Delannoy numbers and a combinatorial proof of the orthogonality of the Jacobi polynomials with natural number parameters

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Delannoy numbers

$$d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1}$$

$d_{i,j} :=$

$i \backslash j$	0	1	2	3	4
0	1	1	1	1	1
1	1	3	5	7	9
2	1	5	13	25	41
3	1	7	25	63	129
4	1	9	41	129	321

They count the number of lattice paths from $(0,0)$ to (m,n) using only steps $(1,0)$, $(0,1)$, and $(1,1)$.

$\Rightarrow d_{n,n} = \sum_{j=0}^{n} \binom{n}{j}\binom{n+j}{j}$.

(Defined by Henri Delannoy (1895), Sulanke has ≥ 29 interpretations.)

A mysterious relation with the Legendre polynomials

Good (1958), Lawden (1952), Moser and Zayachkowski (1963) observed that

$$d_{n,n} = P_n(3),$$

where $P_n(x)$ is the n-th Legendre polynomial.

There has been a consensus that this link is not very relevant.

Banderier and Schwer (2004): "there is no "natural" correspondence between Legendre polynomials and these lattice paths."

Sulanke (2003): "the definition of Legendre polynomials does not appear to foster any combinatorial interpretation leading to enumeration".

Jacobi and Legendre polynomials

Usual definition of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta} \\ & \quad \frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right). \end{aligned}$$

$\alpha, \beta > -1$ "for integrability purposes", $\alpha = \beta = 0$ gives Legendre.

The formula below extends to all $\alpha, \beta \in \mathbb{C}$ (see Szegő (4.21.2)):

$$P_n^{(\alpha,\beta)}(x) = \sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{n-j}\left(\frac{x-1}{2}\right)^j.$$

Substitute $\alpha = \beta = 0$:

$$P_n^{(0,0)}(x) = \sum_j \binom{n+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j$$

is the n-th Legendre polynomial.

Properties of Jacobi polynomials

For $\alpha, \beta > -1$ the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ form an orthogonal basis with respect to the inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(x) \cdot g(x) \cdot (1-x)^{\alpha}(1+x)^{\beta}\, dx.$$

"Swapping rule:"

$$(-1)^n P_n^{(\alpha,\beta)}(-x) = P_n^{(\beta,\alpha)}(x),$$

Asymmetric Delannoy numbers

$\widetilde{d}_{m,n} :=$

$m \backslash n$	0	1	2	3	4
0	1	2	4	8	16
1	1	3	8	20	48
2	1	4	13	38	104
3	1	5	19	63	192
4	1	6	26	96	321

$\widetilde{d}_{m,n}$ is the number of lattice paths from $(0,0)$ to $(m, n+1)$ having steps $(x,y) \in \mathbb{N} \times \mathbb{P}$.

(Variant of A049600 in the On-Line Encyclopedia of Integer Sequences.)

Lemma 1 *The asymmetric Delannoy numbers satisfy*

$$\widetilde{d}_{m,n} = \sum_{j=0}^{n} \binom{n}{j} \binom{m+j}{j}.$$

Proof: We are enumerating sequences $(0,0) = (x_0, y_0), (x_1, y_1), \ldots, (x_j, y_j), (x_{j+1}, y_{j+1}) = (m, n+1)$, where $0 \le j \le n$, $0 = x_0 \le x_1 \le \cdots \le x_j \le x_{j+1} = m$, and $0 = y_0 < y_1 < \cdots < y_j < y_{j+1} = n+1$. For a given j there are $\binom{m+j}{j}$ ways to choose $0 = x_0 \le x_1 \le \cdots \le x_j \le x_{j+1} = m$ and $\binom{n}{j}$ ways to choose $0 = y_0 < y_1 < \cdots < y_j < y_{j+1} = n+1$. ◇

Since

$$P_n^{(0,\beta)}(x) = \sum_j \binom{n+\beta+j}{j} \binom{n}{j} \left(\frac{x-1}{2}\right)^j,$$

we get

$$\widetilde{d}_{n+\beta,n} = P_n^{(0,\beta)}(3) \quad \text{for } m \ge n$$

because $\dfrac{3-1}{2} = 1$.

