quad r=0,1;\ n=i+3k,\ k\in\mathbb{N},$$

where

$$\varphi_{0,2}(\xi)=\frac{3\Gamma(\frac{\nu+1}{2})\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}\xi^{-1}G_{2,2}^{2,0}\left(4\xi^3\,\middle|\,\begin{matrix}\frac{\nu+1}{2},\frac{\nu+2}{2}\\ \frac{1}{3},\frac{2}{3}\end{matrix}\right),$$

$$\varphi_{1,2}(\xi)=\frac{\Gamma(\frac{\nu+2}{2})\Gamma(\frac{\nu+3}{2})}{\nu\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})}\xi^{-2}G_{2,2}^{2,0}\left(4\xi^3\,\middle|\,\begin{matrix}\frac{\nu+2}{2},\frac{\nu+3}{2}\\ \frac{2}{3},\frac{4}{3}\end{matrix}\right).$$

Thus, by using the transformations (2.28) and the identity [24, p. 275]

$$G_{2,2}^{2,0}\left(t \,\middle|\, \begin{matrix}\gamma_1+\delta_1, \gamma_2+\delta_2\\ \gamma_1, \gamma_2\end{matrix}\right) = \frac{t^{\gamma_2}(1-t)^{\delta_1+\delta_2-1}}{\Gamma(\delta_1+\delta_2)} {}_2F_1\left(\begin{matrix}\gamma_2+\delta_2-\gamma_1, \delta_1\\ \delta_1+\delta_2\end{matrix}; 1-t\right), \quad t<1, \tag{2.32}$$

we obtain (2.31). □

Remark 2.6. It is worthy to note the non-negativity of the weight functions given by (2.31) for $\nu > 0$.

2.3.3. Chebyshev type d-orthogonal polynomials ($\nu = 1$)

As solution of a characterization problem related to the associated polynomials of a given PS, Douak and Maroni consider the Chebyshev type d-OPS of the second kind $\{U_n(.;d)\}_{n\geqslant 0}$ generated by [18]

$$\left(1-xt+bt^{d+1}\right)^{-1} = \sum_{n\geqslant 0} U_n(x;d)t^n, \quad b\neq 0, \tag{2.33}$$

which for $d=b=1$ reduces to the classical Chebyshev PS of the second kind $\{U_n(.)\}_{n\geqslant 0}$. They have stated some of their properties. For the case $d=2$, they derive integral representations of the linear functionals with respect to which the polynomials are 2-orthogonal using the fact that these polynomials are "classical." These integral representations were also given by Ben Cheikh and Ben Romdhane [6] using a different approach based on lowering and transfer operators relative to the involved polynomials. Next, we use Theorem 2.4 to extend this result to arbitrary positive integer d. In fact, by letting $\nu = 1$ in the identity (1.1), we obtain

$$U_n(x;d) = b^{\frac{n}{d+1}} h^1_{n,d+1}\left(\frac{x}{(d+1)b^{\frac{1}{d+1}}}\right), \quad n\geqslant 0. \tag{2.34}$$

In this case, Theorem 2.4 is reduced to the following.

Corollary 2.7. *The Chebyshev type PS $\{U_n(.;d)\}_{n\geqslant 0}$ defined by* (2.33) *is a d-OPS with respect to the d-dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \ldots, u_{d-1})$ given by their moments*:

$$\langle u_r, x^n\rangle = 0 \quad \text{if } n<r, \tag{2.35}$$

and, if $n\geqslant r$,

$$\langle u_r, x^n\rangle = \delta_{r,i} \int_0^{(d+1)(\frac{b}{d})^{\frac{1}{d+1}}} \xi^n \psi_{r,d}(\xi)\, d\xi, \tag{2.36}$$

where $n = i+(d+1)k$, $k\in\mathbb{N}$, $i = 0,1,\ldots,d$, $r = 0,1,\ldots,d-1$ and

$$\psi_{r,d}(\xi) = (d+1)\frac{\prod_{j=1}^{d}\Gamma(\frac{1+r+j}{d})}{\prod_{j=1}^{d+1}\Gamma(\frac{r+j}{d+1})}\xi^{-(r+1)} G^{d+1,0}_{d+1,d+1}\left(\frac{d^d\xi^{d+1}}{b(d+1)^{d+1}} \,\middle|\, \begin{matrix}\frac{r+2}{d}, \ldots, \frac{r+1+d}{d}, 1\\ \frac{r+1}{d+1}, \ldots, \frac{r+(d+1)}{d+1}\end{matrix}\right).$$

Two particular cases are worthy to note.

• *Case* 1: $d=b=1$.

This case corresponds to Chebyshev PS of the second kind $\{U_n(\frac{x}{2})\}_{n\geqslant 0}$, which is orthogonal with respect to the well-known weight function $\psi_{0,1}(\xi) = \frac{2}{\pi}(1-\xi^2)^{\frac{1}{2}}$ on the interval $-1\leqslant\xi\leqslant 1$ (see, for instance, [25]). This weight function is also given by (2.36) for $d=1$. Indeed, from (2.36) we have

$$\langle u_0, x^n\rangle = \delta_{0,i}\frac{2}{\sqrt{\pi}}\int_0^2 \xi^{2k-1} G^{1,0}_{1,1}\left(\left(\frac{\xi}{2}\right)^2 \,\middle|\, \begin{matrix}2\\ \frac{1}{2}\end{matrix}\right) d\xi, \tag{2.37}$$

where $n = i+2k$, $k\in\mathbb{N}$ and $i = 0,1$.

Under the transformation given by (2.16), the identity (2.37) becomes

$$\langle u_0, x^n\rangle = \frac{\delta_{0,i}}{2\sqrt{\pi}}\int_0^2 \xi^{2k+1} G_{1,1}^{1,0}\left(\left(\frac{\xi}{2}\right)^2 \,\middle|\, \begin{matrix} 1 \\ -\frac{1}{2}\end{matrix}\right) d\xi.$$

Applying (2.26) and the change of variable $t = \frac{\xi}{2}$, we get

$$\left\langle u_0, \left(\frac{x}{2}\right)^n\right\rangle = \delta_{0,i}\int_{-1}^1 \xi^n \frac{2}{\pi}\left(1-\xi^2\right)^{\frac{1}{2}}\, d\xi.$$

• *Case* 2: $d = 2$, $b = \frac{4}{27}$.
From Corollary 2.7, we deduce the following.

Corollary 2.8. *(See Douak and Maroni [18].) The pair of functionals u_0 and u_1 for which the Chebyshev type PS $\{U_n(.,2)\}_{n\geqslant 0}$ is 2-orthogonal have the following moments*:

$$\langle u_r, x^n\rangle = \delta_{r,i}\int_0^1 \xi^n \psi_{r,2}(\xi)\, d\xi, \tag{2.38}$$

where $n = i + 3k$, $k \in \mathbb{N}$, $i = 0, 1, 2$, $r = 0, 1$ and

$$\begin{aligned}\psi_{0,2}(\xi) &= \frac{9\sqrt{3}}{4\pi}\left[\left(1+\sqrt{1-\xi^3}\right)^{\frac{1}{3}} - \left(1-\sqrt{1-\xi^3}\right)^{\frac{1}{3}}\right],\\ \psi_{1,2}(\xi) &= \frac{27\sqrt{3}}{8\pi}\left[\left(1+\sqrt{1-\xi^3}\right)^{\frac{2}{3}} - \left(1-\sqrt{1-\xi^3}\right)^{\frac{2}{3}}\right].\end{aligned} \tag{2.39}$$

Proof. Put $d = 2$ and $b = \frac{4}{27}$ in (2.36) to obtain (2.38) with

$$\begin{aligned}\psi_{0,2}(\xi) &= \frac{3\sqrt{3}}{4\sqrt{\pi}}\xi^{-1} G_{2,2}^{2,0}\left(\xi^3 \,\middle|\, \begin{matrix} 1, \frac{3}{2} \\ \frac{1}{3}, \frac{2}{3}\end{matrix}\right),\\ \psi_{1,2}(\xi) &= \frac{9\sqrt{3}}{4\sqrt{\pi}}\xi^{-2} G_{2,2}^{2,0}\left(\xi^3 \,\middle|\, \begin{matrix} 2, \frac{3}{2} \\ \frac{2}{3}, \frac{4}{3}\end{matrix}\right).\end{aligned}$$

Using the identity (2.32), we deduce

$$\begin{aligned}\psi_{0,2}(\xi) &= \frac{3\sqrt{3}}{2\pi}\xi\left(1-\xi^3\right)^{\frac{1}{2}} {}_2F_1\left(\begin{matrix}\frac{7}{6}, \frac{2}{3} \\ \frac{3}{2}\end{matrix}; 1-\xi^3\right),\\ \psi_{1,2}(\xi) &= \frac{9\sqrt{3}}{2\pi}\xi^2\left(1-\xi^3\right)^{\frac{1}{2}} {}_2F_1\left(\begin{matrix}\frac{4}{3}, \frac{5}{6} \\ \frac{3}{2}\end{matrix}; 1-\xi^3\right).\end{aligned} \tag{2.40}$$

On the other hand, from the following identity [31, p. 70]:

$$\begin{aligned}{}_2F_1\left(\begin{matrix}a, b \\ c\end{matrix}; z\right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1\left(\begin{matrix}a, b \\ a+b+1-c\end{matrix}; 1-z\right)\\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} {}_2F_1\left(\begin{matrix}c-a, c-b \\ c-a-b+1\end{matrix}; 1-z\right),\end{aligned}$$

we have

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}\frac{7}{6},\frac{2}{3}\\ \frac{3}{2}\end{matrix};1-\xi^3\right)&=-\frac{3}{2^{\frac{4}{3}}}{}_2F_1\left(\begin{matrix}\frac{7}{6},\frac{2}{3}\\ \frac{4}{3}\end{matrix};\xi^3\right)+\frac{3}{2^{\frac{2}{3}}}\xi^{-1}{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{5}{6}\\ \frac{2}{3}\end{matrix};\xi^3\right),\\
{}_2F_1\left(\begin{matrix}\frac{4}{3},\frac{5}{6}\\ \frac{3}{2}\end{matrix};1-\xi^3\right)&=-\frac{3}{2^{\frac{8}{3}}}{}_2F_1\left(\begin{matrix}\frac{4}{3},\frac{5}{6}\\ \frac{5}{3}\end{matrix};\xi^3\right)+\frac{3}{2^{\frac{4}{3}}}\xi^{-2}{}_2F_1\left(\begin{matrix}\frac{1}{6},\frac{2}{3}\\ \frac{1}{3}\end{matrix};\xi^3\right).
\end{aligned}
$$

That, under the following transformation [31, p. 70]:

$$
{}_2F_1\left(\begin{matrix}a,a+\frac{1}{2}\\ 2a\end{matrix};z\right)=(1-z)^{-\frac{1}{2}}2^{2a-1}\left(1+\sqrt{1-z}\right)^{1-2a},
$$

leads to

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}\frac{7}{6},\frac{2}{3}\\ \frac{3}{2}\end{matrix};1-\xi^3\right)&=\frac{3}{2}\xi^{-1}\left(1-\xi^3\right)^{-\frac{1}{2}}\left[\left(1+\sqrt{1-\xi^3}\right)^{\frac{1}{3}}-\left(1-\sqrt{1-\xi^3}\right)^{\frac{1}{3}}\right],\\
{}_2F_1\left(\begin{matrix}\frac{4}{3},\frac{5}{6}\\ \frac{3}{2}\end{matrix};1-\xi^3\right)&=\frac{3}{4}\xi^{-2}\left(1-\xi^3\right)^{-\frac{1}{2}}\left[\left(1+\sqrt{1-\xi^3}\right)^{\frac{2}{3}}-\left(1-\sqrt{1-\xi^3}\right)^{\frac{2}{3}}\right].
\end{aligned}
\tag{2.41}
$$

Substituting (2.41) in (2.40) to obtain (2.39). □

Remark 2.9. Notice that the weight functions given by (2.39) are non-negative.

2.3.4. Legendre type polynomials ($\nu=\frac{1}{2}$)

As far as we know there do not exist in the literature works dealing with d-OPS of Legendre type. In this subsection, we define the d-OPS of Legendre type $\{L_n(.;d)\}_{n\geqslant 0}$ by the Humbert PS $\{h^{\nu}_{n,d+1}(x)\}_{n\geqslant 0}$ with $\nu=\frac{1}{2}$, i.e. $L_n(x;d)=h^{\frac{1}{2}}_{n,d+1}(x)$. This PS is a natural extension of the Legendre ones [20], since, in the case when $d=1$ we meet the Legendre OPS $\{L_n(x)\}_{n\geqslant 0}$. Also, these polynomials satisfy the following recurrence relation given by (1.2) with $\nu=\frac{1}{2}$:

$$
2(n+1)L_{n+1}(x;d)=2(d+1)(2n+1)xL_n(x;d)-(2n+1-d)L_{n-d}(x;d),
$$

which is analogous to the Legendre ones [25]:

$$
(n+1)L_{n+1}(x)=2(2n+1)xL_n(x)-nL_{n-1}(x).
$$

The use of Theorem 2.4 with $\nu=\frac{1}{2}$, leads to the following:

Corollary 2.10. *The PS $\{L_n(.;d)\}_{n\geqslant 0}$ is a d-OPS with respect to the d-dimensional functional vector $\mathcal{U}={}^t(u_0,u_1,\dots,u_{d-1})$ given by their moments*:

$$
\langle u_r,x^n\rangle=0\quad \text{if } n<r,\tag{2.42}
$$

and, if $n\geqslant r$,

$$
\langle u_r,x^n\rangle=\delta_{r,i}\int_0^{d^{-\frac{d}{d+1}}}\xi^n\varphi_{r,d}(\xi)\,d\xi,\tag{2.43}
$$

where $n=i+(d+1)k$, $k\in\mathbb{N}$, $i=0,1,\dots,d$, $r=0,1,\dots,d-1$ and

$$
\varphi_{r,d}(\xi)=\frac{r!}{(d+1)^{r-1}(\frac{1}{2})_r}\frac{\prod_{j=1}^{d}\Gamma(\frac{1}{2d}+\frac{r+j}{d})}{\prod_{j=1}^{d+1}\Gamma(\frac{r+j}{d+1})}\xi^{-(r+1)}G^{d+1,0}_{d+1,d+1}\left(d^d\xi^{d+1}\,\middle|\,\begin{matrix}\frac{1}{2d}+\frac{r+1}{d},\dots,\frac{1}{2d}+\frac{r+d}{d},1\\ \frac{r+1}{d+1},\dots,\frac{r+(d+1)}{d+1}\end{matrix}\right).\tag{2.44}
$$

Next, we consider two particular cases.

• *Case* 1: $d = 1$.

In this case the identity (2.44) is reduced to (2.24) with $\nu = \frac{1}{2}$. That leads to the well-known weight function [25] $\varphi_{0,1}(\xi) = 1$; $-1 \leqslant x \leqslant 1$; with respect to which the Legendre polynomials $L_n(x)$ are orthogonal.

• *Case* 2: $d = 2$.

In this case Corollary 2.10 is reduced to Corollary 2.5 with $\nu = \frac{1}{2}$, which may be written as follows.

Corollary 2.11. *The moments of the* 2*-dimensional vector* $\mathcal{U} = {}^t(u_0, u_1)$ *ensuring the* 2*-orthogonality of the Legendre type polynomials* $\{L_n(.; d)\}_{n \geqslant 0}$ *have the following integral representations*:

$$\langle u_r, x^n \rangle = \delta_{r,i} \int_0^{2^{\frac{-2}{3}}} \xi^n \varphi_{r,2}(\xi)\, d\xi, \quad r = 0, 1,\ n = i + 3k,\ k \in \mathbb{N}, \tag{2.45}$$

where the weight functions $\varphi_{r,2}(\xi)$; $r = 0, 1$; *are given by*

$$\begin{aligned} \varphi_{0,2}(\xi) &= \frac{3\sqrt{3}}{2^{\frac{7}{6}}} \xi\, {}_2F_1\left(\begin{matrix} \frac{11}{12}, \frac{5}{12} \\ 1 \end{matrix}; 1 - 4\xi^3\right), \\ \varphi_{1,2}(\xi) &= \frac{9\sqrt{3}}{2^{\frac{5}{6}}} \xi^2\, {}_2F_1\left(\begin{matrix} \frac{13}{12}, \frac{7}{12} \\ 1 \end{matrix}; 1 - 4\xi^3\right). \end{aligned} \tag{2.46}$$

Remark 2.12. It is worthy to note the non-negativity of the weight functions given by (2.46).

Remark 2.13. For $\nu = \frac{1}{d+1}$, the Humbert PS $\{h^{\nu}_{n,d+1}(x)\}_{n \geqslant 0}$ is reduced to the Kinney ones [20] $\{K_n(.; d)\}_{n \geqslant 0}$. That include as particular cases, the Legendre polynomials $\{L_n(x)\}_{n \geqslant 0}$ $(d = 1)$, and the Pincherle type polynomials $\{P_n^{\frac{1}{3}}(x)\}_{n \geqslant 0}$ $(d = 2)$. These polynomials have some properties analogous to those of Legendre. Consequently, they can be also viewed as Legendre type d-orthogonal polynomials.

3. Jacobi type polynomials

3.1. Components of Humbert polynomials

Throughout this subsection, we need the following notion.

For any arbitrary positive integer s, we denote by $w_s = e^{\frac{2i\pi}{s}}$ the complex sth root of unity.

Let f be a function of the complex variable z holomorphic in Ω a circular neighborhood in the origin, there exists a unique sequence $\{f_{[s,\mu]}\}_{\mu \in \mathbb{N}_s}$, such that [5]

$$f = \sum_{\mu=0}^{s-1} f_{[s,\mu]}, \tag{3.1}$$

where

$$f_{[s,\mu]}(z) = \Pi_{[s,\mu]} f(z) = \frac{1}{s} \sum_{j=0}^{s-1} w_s^{-\mu j} f\left(w_s^j z\right), \quad \mu \in \mathbb{N}_s. \tag{3.2}$$

$\Pi_{[s,\mu]}$ being the projection operator.

The identity (3.1) is called the decomposition of the function f with respect to the cyclic group of order s and we refer to the functions $f_{[s,\mu]}$, $\mu \in \mathbb{N}_s$, as the components with respect to the cyclic group of order s of f. It is clear that $f_{[s,\mu]}(w_s z) = w_s^{\mu} f_{[s,\mu]}(z)$. For $s = 2$, $f_{[2,0]}$ and $f_{[2,1]}$ are respectively the even and odd parts of the function f.

Our interest here is to determinate the components of the Humbert polynomials $\{h^{\nu}_{n,d+1}(x)\}_{n \geqslant 0}$ and the corresponding d-dimensional functional vector.

Theorem 3.1. *The components of the Humbert PS* $\{h^{\nu}_{n,d+1}(\frac{x}{d+1})\}_{n\geqslant 0}$ *are the PSs* $\{\frac{(\nu)_\mu}{\mu!}P^{\nu+\mu}_n(x;(\alpha_{d,\mu}))\}_{n\geqslant 0}$; $\mu\in\mathbb{N}_{d+1}$; *defined by*

$$P^{\nu+\mu}_n\big(x;(\alpha_{d,\mu})\big)=\frac{(-1)^n(\nu+\mu)_n}{n!}\,{}_{d+1}F_d\left(\begin{matrix}-n,\,\Delta(d,n+\nu+\mu)\\ (\alpha_{d,\mu})\end{matrix};\frac{d^dx}{(d+1)^{d+1}}\right),\tag{3.3}$$

where $(\alpha_{d,\mu})$ *designates the set given by*

$$\left\{\frac{\mu+1+j}{d+1};\ j=0,\dots,d \text{ and } j\neq d-\mu\right\},\tag{3.4}$$

and $\Delta(p,a)$ *abbreviates the array of* p *parameters* $\frac{a+j-1}{p}$, $j=1,\dots,p$.