Shifted Jacobi and Legendre polynomials

Shifted Legendre polynomials appear even in Abramowitz-Stegun:

$$\widetilde{P}_n(x) := P_n(2x-1).$$

Shifted Jacobi polynomials seem to be less widely used:

$$\widetilde{P}_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(2x-1).$$

Well-known:

$$\begin{aligned}\widetilde{P}_n(x) &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{n}x^k \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}\binom{2k}{k}x^k.\end{aligned}$$

Generalization for shifted Jacobi polynomials ($\alpha \in \mathbb{N}$, $\beta \in \mathbb{C}$):

$$(x-1)^{\alpha}\widetilde{P}_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n+\alpha}(-1)^{n+\alpha-k}x^k\binom{n+\alpha}{k}\binom{n+\beta+k}{n}.$$

$$\Rightarrow \widetilde{P}_n^{(0,\beta)}(x) = \sum_{k=0}^{n}(-1)^{n-k}x^k\binom{n}{k}\binom{n+\beta+k}{n}.$$

Weighted Delannoy numbers

Let u, v, w be commuting variables. We define the *weighted Delannoy numbers* $d_{m,n}^{u,v,w}$ as the total weight of all Delannoy paths from $(0,0)$ to (m,n), where each step $(0,1)$ has weight u, each step $(1,0)$ has weight v, and each step $(1,1)$ has weight w. The weight of a lattice path is the product of the weights of its steps.

Easy to show:

$$d_{n,n}^{u,v,w} = \sum_{k=0}^{n} \binom{2n-k}{k} \binom{2n-2k}{n-k} u^{n-k} v^{n-k} w^k.$$

Since

$$d_{n,n}^{u,v,w} = (-w)^n d_{n,n}^{u,-v/w,-1} = (-w)^n d_{n,n}^{1,-uv/w,-1}$$

we have

$$d_{n,n}^{1,-uv/w,-1} = \sum_{k=0}^{n} \binom{2n-k}{k} \binom{2n-2k}{n-k} \left(-\frac{uv}{w}\right)^{n-k} (-1)^k.$$

$$d_{n,n}^{u,v,w} = (-w)^n \widetilde{P}_n\left(-\frac{uv}{w}\right).$$

Now

$$d_{n,n} = d_{n,n}^{1,1,1} = (-1)^n \widetilde{P}_n(-1) = (-1)^n P_n(-3) = P_n(3)$$

since $(-1)^n P_n(-x) = P_n(x)$.

Generalization to shifted Jacobi polynomials

$$d_{m,n}^{u,v,w} = \sum_{k=0}^{n} \binom{m+n-k}{k} \binom{m+n-2k}{n-k} u^{m-k} v^{n-k} w^{k}.$$

$$d_{n+\beta,n}^{u,v,w} = u^{\beta}(-w)^{n} \widetilde{P}_{n}^{(0,\beta)} \left(-\frac{uv}{w}\right).$$

Here $\beta \in \mathbb{Z}$ is any integer satisfying $\beta \geq -n$.

$$d_{n+\beta,n} = (-1)^{n} \widetilde{P}_{n}^{(0,\beta)}(-1) = (-1)^{n} P_{n}^{(0,\beta)}(-3).$$

Using the "swapping rule"

$$(-1)^{n} P_{n}^{(\alpha,\beta)}(-x) = P_{n}^{(\beta,\alpha)}(x),$$

we get

$$d_{n+\beta,n} = P_{n}^{(\beta,0)}(3).$$

"Swapped" variant of the formula for weighted Delannoy numbers:

$$d_{n+\beta,n}^{u,v,w} = u^{\beta} w^{n} \widetilde{P}_{n}^{(\beta,0)} \left(\frac{uv}{w} + 1\right).$$

Many arrays, same diagonal

$$d_{n,n} = d_{n,n}^{r,2/r,-1} \quad \text{for all } r \in \mathbb{R} \setminus \{0\},$$

and

$$d_{n,n} = d_{n,n}^{r,1/r,1} \quad \text{for all } r \in \mathbb{R} \setminus \{0\}.$$

Lattice path model for the shifted Legendre and Jacobi polynomials

$$\widetilde{P}_n(x) = d_{n,n}^{1,x-1,1} = d_{n,n}^{1,x,-1}$$

$$\widetilde{P}_n^{(0,\beta)}(x) = d_{n+\beta,n}^{1,x,-1}$$

Fact: The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ form an orthogonal basis with respect to the inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(x) \cdot g(x) \cdot (1-x)^{\alpha}(1+x)^{\beta}\, dx.$$

Goal: to provide a combinatorial, non-inductive proof of this fact for all $\alpha, \beta \in \mathbb{N}$

A linear substitution gives the following equivalent form.

The shifted Jacobi polynomials $\widetilde{P}_n^{(\alpha,\beta)}(x)$ form an orthogonal basis with respect to the inner product

$$\langle f, g \rangle := \int_{0}^{1} f(x) \cdot g(x) \cdot (1-x)^{\alpha} x^{\beta}\, dx.$$

The case $\alpha = 0$

Assume $m < n$.

$$\begin{aligned} & (n+m+\beta+1)! \int_0^1 x^{m+\beta} \widetilde{P}_n^{(0,\beta)}(x)\, dx \\ = \quad & \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+\beta+k}{n} \frac{(n+m+\beta+1)!}{m+\beta+k+1} \end{aligned}$$

total weight of all pairs (L, σ) where L is a Delannoy path from $(0,0)$ to $(n+\beta, n)$ and σ is a bijection $\{r, a_1, \ldots, a_{n+\beta}, b_1, \ldots, b_m\} \to \{1, \ldots, m+n+\beta+1\}$, subject to:

(i) $\sigma(r) < \sigma(a_i)$ holds for all i such that there is an east step in L from $(i-1, y)$ to (i, y) for some y;

(ii) $\sigma(r) < \sigma(b_j)$ holds for $j = 1, 2, \ldots, m$.