Proof. According to Lemma 1.5, we have to show

$$h^{\nu}_{n(d+1)+\mu,d+1}(x)=\frac{(\nu)_\mu((d+1)x)^\mu}{\mu!}P^{\nu+\mu}_n\big(((d+1)x)^{d+1};(\alpha_{d,\mu})\big).\tag{3.5}$$

Recall that the Humbert polynomials $h^{\nu}_{n,d+1}(x)$ are d-symmetric and generated by (1.1), which can be written in the form

$$\big(1+t^{d+1}\big)^{-\nu}{}_1F_0\left(\begin{matrix}\nu\\ -\end{matrix};x\frac{(d+1)t}{1+t^{d+1}}\right)=\sum_{n\geqslant 0}h^{\nu}_{n,d+1}(x)t^n.\tag{3.6}$$

Applying the projection operator $\Pi_{[d+1,\mu]}$; $\mu\in\mathbb{N}_{d+1}$; to the two members of the identity (3.6) considered as functions of the variables x. Using the fact that the Humbert polynomials $h^{\nu}_{n,d+1}(x)$ are d-symmetric, the identity (2.9) and the Osler–Srivastava identity [5]:

$$\Pi_{[s,\mu]}\left(z\mapsto {}_pF_q\left(\begin{matrix}(a_p)\\ (b_q)\end{matrix};z\right)\right)=\frac{[a_p]_\mu}{[b_q]_\mu}\frac{z^\mu}{\mu!}\times{}_{sp}F_{sq+n-1}\left(\begin{matrix}\Delta(s,a_1+\mu),\dots,\Delta(s,a_p+\mu)\\ \Delta^*(s,\mu+1),\Delta(s,b_1+\mu),\dots,\Delta(s,b_q+\mu)\end{matrix};\frac{z^s}{s^{(1-p+q)s}}\right),$$

where $\Delta^*(s,\mu+1)=\{\frac{\mu+1+j}{s},\ j\in\mathbb{N}_s\}\setminus\{1\}$, we obtain

$$\begin{aligned}
&\frac{\mu!}{(\nu)_\mu((d+1)x)^\mu}\sum_{n=0}^{\infty}h^{\nu}_{n(d+1)+\mu,d+1}(x)t^{n(d+1)+\mu}\\
&=t^\mu\big(1+t^{d+1}\big)^{-\nu-\mu}{}_{d+1}F_d\left(\begin{matrix}\Delta(d+1,\mu+\nu)\\ (\alpha_{d,\mu})\end{matrix};\left(\frac{(d+1)xt}{1+t^{d+1}}\right)^{d+1}\right)\\
&=\sum_{n,k=0}^{\infty}\frac{(-1)^n(\nu+\mu+(d+1)k)_n}{n!}\frac{\prod_{j=0}^{d}(\frac{\nu+\mu+j}{d+1})_k}{[\alpha_{d,\mu}]_k}\frac{((d+1)x)^{(d+1)k}}{k!}t^{(d+1)(n+k)+\mu}\\
&=\sum_{n,k=0}^{\infty}\frac{(-1)^{n-k}(\nu+\mu+(d+1)k)_{n-k}}{(n-k)!}\frac{\prod_{j=0}^{d}(\frac{\nu+\mu+j}{d+1})_k}{[\alpha_{d,\mu}]_k}\frac{((d+1)x)^{(d+1)k}}{k!}t^{(d+1)n+\mu}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n(\nu+\mu)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k\prod_{j=1}^{d}(\frac{\nu+\mu+n+j-1}{d})_k}{[\alpha_{d,\mu}]_k}\frac{(d^dx^{d+1})^k}{k!}t^{(d+1)n+\mu}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n(\nu+\mu)_n}{n!}{}_{d+1}F_d\left(\begin{matrix}-n,\,\Delta(d,n+\nu+\mu)\\ (\alpha_{d,\mu})\end{matrix};\frac{d^dx}{(d+1)^{d+1}}\right)t^{(d+1)n+\mu}\\
&=\sum_{n=0}^{\infty}P^{\nu+\mu}_n\big(((d+1)x)^{d+1};(\alpha_{d,\mu})\big)t^{(d+1)n+\mu}.
\end{aligned}$$

That, by identification, leads to (3.5). $\square$

Remark 3.2. Notice that for $d=1$, the identity (3.5) is reduced to the well-known relationship between Gegenbauer and Jacobi polynomials given by (1.6).

Next, we study the d-orthogonality of the components of Humbert PS. We state the following.

Theorem 3.3. *For $\nu > \frac{-1}{2}$, $\mu \in \mathbb{N}_{d+1}$ and $(\alpha_{d,\mu})$ the set given by* (3.4)*, the PS $\{P_n^{\nu+\mu}(x,(\alpha_{d,\mu}))\}_{n\geqslant 0}$ defined by* (3.3) *is a d-OPS with respect to the d-dimensional functional vector $\mathcal{U}^\mu = {}^t(u_0^\mu, u_1^\mu, \ldots, u_{d-1}^\mu)$ given by their moments*:

$$\langle u_r, x^n\rangle = 0 \quad \textit{if } n < r, \tag{3.7}$$

and, if $n \geqslant r$,

$$\langle u_r^\mu, x^n\rangle = \int_0^{d(1+\frac{1}{d})^{d+1}} \xi^n \varphi_{r,d}(\xi)\, d\xi, \tag{3.8}$$

where

$$\varphi_{r,d}(\xi) = \vartheta_{r,d}(\nu,\mu)\xi^{-1} G_{d+1,d+1}^{d+1,0}\left(\frac{\xi}{d(1+\frac{1}{d})^{d+1}}\,\middle|\, \begin{matrix} \frac{\nu+\mu+r+1}{d}, \ldots, \frac{\nu+\mu+r+d}{d}, 1-r \\ \frac{\mu+1}{d+1}, \ldots, \frac{\mu+(d+1)}{d+1} \end{matrix}\right),$$

with

$$\vartheta_{r,d}(\nu,\mu) = \frac{(\nu+\mu+r(d+1))}{(\nu+\mu)_{r+1}} \frac{\prod_{j=1}^{d}\Gamma(\frac{\nu+\mu+r+j}{d})}{\prod_{j=1}^{d+1}\Gamma(\frac{\mu+j}{d+1})}. \tag{3.9}$$

Proof. According to Theorem 2.3, the dual sequence $\{u_r\}_{r\geqslant 0}$ associated with the Humbert PS $\{h_{n,d+1}^{\nu}(\frac{x}{d+1})\}_{n\geqslant 0}$ is given by

$$\langle u_r, x^n\rangle = 0, \quad r > n,$$

$$\langle u_r, x^n\rangle = \delta_{i',i}\int_0^{d^{-\frac{d}{d+1}}} \big((d+1)\xi\big)^n \varphi_{r,d}(\xi)\, d\xi, \quad r \leqslant n,$$

where $\varphi_{r,d}(\xi)$ is the weight function given by (2.5), and i, i' are two integers defined by (2.4).

That, upon the change of variable $t = (d+1)\xi$, leads to

$$\langle u_r, x^n\rangle = \delta_{i',i}\int_0^{d^{-\frac{d}{d+1}}(d+1)} \xi^n (d+1)^{-1}\varphi_{r,d}\left(\frac{\xi}{d+1}\right) d\xi. \tag{3.10}$$

On the other hand, from Lemma 1.6, we deduce that the PS $\{P_n^{\nu+\mu}(x,(\alpha_{d,\mu}))\}_{n\geqslant 0}$ is d-orthogonal with respect to the d-dimensional functional vector $\mathcal{U}^\mu = {}^t(u_0^\mu, u_1^\mu, \ldots, u_{d-1}^\mu)$ given by

$$\langle u_r^\mu, x^n\rangle = \frac{(\nu)_\mu}{\mu!}\langle u_{\mu+r(d+1)}, x^{\mu+n(d+1)}\rangle. \tag{3.11}$$

Substituting (3.10) in (3.11) and using the transformation (2.16), we get

$$\begin{aligned}\langle u_r^\mu, x^n\rangle &= \frac{(\nu)_\mu}{\mu!}\frac{(\mu+r(d+1))!}{(\nu)_{\mu+r(d+1)}}\frac{d^{rd}}{(d+1)^{r(d+1)-1}}\frac{\prod_{j=1}^{d}\Gamma(\frac{\nu+\mu+r(d+1)+j}{d})}{\prod_{j=1}^{d+1}\Gamma(\frac{\mu+r(d+1)+j}{d+1})} \\ &\quad\times \int_0^{d^{-\frac{d}{d+1}}(d+1)} \xi^{n(d+1)-1} G_{d+1,d+1}^{d+1,0}\left(\frac{d^d\xi^{d+1}}{(d+1)^{d+1}}\,\middle|\, \begin{matrix}\frac{\nu+\mu+r+1}{d}, \ldots, \frac{\nu+\mu+r+d}{d}, 1-r \\ \frac{\mu+1}{d+1}, \ldots, \frac{\mu+(d+1)}{d+1}\end{matrix}\right) d\xi.\end{aligned}$$

Applying the change of variable $t = \xi^{d+1}$ and the transformation (2.9), we obtain the desired result. $\square$

Two particular cases are worthy to note.

• *Case* 1: $d = 1$.

In this case $(\alpha_{d,\mu}) = \{\mu + \frac{1}{2},\ \mu = 0, 1\}$, and the PSs $\{P_n^{\nu+\mu}(x, (\alpha_{d,\mu}))\}_{n\geqslant 0}$; $\mu = 0, 1$; are reduced to the shifted Jacobi PSs [32] $\{\frac{(\mu+\nu)_n}{(\mu+\frac{1}{2})_n} P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\frac{x}{2} - 1)\}_{n\geqslant 0}$; $\mu = 0, 1$. Moreover, the identity (3.8) becomes

$$\langle u_0, x^n\rangle = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+\frac{1}{2})} \int_0^4 \xi^{n-1} G_{1,1}^{1,0}\left(\frac{\xi}{4} \,\middle|\, \begin{matrix} \mu+\nu+1 \\ \mu+\frac{1}{2} \end{matrix}\right) d\xi.$$

According to (2.26) we get

$$\langle u_0, x^n\rangle = \frac{\Gamma(\nu+\mu+1)}{4\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_0^4 \xi^n \left(\frac{\xi}{4}\right)^{\mu-\frac{1}{2}} \left(1-\frac{\xi}{4}\right)^{\nu-\frac{1}{2}} d\xi.$$

That, upon the change of variable $t = \frac{\xi}{2} - 1$, leads to

$$\left\langle u_0, \left(\frac{x}{2}\right)^n\right\rangle = \frac{\Gamma(\nu+\mu+1)}{2^{\mu+\nu}\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_{-1}^1 (\xi+1)^n (\xi+1)^{\mu-\frac{1}{2}} (1-\xi)^{\nu-\frac{1}{2}}\, d\xi.$$

Consequently

$$\begin{aligned} \left\langle u_0, \left(\frac{x}{2}-1\right)^n\right\rangle &= \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \left\langle u_0, \left(\frac{x}{2}\right)^j\right\rangle \\ &= \frac{\Gamma(\nu+\mu+1)}{2^{\mu+\nu}\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_{-1}^1 \xi^n (1-\xi)^{\nu+\frac{1}{2}} (\xi+1)^{\mu-\frac{1}{2}}\, d\xi. \end{aligned} \tag{3.12}$$

From (3.12), we deduce the well-known weight functions associated to the shifted Jacobi polynomials $\{\frac{(\mu+\nu)_n}{(\mu+\frac{1}{2})_n} P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\frac{x}{2} - 1)\}_{n\geqslant 0}$ given by [32]:

$$\int_{-1}^1 (1-\xi)^\alpha (1+\xi)^\beta P_m^{(\alpha,\beta)}(\xi) P_n^{(\alpha,\beta)}(\xi)\, d\xi = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta)n!\Gamma(n+\alpha+\beta+1)}.$$

• *Case* 2: $d = 2$.

From Theorem 3.1, we deduce that the components of Pincherle PS $\{P_n^\nu(\frac{x}{2})\}_{n\geqslant 0}$ are the PSs $\{\frac{(\nu)_\mu}{\mu!} P_n^{\nu+\mu}(x, (\alpha_{d,\mu}))\}_{n\geqslant 0}$; $\mu \in \mathbb{N}_3$; given by

$$P_n^{\mu+\nu}(x; \alpha_1, \alpha_2) = \frac{(-1)^n(\nu+\mu)_n}{n!}\, {}_3F_2\left(\begin{matrix} -n, \frac{n+\mu+\nu}{2}, \frac{n+\mu+\nu+1}{2} \\ \alpha_1, \alpha_2 \end{matrix}; \frac{4x}{27}\right), \tag{3.13}$$

where $\{\alpha_1, \alpha_2\} = \{\frac{\mu+1}{3}, \frac{\mu+2}{3}, \frac{\mu+3}{3}\} \setminus \{1\}$.

These components were first obtained by Baker [2], by solving a differential equation satisfied by the Pincherle polynomials. He found the following relation between the Pincherle polynomials and some generalized hypergeometric functions:

$$P_n^\nu(x) = \frac{1}{3} \frac{(-1)^n\Gamma(n+\nu)}{4^{\frac{n}{3}-1}\Gamma(\nu)\Gamma(\frac{n+\nu}{2})\Gamma(\frac{n+\nu+1}{2})} \left[\frac{\Gamma(\frac{n}{6}+\frac{\nu}{2})\Gamma(\frac{n}{6}+\frac{\nu+1}{2})}{\Gamma(\frac{n+3}{3})} \right.$$

$$\times \sin\left((n-1)\frac{\pi}{3}\right)\sin\left((n-2)\frac{\pi}{3}\right){}_3F_2\left(\begin{matrix}-\frac{n}{3}, \frac{n}{6}+\frac{\nu}{2}, \frac{n}{6}+\frac{\nu+1}{2}\\ \frac{1}{3}, \frac{2}{3}\end{matrix}; 4x^3\right)$$
$$+\frac{\Gamma(\frac{n}{6}+\frac{\nu}{2}+\frac{1}{3})\Gamma(\frac{n}{6}+\frac{\nu+1}{2}+\frac{1}{3})}{\frac{1}{3}\Gamma(\frac{n+2}{3})}\sin\left((n-2)\frac{\pi}{3}\right)\sin\left(n\frac{\pi}{3}\right)$$
$$\times {}_3F_2\left(\begin{matrix}-\frac{n}{3}+\frac{1}{3}, \frac{n}{6}+\frac{\nu}{2}+\frac{1}{3}, \frac{n}{6}+\frac{\nu+1}{2}+\frac{1}{3}\\ \frac{2}{3}, \frac{4}{3}\end{matrix}; 4x^3\right)\left(4x^3\right)^{\frac{1}{3}}$$
$$+\frac{\Gamma(\frac{n}{6}+\frac{\nu}{2}+\frac{2}{3})\Gamma(\frac{n}{6}+\frac{\nu+1}{2}+\frac{2}{3})}{\frac{1}{3}\frac{2}{3}\Gamma\left(\frac{n+1}{3}\right)}\sin\left((n-1)\frac{\pi}{3}\right)\sin\left(n\frac{\pi}{3}\right)$$
$$\times {}_3F_2\left(\begin{matrix}-\frac{n}{3}+\frac{2}{3}, \frac{n}{6}+\frac{\nu}{2}+\frac{2}{3}, \frac{n}{6}+\frac{\nu+1}{2}+\frac{2}{3}\\ \frac{4}{3}, \frac{5}{3}\end{matrix}; 4x^3\right)\left(4x^3\right)^{\frac{2}{3}}\Bigg].$$

Replacing in this identity x by $\frac{x}{3}$ and n by $3n$ (respectively $3n+1$ and $3n+2$), we obtain the components given by (3.13) for $\mu=0$ (respectively $\mu=1$ and $\mu=2$).

Notice that, the second-order recurrence relations ensuring the 2-orthogonality of these components were given by Douak and Maroni [16, pp. 89, 93, 97]. Further results related to the 2-orthogonality of these components may be deduced from Theorem 3.3 which, in view of the transformation (2.32), leads to

Corollary 3.4. *The components of the Humbert polynomials* $\{P_n^{\nu+\mu}(x,(\alpha_{d,\mu}))\}_{n\geqslant 0}$; $\mu=0,1$; *defined by* (3.13) *are* 2-*orthogonal with respect to the* 2-*dimensional functional vector* $\mathcal{U}={}^t(u_0,u_1)$ *given by their moments*:

$$\langle u_r, x^n\rangle=\int_0^{\frac{27}{4}}\xi^n\varphi_{r,2}(\xi)\,d\xi,\quad r=0,1,$$

with

$$\varphi_{0,2}(\xi)=\frac{\Gamma(\frac{\nu+\mu+1}{2})\Gamma(\frac{\nu+\mu+2}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\mu+\nu+\frac{3}{2}-(\alpha_1+\alpha_2))}\left(\frac{4}{27}\right)^{\alpha_2}\xi^{\alpha_2-1}\left(1-\frac{4\xi}{27}\right)^{\mu+\nu+\frac{1}{2}-(\alpha_1+\alpha_2)}$$
$$\times {}_2F_1\left(\begin{matrix}\frac{\mu+\nu+2}{2}-\alpha_1, \frac{\mu+\nu+1}{2}-\alpha_1\\ \mu+\nu+\frac{3}{2}-(\alpha_1+\alpha_2)\end{matrix}; 1-\frac{4\xi}{27}\right), \tag{3.14}$$

$$\varphi_{1,2}(\xi)=\frac{(\mu+\nu+3)}{(\mu+\nu)_2}\frac{\Gamma(\frac{\nu+\mu+2}{2})\Gamma(\frac{\nu+\mu+3}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\xi^{-1}G_{3,3}^{3,0}\left(\frac{4\xi}{27}\,\middle|\,\begin{matrix}\frac{\mu+\nu+2}{2}, \frac{\mu+\nu+3}{2}, 0\\ 1, \alpha_1, \alpha_2\end{matrix}\right). \tag{3.15}$$

Remark 3.5. It is worthy to note the non-negativity of the weight function given by (3.14) for some conditions on the involved parameters.

3.2. An extension of (3.3)

3.2.1. Definition

As an extension of the components of Humbert polynomials, we consider the generalized hypergeometric polynomials defined by

$$P_n^{\nu}\big(x;(\alpha_d)\big):=P_n^{\nu}(x;\alpha_1,\ldots,\alpha_d)=\frac{(-1)^n(\nu)_n}{n!}{}_{d+1}F_d\left(\begin{matrix}-n, \Delta(d,n+\nu)\\ (\alpha_d)\end{matrix}; \frac{d^dx}{(d+1)^{d+1}}\right), \tag{3.16}$$

where $\alpha_j\neq 0,-1,-2,\ldots,$ and $\nu+d(1-\alpha_j)\neq 0,-1,-2,\ldots,$ for $j=1,\ldots,d$.

Remark 3.6. If we replace in (3.16) (α_d) by $(\alpha_{d,\mu})$ and ν by $\nu+\mu$, we meet the components of the Humbert polynomials given by (3.3).

For $d=1$, the PS $\{P_n^\nu(x;(\alpha_d))\}_{n\geqslant 0}$ is reduced to the shifted Jacobi ones [32] $\{\frac{(\nu)_n}{(\alpha_1)_n}P_n^{(\nu-\alpha_1,\alpha_1-1)}(\frac{x}{2}-1)\}_{n\geqslant 0}$.

Next, we derive for the polynomials defined by (3.16) some properties analogous to those of the Jacobi ones. So we refer to these polynomials as Jacobi type polynomials.

3.2.2. Generating function

Proposition 3.7. *The Jacobi type polynomials defined by* (3.16) *are generated by*

$$(1+t)^{-\nu}{}_{d+1}F_d\left(\begin{matrix}\Delta(d+1,\nu)\\(\alpha_d)\end{matrix};\frac{xt}{(1+t)^{d+1}}\right)=\sum_{n\geqslant 0}P_n^\nu\big(x;(\alpha_d)\big)t^n. \tag{3.17}$$

Proof. Recall that [23, p. 178]

$$(1-t)^{-\lambda}{}_pF_q\left(\begin{matrix}(a)_p\\(b)_q\end{matrix};-\frac{r^rxt}{(1-t)^r}\right)=\sum_{n\geqslant 0}\frac{(\lambda)_n}{n!}{}_{p+r}F_{q+r}\left(\begin{matrix}-n,\Delta(r-1,n+\lambda),(a)_p\\\Delta(r,\lambda),(b)_q\end{matrix};(r-1)^{r-1}x\right)t^n. \tag{3.18}$$

Replacing in the identity (3.18) r by $d+1$, λ by ν, (b_q) by (α_d), (a_p) by $\Delta(d+1,\nu)$, x by $\frac{x}{(d+1)^{d+1}}$ and t by $-t$, we obtain

$$(1+t)^{-\nu}{}_{d+1}F_d\left(\begin{matrix}\Delta(d+1,\nu)\\(\alpha_d)\end{matrix};\frac{xt}{(1+t)^{d+1}}\right)=\sum_{n\geqslant 0}\frac{(-1)^n(\nu)_n}{n!}{}_{d+1}F_d\left(\begin{matrix}-n,\Delta(d,n+\nu)\\(\alpha_d)\end{matrix};\frac{d^dx}{(d+1)^{d+1}}\right)t^n.$$