Diagonal steps contribute a factor of (-1), all others contribute 1.

Cancelling terms

Cancel the diagonal steps with the $((1,0),(0,1))$ sequences, when possible. You will be left with pairs of lattice paths and permutations such that

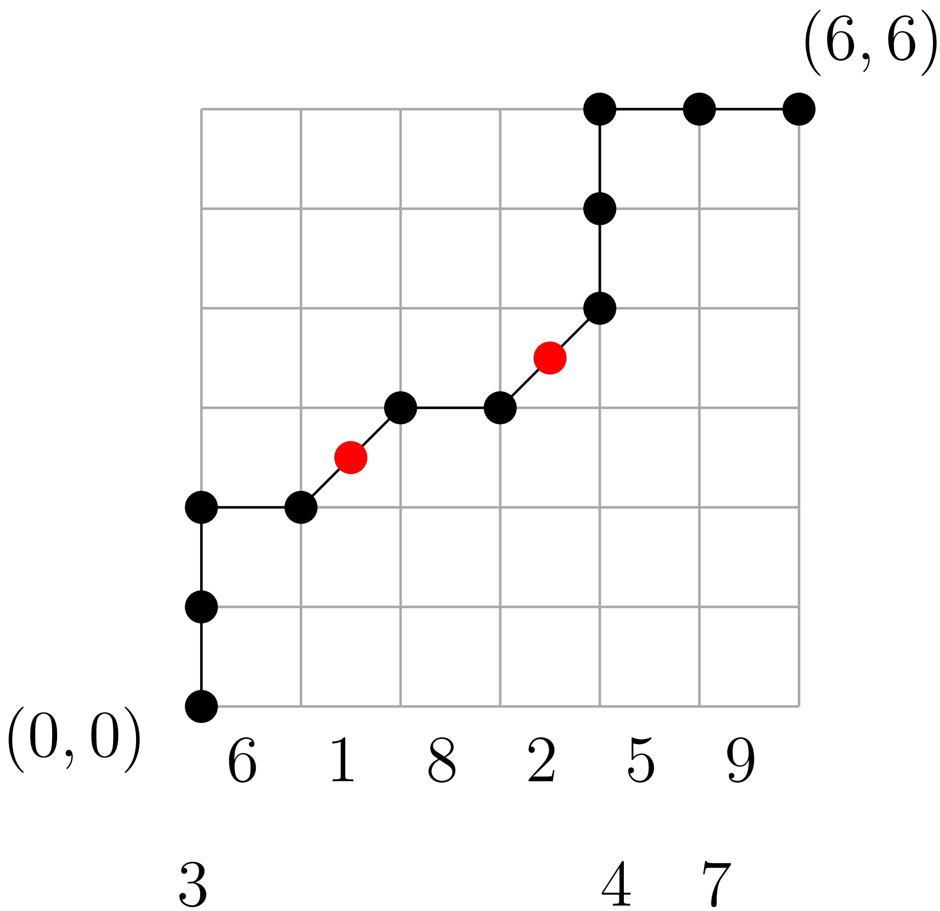
(a) $((1,0),(0,1))$ is forbidden;

(b) $\sigma(r) > \sigma(a_i)$ holds for all i such that there is a northeast east step in L from $(i-1,y)$ to $(i,y+1)$ for some y.

(b) makes $\sigma(r)$ unique, (a) makes the lattice path depend on the position of the diagonal steps only ($\sim$ "rook placements").

Example

$\alpha = 0$, $n = 6$, $m = 2$.



Connection to the orthogonality of Laguerre polynomials

We obtained

$$(n+m+\beta+1)! \cdot \int_0^1 x^{m+\beta} \cdot \widetilde{P}_n^{(0,\beta)}(x)\ dx$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n+\beta}{k} \binom{n}{k} \cdot k!(n+m+\beta-k)!$$

The right hand side is

$$\int_0^\infty x^m l_n^{(\beta)}(x) x^\beta e^{-x}\ dx \quad \text{for all } m, n \in \mathbb{N}.$$

Here

$$l_n^{(\beta)}(x) := \sum_{k=0}^{n} (-1)^k \binom{n+\beta}{k} \binom{n}{k} k! x^{n-k}$$

is the n-th *generalized Laguerre polynomial* associated to the rectangular board $[n+\beta] \times [n]$.