That, by virtue of the identity (3.16), leads to (3.17). □

Remark 3.8. For $d=1$, the identity (3.17) is reduced to the well-known generating function associated with the shifted Jacobi polynomials [25, p. 39]

$$(1+t)^{-\nu}{}_2F_1\left(\begin{matrix}\frac{\nu}{2},\frac{\nu+1}{2}\\\alpha_1\end{matrix};\frac{xt}{(1+t)^2}\right)=\sum_{n=0}^{\infty}\frac{(\nu)_n}{(\alpha_1)_n}P_n^{(\nu-\alpha_1,\alpha_1-1)}\left(\frac{x}{2}-1\right)t^n.$$

3.2.3. A $(d+1)$-order differential equation

Theorem 3.9. *The Jacobi type polynomials defined by* (3.16) *verify the following differential equation*:

$$\begin{aligned}&\Bigg\{(x-1)x^dD^{d+1}+\left[\left(\nu+\frac{d^2+2d-1}{2}\right)x-\frac{d(d-1)}{2}-\sum_{j=1}^{d}\alpha_j\right]x^{d-1}D^d\\&\quad+\sum_{m=1}^{d-1}\big(c_{d+1,m}x-c'_{d+1,m}\big)x^{m-1}D^m-\frac{n(\nu+n)_d}{d^d}\Bigg\}y=0,\end{aligned} \tag{3.19}$$

where $y=P_n^\nu(\frac{(d+1)^{d+1}}{d^d}x;(\alpha_d))$, $D=\frac{d}{dx}$,

$$c_{d+1,m}=\sum_{k=0}^{d+1-m}\binom{d+1-k}{d+1-k-m}B_{d+1-k-m}^{(-m)}S_k(a_{d+1}), \tag{3.20}$$

$$c'_{d+1,m}=\sum_{k=0}^{d+1-m}\binom{d+1-k}{d+1-k-m}B_{d+1-k-m}^{(-m)}S_k(b_{d+1}), \tag{3.21}$$

$$\begin{aligned}&a_{d+1}=-n,\qquad a_j=\frac{\nu+n+j-1}{d};\quad j=1,\dots,d;\\&b_{d+1}=0,\qquad b_j=\alpha_j-1;\quad j=1,\dots,d;\end{aligned} \tag{3.22}$$

$B_n^{(a)}$ *are the Bernoulli numbers* [26], $S_0(a_{d+1}) = 1$, *and for* $k > 0$, $S_k(a_{d+1})$ *are the symmetric polynomials*

$$S_k(a_{d+1}) = \sum a_{i_k} a_{i_{k-1}} \cdots a_{i_1}, \quad a_{i_1} < a_{i_2} < \cdots < a_{i_k},\ k_j = 1, 2, \ldots, d+1,\ j = 1, 2, \ldots, k. \tag{3.23}$$

Proof. Let us recall that an hypergeometric function

$$f(x) = {}_pF_q\left(\begin{matrix}(\sigma_p)\\(\delta_q)\end{matrix}; x\right)$$

is a solution of the following differential equation [31, p. 75]:

$$\left[\theta \prod_{j=1}^{q} (\theta + \delta_j - 1) - x \prod_{j=1}^{p} (\theta + \sigma_j)\right] y = 0, \quad \theta = x\frac{d}{dx}. \tag{3.24}$$

Using (3.16) and (3.24), we deduce that the Jacobi type polynomials verify the following differential equation:

$$\left[x \prod_{j=1}^{d+1} (\theta + a_j) - \prod_{j=1}^{d+1} (\theta + b_j)\right] y = 0, \tag{3.25}$$

where $y = P_n^{\nu}(\frac{(d+1)^{d+1}}{d^d} x; (\alpha_d))$, and a_j, b_j; $j = 1, \ldots, d+1$; are the real numbers given by (3.22).

On the other hand, we have [26, p. 25]

$$\prod_{j=1}^{p} (\theta + \sigma_j) = \sum_{m=0}^{p} c_{p,m} x^m D^m, \tag{3.26}$$

$$c_{p,m} = \sum_{k=0}^{p-m} \binom{p-k}{p-k-m} B_{p-k-m}^{(-m)} S_k(\sigma_p), \tag{3.27}$$

where B_n^a designate the Bernoulli numbers [26], $S_0(\sigma_p) = 1$, and for $k > 0$, $S_k(\sigma_p)$ are the symmetric polynomials given by (3.23).

Combining (3.25) and (3.26), we obtain

$$\begin{aligned}&\Big[\big(c_{d+1,d+1}x - c'_{d+1,d+1}\big)x^{d+1}D^{d+1} + \big(c_{d+1,d}x - c'_{d+1,d}\big)x^d D^d \\ &\quad + \sum_{m=1}^{d-1} \big(c_{d+1,m}x - c'_{d+1,m}\big)x^m D^m + \big(c_{d+1,0}x - c'_{d+1,0}\big)\Big] y = 0,\end{aligned} \tag{3.28}$$

where $c_{d+1,m}$ and $c'_{d+1,m}$; $m = 1, \ldots, d+1$; are the real numbers given by (3.20) and (3.21), respectively. Furthermore, from (3.27) and (3.22), it can be readily shown that

$$\begin{aligned}&c_{d+1,d+1} = c'_{d+1,d+1} = 1,\\ &c_{d+1,0} = \frac{-n(\nu+n)_d}{d^d}, \qquad c'_{d+1,0} = 0,\\ &c_{d+1,d} = \nu + \frac{d^2+2d-1}{2}, \qquad c'_{d+1,d} = \frac{d(d-1)}{2} + \sum_{j=1}^{d} \alpha_j.\end{aligned}$$

That, by virtue of (3.28), leads to (3.19). □

Remark 3.10. For $d = 1$, $\nu = \alpha + \beta + 1$ and $\alpha_1 = \beta + 1$, the differential equation (3.19) is reduced to the well-known differential equation satisfied by the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [25]

$$\big(1 - x^2\big)y'' + \big[\beta - \alpha - (\alpha + \beta + 2)x\big]y' - n(\alpha + \beta + 1 + n)y = 0, \quad y = P_n^{(\alpha,\beta)}(x). \tag{3.29}$$

Indeed, for $d=1$, the identity (3.19) provides the following:

$$\big[(x-1)xD^2+\big[(\nu+1)x-\alpha_1\big]D-n(\nu+n)\big]y=0, \quad y=P_n^{(\nu-\alpha_1,\alpha_1-1)}(2x-1). \tag{3.30}$$

By letting $\nu=\alpha+\beta+1$, $\alpha_1=\beta+1$, and replacing x by $\frac{x+1}{2}$ in this identity, we obtain (3.29).

3.2.4. d-Orthogonality and "classical" property

Theorem 3.11. *The Jacobi type polynomials defined by* (3.16) *are "classical" d-orthogonal polynomials.*

Proof. In order to derive the d-orthogonality of the Jacobi type polynomials defined by (3.16), we show that the polynomials $P_n(x)=\frac{n!}{(-1)^n(\nu)_n}P_n^{\nu}((d+1)^{d+1}x;(\alpha_d))$ verify a $(d+1)$-order recurrence relation of type (1.5) and we use Lemma 1.4.

According to (3.16), we have

$$P_n(x)=\sum_{k=0}^{n}\frac{(-n)_k(n+\nu)_{dk}}{k![\alpha_d]_k}x^k.$$

Therefore

$$\begin{aligned} xP_n(x)&=\sum_{k=1}^{n+1}\frac{(-n)_{k-1}(n+\nu)_{d(k-1)}}{(k-1)![\alpha_d]_{k-1}}x^k \\ &=\sum_{k=1}^{n+1}F_{d,\nu}(k)\frac{x^k}{k![\alpha_d]_k}, \end{aligned}\tag{3.31}$$

where $F_{d,\nu}(k)=(-n)_{k-1}(n+\nu)_{d(k-1)}k\pi(k-1)$; $1\leqslant k\leqslant n+1$; with $\pi(x)=\prod_{j=1}^{d}(x+\alpha_j)$.

Since, for $n\geqslant d$, the polynomial family $\{(x-n-1)_{d+1-r}(dx+n-d+\nu)_r\}_{0\leqslant r\leqslant d+1}$ is a basis of the subspace of polynomials of degree less or equal to $d+1$, then there exist $(d+2)$ complex numbers $\alpha'_{r,d}(n)$; $r=0,\dots,d+1$; such that

$$x\pi(x-1)=\sum_{r=0}^{d+1}\alpha'_{r,d}(n)(x-n-1)_{d+1-r}(dx+n-d+\nu)_r, \quad \forall x\in\mathbb{C}. \tag{3.32}$$

By letting $\alpha_{r,d}(n)=\frac{(-n)_d(n-d+\nu)_r}{(-n-r+d)_r(n-d+\nu)_d}\alpha'_{r,d}(n)$, we deduce that, for $k\in\mathbb{N}$,

$$k\pi(k-1)=\sum_{r=0}^{d+1}\frac{(-n-r+d)_r(n-d+\nu)_d}{(-n)_d(n-d+\nu)_r}\alpha_{r,d}(n)(k-n-1)_{d+1-r}(dk+n-d+\nu)_r. \tag{3.33}$$

For $1\leqslant k\leqslant d+1$, multiplying both sides of the equality (3.33) by $\frac{(n-d+\nu)_{dk}}{(k-n-1)_{d+1-k}}$ and using the following identities:

$$\begin{aligned} &(-n)_{k-1}=\frac{(-n)_d}{(-n-1+k)_{d+1-k}}, &&(-n+d-r)_k=\frac{(-n-1+k)_{d+1-r}(-n-r+d)_r}{(-n-1+k)_{d+1-k}}, \\ &(n+\nu)_{d(k-1)}=\frac{(n+\nu-d)_{dk}}{(n+\nu-d)_d}, &&(n-d+r+\nu)_{dk}=\frac{(n+\nu+d(k-1))_r(n-d+\nu)_{kd}}{(n-d+\nu)_r}, \end{aligned}$$

we obtain

$$k\pi(k-1)(-n)_{k-1}(n+\nu)_{d(k-1)}=\sum_{r=0}^{d+1}\alpha_{r,d}(n)(-n+d-r)_k(n-d+r+\nu)_{kd}. \tag{3.34}$$

For $d+1\leqslant k\leqslant n+1$, multiplying both sides of the equality (3.33) by $(-n+d)_{k-1-d}(n+1+\nu)_{d(k-1)-1}$ and using the following identities:

$$\begin{aligned}
&(-n)_{k-1}=(-n)_d(-n+d)_{k-d-1}, \qquad (n+\nu)_{d(k-1)}=(n+\nu)(n+1+\nu)_{d(k-1)-1},\\
&(n-d+r+\nu)_{dk}=(n-d+r+\nu)_{d+1-r}(n+1+\nu)_{d(k-1)-1}(dk+n-d+\nu)_r,\\
&(-n+d-r)_k=(-n+d-r)_r(-n+d)_{k-1-d}(-n-1+k)_{d+1-r},\\
&(n-d+\nu)_d(n+\nu)=(n-d+\nu)_r(n-d+r+\nu)_{d+1-r},
\end{aligned}\tag{3.35}$$

we obtain (3.34). Consequently (3.34) is valid for $1\leqslant k\leqslant n+1$. That means

$$F_{d,\nu}(k)=\sum_{r=0}^{d+1}\alpha_{r,d}(n)(-n+d-r)_k(n-d+r+\nu)_{kd}, \quad 1\leqslant k\leqslant n+1. \tag{3.36}$$

By substituting (3.36) in (3.31), we deduce

$$xP_n(x)=\sum_{k=1}^{n+1}\left[\sum_{r=0}^{d+1}\alpha_{r,d}(n)(-n+d-r)_k(n-d+r+\nu)_{kd}\right]\frac{x^k}{k![\alpha_d]_k}.$$

On the other hand, from (3.33), it is easy to verify that $\sum_{r=0}^{d+1}\alpha_{r,d}(n)=0$. Hence,

$$\begin{aligned}
xP_n(x)&=\sum_{k=0}^{n+1}\left[\sum_{r=0}^{d+1}\alpha_{r,d}(n)(-n+d-r)_k(n-d+r+\nu)_{kd}\right]\frac{x^k}{k![\alpha_d]_k}\\
&=\sum_{r=0}^{d+1}\alpha_{r,d}(n)\left[\sum_{k=0}^{n-d+r}\frac{(-n+d-r)_k(n-d+r+\nu)_{dk}}{k![\alpha_d]_k}x^k\right]\\
&=\sum_{r=0}^{d+1}\alpha_{r,d}(n)P_{n-d+r}(x).
\end{aligned}\tag{3.37}$$

Then the PS $\{P_n(x)\}_{n\geqslant 0}$ verifies a $(d+1)$-order recurrence relation. Now, we shall show that $\alpha_{0,d}(n)\alpha_{d+1,d}(n)\neq 0$; $n\geqslant d$.

Replacing successively in (3.32), x by $(n+1)$ and $(1-\frac{n+\nu}{d})$, we get

$$\alpha_{d+1,d}(n)=-\frac{(n+\nu)}{((d+1)n+\nu)_{d+1}}\pi(n), \tag{3.38}$$

$$\alpha_{0,d}(n)=-\frac{(-n)_d}{d(n+1-d+\nu)_{d-1}}\frac{\pi(-\frac{n+\nu}{d})}{(-\frac{(d+1)n+\nu}{d})_{d+1}}. \tag{3.39}$$

However $\pi(n)\neq 0$ and $\pi(-\frac{n+\nu}{d})\neq 0$, since $\alpha_j\neq 0,-1,-2,\ldots,$ and $\nu+d(1-\alpha_j)\neq 0,-1,-2,\ldots,$ for $j=1,\ldots,d$. Consequently $\alpha_{0,d}(n)\alpha_{d+1,d}(n)\neq 0$.

Then, according to Lemma 1.4, the Jacobi type PS is d-orthogonal. In order to verify that these polynomials are "classical," let us recall the identity [31, p. 107]

$$D\,{}_pF_q\left(\begin{matrix}(a_p)\\(b_q)\end{matrix};x\right)=\frac{\prod_{j=1}^p a_j}{\prod_{j=1}^q b_j}\,{}_pF_q\left(\begin{matrix}(a_p+1)\\(b_q+1)\end{matrix};x\right), \tag{3.40}$$

where D is the derivative operator $\frac{d}{dx}$.

From (3.40) and (3.16), we deduce

$$DP_{n+1}^{\nu}\big(x;(\alpha_d)\big)=\mathscr{T}_nP_n^{\nu+d+1}\big(x,(\alpha_d)+1\big),$$

where

$$\mathscr{T}_n=\frac{(\nu)_{n+1}}{(\nu+d+1)_n}\frac{\prod_{j=1}^d(\frac{\nu+n+j}{\alpha_j})}{(d+1)^{d+1}} \quad\text{and}\quad (\alpha_d)+1=\{\alpha_1+1,\ldots,\alpha_d+1\}.$$

So the sequence $\{DP_{n+1}^{\nu}(x;(\alpha_d))\}_{n\geqslant 0}$ is also d-orthogonal. The desired result follows from Definition 1.2. □

Remark 3.12. From Theorem 3.11 and Remark 3.6, we deduce that all the components of Humbert polynomials given by (3.3) are "classical" d-orthogonal polynomials. The classical property of the first component of the Humbert polynomials may be also deduced from a general result stated by Douak and Maroni [16, Corollary 5.4].

3.3. Particular case: $d=2$

3.3.1. Definition

The Jacobi type 2-orthogonal polynomials are defined by

$$P_n^{\nu}(x;\alpha_1,\alpha_2)=\frac{(-1)^n(\nu)_n}{n!}\,{}_3F_2\left(\begin{matrix}-n,\frac{n+\nu}{2},\frac{n+\nu+1}{2}\\ \alpha_1,\alpha_2\end{matrix};\frac{4x}{27}\right). \tag{3.41}$$

These polynomials have the following properties.

3.3.2. Generating function

Using (3.17), we deduce that the Jacobi type 2-OPS is generated by

$$(1+t)^{-\nu}\,{}_3F_2\left(\begin{matrix}\frac{\nu}{3},\frac{\nu+1}{3},\frac{\nu+2}{3}\\ \alpha_1,\alpha_2\end{matrix};\frac{xt}{(1+t)^3}\right)=\sum_{n\geqslant 0}P_n^{\nu}(x;\alpha_1,\alpha_2)t^n. \tag{3.42}$$

3.3.3. A third-order differential equation

For $d=2$, Theorem 3.9 is reduced to the following.

Corollary 3.13. *The Jacobi type 2-OPS defined by* (3.41) *satisfies the following differential equation*:

$$\begin{aligned}&\left\{(x-1)x^2D^3+\left[\left(\nu+\frac{7}{2}\right)x-(\alpha_1+\alpha_2+1)\right]xD^2\right.\\&\left.\quad+\left[\left(-\frac{3}{4}n^2-\frac{1+2\nu}{4}n+\frac{\nu(\nu+5)}{4}+\frac{3}{2}\right)x-\alpha_1\alpha_2\right]D-\frac{n(\nu+n)(\nu+n+1)}{4}\right\}y=0,\end{aligned} \tag{3.43}$$

where $y=P_n^{\nu}(\frac{27}{4}x;\alpha_1,\alpha_2)$ *and* $D=\frac{d}{dx}$.

3.3.4. A third-order recurrence relation

Proposition 3.14. *The Jacobi type 2-OPS defined by* (3.41) *verifies the following recurrence relation*:

$$\begin{aligned}xP_n^{\nu}(x;\alpha_1,\alpha_2)&=27\frac{(\nu+n-2)_2}{n(n-1)}\alpha_{0,2}(n)P_{n-2}^{\nu}(x;\alpha_1,\alpha_2)-27\frac{(\nu+n-1)}{n}\alpha_{1,2}(n)P_{n-1}^{\nu}(x;\alpha_1,\alpha_2)\\&\quad+27\alpha_{2,2}(n)P_n^{\nu}(x;\alpha_1,\alpha_2)-27\alpha_{3,2}(n)\frac{n+1}{\nu+n}P_{n+1}^{\nu}(x;\alpha_1,\alpha_2),\end{aligned} \tag{3.44}$$

$$P_0^{\nu}(x;\alpha_1,\alpha_2)=1,\qquad P_1^{\nu}(x;\alpha_1,\alpha_2)=\frac{(\nu)_3}{27\alpha_1\alpha_2}x-\nu,$$

$$P_2^{\nu}(x;\alpha_1,\alpha_2)=\frac{(\nu)_6}{27^2(\alpha_1)_2(\alpha_2)_2}x^2-\frac{(\nu)_4}{27\alpha_1\alpha_2}x+\frac{(\nu)_2}{2},$$

where

$$\alpha_{0,2}(n)=-\frac{(-n)_2(\alpha_1-\frac{n+\nu}{2})(\alpha_2-\frac{n+\nu}{2})}{2(n-1+\nu)(-\frac{3n+\nu}{2})_3}, \tag{3.45}$$

$$\alpha_{2,2}(n)=\frac{(n+1)(\alpha_1+n)(\alpha_2+n)}{(3n+\nu+1)_2}-\frac{n(\alpha_1+n-1)(\alpha_2+n-1)}{(3n+\nu-2)_2}, \tag{3.46}$$

$$\alpha_{3,2}(n)=-\frac{(n+\nu)(\alpha_1+n)(\alpha_2+n)}{(3n+\nu)_3}, \tag{3.47}$$

$$\alpha_{1,2}(n) = \frac{n}{2}\left[\frac{(n-2+\nu)(3n-3+\nu)(n+\nu-2\alpha_1)(n+\nu-2\alpha_2)}{(n-1+\nu)(3n+\nu)(3n+\nu-2)(3n+\nu-4)}\right. \tag{3.48}$$

$$\left. - \frac{(n+1+\nu-2\alpha_1)(n+1+\nu-2\alpha_2)}{(3n+\nu-1)(3n+\nu+1)}\right]. \tag{3.49}$$

Proof. Put $P_n(x) = \frac{n!}{(-1)^n(\nu)_n} P_n^\nu(27x; \alpha_1, \alpha_2)$. From (3.37), it follows

$$x P_n(x) = \sum_{r=0}^{3} \alpha_{r,2}(n) P_{n-2+r}(x),$$

where $\alpha_{0,2}(n)$ and $\alpha_{3,2}(n)$ are given by (3.38)–(3.39).

Consequently, the Jacobi type polynomials $\{P_n^\nu(x; \alpha_1, \alpha_2)\}_{n\geqslant 0}$ satisfy the recurrence relation (3.44).

To determinate $\alpha_{2,2}(n)$, we replace k by n and d by 2 in (3.33). That leads to (3.46).

Using (3.33), it is easy to verify that $\sum_{r=0}^{d+1} \alpha_{r,d}(n) = 0$. Hence, for $d = 2$, we have $\alpha_{1,2}(n) = -[\alpha_{0,2}(n) + \alpha_{2,2}(n) + \alpha_{3,2}(n)]$. □

From Proposition 3.14, we deduce the following.

Corollary 3.15. *The monic PS* $\{p_n(x)\}_{n\geqslant 0}$ *defined by*

$$p_n(x) = \frac{n!(\alpha_1)_n(\alpha_2)_n}{(\nu)_{3n}} P_n^\nu(27x; \alpha_1, \alpha_2),$$

fulfils the recurrence relation

$$\begin{cases} p_{n+1}(x) = (x-\beta_n)p_n(x) - \gamma_n^1 p_{n-1}(x) - \gamma_{n-1}^0 p_{n-2}(x), \quad n \geqslant 2, \\ p_0(x) = 1, \quad p_1(x) = x - \dfrac{\alpha_1\alpha_2}{(1+\nu)_2}, \quad p_2(x) = x^2 + 2\dfrac{(\alpha_1+1)(\alpha_2+1)}{(4+\nu)_2}x + \dfrac{(\alpha_1)_2(\alpha_2)_2}{(2+\nu)_4}, \end{cases} \tag{3.50}$$

where

$$\beta_n = \frac{(n+1)(n+\alpha_1)(n+\alpha_2)}{(3n+\nu+1)_2} - \frac{n(n-1+\alpha_1)(n-1+\alpha_2)}{(3n+\nu-2)_2},$$

$$\gamma_{n-1}^0 = \frac{(\alpha_1+n-2)_2(\alpha_2+n-2)_2 n(n-1)(n+\nu-2\alpha_1)(n+\nu-2\alpha_2)(n-2+\nu)}{(3n-6+\nu)(3n-5+\nu)(3n-4+\nu)^2(3n-3+\nu)(3n-2+\nu)^2(3n-1+\nu)(3n+\nu)},$$

$$\gamma_n^1 = \frac{n(n-1+\alpha_1)(n-1+\alpha_2)}{2(3n+\nu-2)(3n+\nu-1)}\left[\frac{(n-1+\nu)(n+1+\nu-2\alpha_1)(n+1+\nu-2\alpha_2)}{(3n-3+\nu)(3n-1+\nu)(3n+1+\nu)}\right.$$

$$\left. - \frac{(n-2+\nu)(n+\nu-2\alpha_1)(n+\nu-2\alpha_2)}{(3n+\nu)(3n-2+\nu)(3n-4+\nu)}\right].$$

As particular cases of polynomials satisfying the recurrence relation (3.50) we quote the components of Humbert polynomials defined by (3.13). For these components the recurrence relations given by (3.50) are reduced to the recurrence relations established by Douak and Maroni [16, pp. 89, 93, 97] if we put $\theta_n^0 = \frac{3\nu+n+2}{3\nu+n}$, $\lambda = \frac{3}{2}\nu$ and $\mu = \frac{3}{2}\nu - \frac{1}{2}$.

3.3.5. Example

For $\alpha_1 = \frac{\nu+1}{3}$, $\alpha_2 = \frac{\nu+2}{3}$ and $\nu = 3\theta$, the Jacobi type PS given by (3.41) is reduced to the PS $\{\frac{(\theta)_n}{n!} B_n(x; \theta, 3, 1)\}_{n\geqslant 0}$ defined by [13]

$$\left(1 - xt + \alpha t^2 + \gamma t^3\right)^{-\theta} = \sum_{n\geqslant 0} \frac{(\theta)_n}{n!} B_n(x; \theta, \alpha, \gamma) t^n. \tag{3.51}$$

Indeed, by letting $\alpha_1 = \frac{\nu+1}{3}$, $\alpha_2 = \frac{\nu+2}{3}$ and $\nu = 3\theta$, the identity (3.42) becomes

$$\left(1-(x-3)t+3t^2+t^3\right)^{-\theta}=\sum_{n\geqslant 0}P_n^{3\theta}\left(x;\theta+\frac{1}{2},\theta+\frac{2}{3}\right)t^n.$$

Replacing in this identity x by $(x-3)$, we obtain

$$\left(1-xt+3t^2+t^3\right)^{-\theta}=\sum_{n\geqslant 0}P_n^{3\theta}\left(x+3;\theta+\frac{1}{2},\theta+\frac{2}{3}\right)t^n.$$

That, by virtue of (3.51), leads to

$$P_n^{3\theta}\left(x+3;\theta+\frac{1}{2},\theta+\frac{2}{3}\right)=\frac{(\theta)_n}{n!}B_n(x;\theta,3,1). \tag{3.52}$$

The PS $\{B_n(x;\theta,\alpha,\gamma)\}_{n\geqslant 0}$ was first introduced by Boukhemis [13, Eq. (4.2.1)] as a PS satisfying a third-order recurrence relation and generalizing the solution of a characterization problem which consists to find all polynomials satisfying a third-order recurrence relation with coefficients independent of n. Later, Maroni [28] showed that these polynomials $B_n(x;\theta,\alpha,\gamma)$ are "classical" 2-orthogonal polynomials analogous to Jacobi ones.

Notice that, by letting $\alpha_1=\frac{\nu+1}{3}$, $\alpha_2=\frac{\nu+2}{3}$ and $\nu=3\theta$, the identities (3.14) and (3.43) are reduced, respectively, to the recurrence relation and the differential equation given by Boukhemis [13, Theorem 4.3.1 and Corollary 4.2.3] for the PS $\{B_n(x;\theta,\alpha,\gamma)\}_{n\geqslant 0}$.

4. A characterization theorem

Recall that the Humbert polynomials have the following generalized hypergeometric representation [32]:

$$h_{n,d+1}^{\nu}(x)=(d+1)^n(\nu)_n\frac{x^n}{n!}{}_{d+1}F_d\left(\begin{matrix}\Delta(d+1,-n)\\ \Delta(d,1-\nu-n)\end{matrix};\frac{d}{(xd)^{d+1}}\right). \tag{4.1}$$

Such polynomials are contained in a general class of generalized hypergeometric polynomials of the form

$$P_n^{\nu}\left(x;m,(a_p),(b_q)\right)=(\nu)_nx^n{}_{m+p}F_{m+q-1}\left(\begin{matrix}\Delta(m,-n),(a_p)\\ \Delta(m-1,1-\nu-n),(b_q)\end{matrix};\frac{m^m}{x^m(m-1)^{m-1}}\right), \tag{4.2}$$

where $m\geqslant 2$, $p,q\geqslant 0$ and $\nu,a_1,\dots,a_p,b_1,\dots,b_q$ are $p+q+2$ complex numbers.

Notice that, a PS having a generalized hypergeometric representation of the form (4.2) is $(m-1)$-symmetric. New d-OPSs may be introduced via a characterization theorem or via a decomposition of a d-symmetric d-OPS (cf. Lemma 1.5). It is then significant to consider the following problem.

P: *Find all d-OPS having generalized hypergeometric representation of type* (4.2).

Such a characterization takes into account the fact that PS which are obtainable from one another by a linear change of variable are assumed equivalent.

A solution of Problem ***P*** is given by the following.

Theorem 4.1. *The Humbert PS $\{h_{n,d+1}^{\nu}(x)\}_{n\geqslant 0}$ given by* (4.1) *is the only d-OPS having generalized hypergeometric representation of type* (4.2).

To prove this theorem we need the following three lemmas.

Lemma 4.2. *Let $\{P_n\}_{n\geqslant 0}$ be an $(m-1)$-symmetric PS and let Q be a polynomial of degree s satisfying*

$$Q(w_mx)=w_m^sQ(x), \tag{4.3}$$

where $w_m=\exp(\frac{2i\pi}{m})$. Then

$$Q(x)=\sum_{j=0}^{[\frac{s}{m}]}\beta_jP_{s-mj}(x).$$

Proof. Put $Q(x)=\sum_{k=0}^{s} q_k x^k$. From the identity (4.3), we deduce

$$\sum_{k=0}^{s} q_k w_m^k x^k = w_m^s \sum_{k=0}^{s} q_k x^k.$$

That, by identification, leads to $q_k=0$, for $k \neq s+mj$, $j\in\mathbb{N}$. Consequently

$$Q(x)=\sum_{j=0}^{[\frac{s}{m}]} q_{s-mj} x^{s-mj}.$$

Using the inversion formula related to the PS $\{P_n\}_{n\geqslant 0}$,

$$x^n=\sum_{i=0}^{[\frac{s}{m}]} c_i P_{n-mi}(x),$$

we obtain

$$Q(x)=\sum_{i=0}^{[\frac{s}{m}]} \beta_i P_{s-mi}(x),$$

where $\beta_i=\sum_{j=0}^{i} c_{i-j} q_{s-mj}$, $i=0,\dots,[\frac{s}{m}]$. □

Lemma 4.3. *Let $\{P_n\}_{n\geqslant 0}$ be an $(m-1)$-symmetric PS and d-OPS defined by*

$$P_n(x)=\sum_{k=0}^{[\frac{n}{m}]} \gamma_{n,m}(k) x^{n-mk}, \tag{4.4}$$

and satisfying the $(d+1)$-order recurrence relation

$$xP_n(x)=\sum_{k=0}^{d+1} \alpha_{k,d}(n) P_{n-d+k}(x), \tag{4.5}$$

where $\alpha_{d+1,d}(n)\alpha_{0,d}(n)\neq 0$, $n\geqslant d$, and by convention, $P_{-n}=0$, $n\geqslant 1$.

Then there exists a positive integer i_0 such that $d=i_0m-1$ and

$$\sum_{i=0}^{\min(k,i_0)} \alpha_{(i_0-i)m,d}(n)\gamma_{n+1-im,m}(k-i)=\gamma_{n,m}(k); \quad n\geqslant i_0m;\ k=0,\dots,\left[\frac{n}{m}\right]. \tag{4.6}$$

Proof. Since $\{P_n\}_{n\geqslant 0}$ is an $(m-1)$-symmetric PS, then $xP_n(x)$ is a polynomial of degree $(n+1)$ satisfying the identity (4.3). That by virtue of Lemma 4.2, leads to

$$xP_n(x)=\sum_{k=0}^{[\frac{n+1}{m}]} \beta_j P_{n+1-mj}(x).$$

According to the identity (4.5), we deduce that there exists a positive integer i_0 such that $d=i_0m-1$ and

$$xP_n(x)=\sum_{i=0}^{i_0} \alpha_{(i_0-i)m,d}(n) P_{n+1-im}(x).$$

Using the identity (4.4), we obtain

$$\sum_{k=0}^{[\frac{n}{m}]} \gamma_{n,m}(k) x^{n+1-mk} = \sum_{i=0}^{i_0} \sum_{k=0}^{[\frac{n+1}{m}]-i} \alpha_{(i_0-i)m,d}(n)\gamma_{n+1-im,m}(k) x^{n+1-(k+i)m}$$

$$= \sum_{i=0}^{i_0} \sum_{k=i}^{[\frac{n+1}{m}]} \alpha_{(i_0-i)m,d}(n)\gamma_{n+1-im,m}(k-i)x^{n+1-km}$$
$$= \sum_{k=0}^{[\frac{n+1}{m}]} \sum_{i=0}^{\min(k,i_0)} \alpha_{(s-i)m,d}(n)\gamma_{n+1-im,m}(k-i)x^{n+1-km}.$$

Then, by identification, we obtain (4.6). □

It is easy to show the following properties of the Pochhammer symbol.

Lemma 4.4. *Let $i \leqslant i_0 \leqslant k \leqslant [\frac{n}{m}]$, we have*

$$(-n-1+im)_{(k-i)m} = (-n-1+im)_{(i_0-i)m}(-n-1+i_0m)_{(k-i_0)m}, \tag{4.7}$$
$$(\nu)_{n+1-i-k(m-1)} = (\nu)_{n+1-i_0-k(m-1)}\big(\nu+n+1-i_0-k(m-1)\big)_{i_0-i}, \tag{4.8}$$
$$(-n)_{km} = (-n)_{i_0m-1}(-n-1+i_0m)_{(k-i_0)m}(-n-1+km), \tag{4.9}$$
$$(a)_{k-i} = (a)_{k-i_0}(a+k-i_0)_{i_0-i}. \tag{4.10}$$

Proof of Theorem 4.1. Put $P_n(x) = P_n^\nu(x; m, (a_p), (b_q))$, where $\{P_n^\nu(x; m, (a_p), (b_q))\}_{n\geqslant 0}$ is a PS of the form (4.2). Then we have

$$P_n(x) = \sum_{k=0}^{[\frac{n}{m}]} \gamma_{n,m}(k)x^{n-mk}, \tag{4.11}$$

where

$$\gamma_{n,m}(k) = \frac{(-1)^{k(m-1)}(-n)_{mk}(\nu)_{n-k(m-1)}}{k!}\frac{[a_p]_k}{[b_q]_k}. \tag{4.12}$$

To prove Theorem 4.1, it is sufficient to show that: if $\{P_n(x)\}_{n\geqslant 0}$ is d-orthogonal, then $d = m-1$ and $p = q = 0$, since that corresponds to Humbert polynomials and the converse follows from Ben Cheikh and Douak result [8].

According to Lemma 4.3, we deduce that there exist a positive integer i_0 and a sequence $\{\alpha_{(i_0-i)m,d}(n)\}_{0\leqslant i\leqslant i_0\leqslant[\frac{n}{m}]}$ such that $d = i_0m-1$, $\alpha_{0,d}(n)\alpha_{i_0m,d}(n) \neq 0$ and

$$\sum_{i=0}^{\min(k,i_0)} \alpha_{(i_0-i)m,d}(n)\gamma_{n+1-im,m}(k-i) = \gamma_{n,m}(k); \quad n \geqslant i_0m;\ k = 0,\dots,\left[\frac{n}{m}\right]. \tag{4.13}$$

Since $i_0 \in \{0,\dots,[\frac{n}{m}]\}$ and k is an arbitrary integer in $\{0,\dots,[\frac{n}{m}]\}$, we can assume that $k \geqslant i_0$. Substituting (4.12) in (4.13) and using the identities (4.7)–(4.10), we obtain

$$\begin{aligned}
&(-n)_{i_0m-1}(-n-1+km)\big(\nu+n+1-i_0-k(m-1)\big)_{i_0-1}\frac{[a_p+k-i_0]_{i_0}}{[b_q+k-i_0]_{i_0}} \\
&\quad= \sum_{i=0}^{i_0}(-1)^{i(m-1)}\alpha_{(i_0-i)m,d}(n)(-n-1+im)_{(i_0-i)m} \\
&\qquad\times \big(\nu+n+1-i_0-k(m-1)\big)_{i_0-i}k^{[i]}\frac{[a_p+k-i_0]_{i_0-i}}{[b_q+k-i_0]_{i_0-i}},
\end{aligned} \tag{4.14}$$

where $x^{[i]}$ denotes the falling factorial polynomials given by

$$x^{[0]} = 1 \quad \text{and} \quad x^{[i]} := i!\binom{x}{i} = x(x-1)\cdots(x-i+1), \quad i = 1,2,\dots.$$

Multiplying both sides of the equality (4.14) by $[b_q+k-i_0]_{i_0}$ and using (2.9) to obtain

$$Q(k) = R(k), \quad i_0 \leqslant k \leqslant \left[\frac{n}{m}\right], \tag{4.15}$$

where R and Q are the polynomials defined by

$$Q(x) = (-n)_{i_0 m-1}(-n-1+mx)\big(\nu+n+1-i_0-(m-1)x\big)_{i_0-1}[a_p+x-i_0]_{i_0},$$

$$R(x) = \sum_{i=0}^{i_0} \alpha_{(i_0-i)m,d}(n)(-n-1+im)_{(i_0-i)m}\big(\nu+n+1-i_0-(m-1)x\big)_{i_0-i}$$
$$\times (-1)^{i(m-1)}[a_p+x-i_0]_{i_0-i}[b_q+x-i]_i x^{[i]}.$$

Observing that: $\deg Q = i_0(p+1)$ and

$$\deg R \leqslant \max_{0\leqslant i\leqslant i_0}\big(i_0(p+1)+i(q-p)\big)$$
$$\leqslant \max\big(i_0(p+1), i_0(q+1)\big).$$

Then $\max(\deg R, \deg Q) \leqslant \max(i_0(p+1), i_0(q+1))$.

Choosing n such that $[\frac{n}{m}] - i_0 \geqslant \max(i_0(p+1), i_0(q+1))$. According to (4.15), we deduce that $Q = R$. In particular, we have

$$Q\left(\frac{\nu+n+1-i_0}{m-1}\right) = R\left(\frac{\nu+n+1-i_0}{m-1}\right).$$

That means,

$$(-1)^{i_0(m-1)}\alpha_{0,d}(n)\left[b_q - i_0 + \frac{\nu+n+1-i_0}{m-1}\right]_{i_0} = (0)_{i_0-1}\left(-n-1+m\frac{\nu+n+1-i_0}{m-1}\right)$$
$$\times (-n)_{i_0 m-1}\left[a_p - i_0 + \frac{\nu+n+1-i_0}{m-1}\right]_{i_0}. \tag{4.16}$$

Since, $\alpha_{0,d}(n) \neq 0$; $n \geqslant i_0 m$; and $[b_q - i_0 + \frac{\nu+n+1-i_0}{m-1}]_{i_0} \sim (\frac{n}{m-1})^{q i_0}$; $n \uparrow \infty$; then there exists a positive integer n_0 such that

$$(-1)^{i_0(m-1)}\alpha_{0,d}(n)\left[b_q - i_0 + \frac{\nu+n+1-i_0}{m-1}\right]_{i_0} \neq 0, \quad n \geqslant n_0.$$

That, by virtue of (4.16), leads to $i_0 = 1$. Hence $d = m-1$.

Now, we shall show that $p = q = 0$.

Replacing k by 0 in (4.13), we get $\alpha_{m,d}(n) = \frac{1}{\nu+n}$. Hence, the identity (4.14) becomes

$$\frac{\prod_{j=1}^{q}(b_j+k-1)}{\prod_{i=1}^{p}(a_i+k-1)} = \left(\frac{m-1}{\nu+n} - \frac{m}{n+1}\right)\frac{(-n-1)_m(-1)^{m-1}}{\alpha_{0,d}(n)}.$$

Taking into account the fact that, the right-hand side of this identity is independent of k, and $b_j \neq a_i$; $1 \leqslant j \leqslant q$, $1 \leqslant i \leqslant p$; we deduce that $p = q = 0$.

We conclude that, if $\{P_n\}_{n\geqslant 0}$ is a d-OPS, then $\{P_n\}_{n\geqslant 0}$ is the Humbert polynomials with $d = m-1$. □

Remark 4.5. The solution of Problem $\boldsymbol{P}$ does not provide us with new d-OPSs and Theorem 4.1 may be viewed as a characterization theorem for the Humbert polynomials.

Two particular cases are worthy to note.

Corollary 4.6. *The only OPS of type* (4.2) *are the Gegenbauer polynomials.*

Corollary 4.7. *The only* 2*-OPS of the form* (4.2) *are the Pincherle type polynomials given by*

$$P_n^{\nu}(x) = x^n {}_3F_2\left(\begin{matrix}\frac{-n}{3}, \frac{-n+1}{3}, \frac{-n+2}{3} \\ \frac{1-\nu-n}{2}, \frac{2-\nu-n}{2}\end{matrix}; \frac{1}{x^3}\right). \tag{4.17}$$

5. Concluding remarks

Remark 5.1. In this paper, we extend the definition (3.3) to introduce a class of generalized hypergeometric polynomials containing the components of the Humbert polynomials. Such polynomials have properties analogous to those of Jacobi. But this class, for $d \geqslant 2$, does not contain the Humbert polynomials as it does for $d = 1$. The question of the existence of an extension of (3.3) satisfying in addition this last condition remains open.

Remark 5.2. Douak and Maroni [16, Corollary 5.4] showed that the first component of "classical" d-symmetric d-OPS is also "classical" d-OPS. The question here is what is about the other components? May be all the components are "classical" d-orthogonal. This suggestion is based on the following:

- The case $d = 1$: The components of the Hermite polynomials are Laguerre polynomials and the components of Gegenbauer polynomials are Jacobi polynomials.
- The Gould–Hopper polynomials: Ben Cheikh and Douak [9] showed that the components of Gould–Hopper polynomials are the Laguerre type polynomials which are "classical" d-orthogonal.
- The Humbert polynomials: see Remark 3.12.

Remark 5.3. The property of the non-negativity of the obtained weight functions in this paper was mentioned for some special cases, when $d = 2$, see Remarks 2.6, 2.9, 2.12 and 3.5. This question for general case, i.e. for weight functions given by Theorems 2.4 and 3.3, remains unsolved.

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