Rook polynomials

Board: $B \subseteq [n] \times [n]$. $S \subseteq B$ *compatible* if no two elements of S agree in either coordinate. The *rook polynomial of* B is

$$r_B(x) := \sum_{k=0}^{n} (-1)^k r_k x^{n-k}$$

where r_k is the number of compatible k-subsets of B. Let $\mathcal{L}$ be the linear functional defined by $\mathcal{L}(x^n) := n!$. Then

$$\mathcal{L}(p(x)) = \int_0^\infty e^{-x} p(x)\, dx$$

and the number of permutations π of $[n] \times [n]$ such that no $(i, \pi(i))$ belongs to B is $\mathcal{L}(r_B(x))$.

The rook polynomial of $[n] \times [n]$ is the *Laguerre polynomial*

$$l_n(x) := \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 k! x^{n-k} \tag{1}$$

$$l_n(x) = (-1)^n n! L_n(x).$$

Laguerre polynomials form an orthogonal basis:

$$\mathcal{L}(l_m(x) l_n(x)) = \delta_{m,n} n!$$

Just for completeness sake

The right hand side is $(m+\beta)!$ times

$$
\begin{aligned}
p(m) :=& \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+\beta)_k (n+m+\beta-k)_{n-k} \\
=& (-1)^n \sum_{k=0}^{n} \binom{n}{k} (n+\beta)_k (-m-\beta-1)_{n-k}.
\end{aligned}
$$

The number $(-1)^n p(-m)$ is then the number of ways to select a k-element subset of an n-element set and injectively color its elements using $n+\beta$ colors, then color the remaining $n-k$ elements injectively, using a disjoint set of $m-\beta-1$ colors. Thus

$$
(-1)^n p(-m) = \binom{n+m-1}{n}
$$

$$
p(m) = (-1)^n \binom{n-m-1}{n}.
$$

The case $\alpha > 0$

$$(x-1)^\alpha \widetilde{P}_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i x^{\alpha-i} \widetilde{P}_n^{(0,\alpha+\beta-i)}(x)$$

since both sides are the total weight of all Delannoy paths from $(0,0)$ to $(n+\alpha+\beta, n+\alpha)$ subject to the restriction that none of the first α steps is an east step.

As a consequence

$$\int_0^1 x^m \cdot \widetilde{P}_n^{(\alpha,\beta)}(x) \cdot (1-x)^\alpha x^\beta \; dx$$

$$= (-1)^\alpha \int_0^1 x^{m+\beta} \cdot \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i x^{\alpha-i} \widetilde{P}_n^{(0,\alpha+\beta-i)}(x) \; dx$$

$$= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^{\alpha+i} \int_0^1 x^{m+(\alpha+\beta-i)} \cdot \widetilde{P}_n^{(0,\alpha+\beta-i)}(x) \; dx.$$

$\widetilde{P}_n^{(0,\beta)}(x)$ **with negative integer** β

For $\beta \in \mathbb{N}$ and $n \geq \beta$ we have

$$\widetilde{P}_n^{(0,-\beta)}(x) = x^\beta \widetilde{P}_{n-\beta}(x).$$

Reason:

$$\widetilde{P}_n^{(0,\beta)}(x) = x^n d_{n+\beta,n}^{1,1,-1/x},$$

and we may swap the horizontal and vertical axis.

$$\widetilde{P}_0^{(0,-6)}(x) = 1 \qquad \widetilde{P}_1^{(0,-6)}(x) = 5 - 4x$$
$$\widetilde{P}_2^{(0,-6)}(x) = 3x^2 - 12x + 10 \qquad \widetilde{P}_3^{(0,-6)}(x) = 3x^2 - 12x + 10$$
$$\widetilde{P}_4^{(0,-6)}(x) = 5 - 4x \qquad \widetilde{P}_5^{(0,-6)}(x) = 1$$
$$\widetilde{P}_6^{(0,-6)}(x) = x^6$$

transformed Jacobi polynomials $\widehat{P}_n^{(\alpha,\beta)}(x)$*:*

$$\widehat{P}_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(2x+1).$$

$$\widehat{P}_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \binom{n+\alpha+\beta+j}{j} \binom{n+\alpha}{n-j} x^j.$$

Claim: For $\beta \in \mathbb{N}$ and $0 \leq n \leq \beta - 1$ we have

$$\widehat{P}_n^{(0,-\beta)}(x) = \widehat{P}_{\beta-1-n}^{(0,-\beta)}(x).$$

A finite orthogonal polynomial sequence

Let $\beta \geq 2$ be any positive integer and let $\mathcal{L}$ be the linear functional defined defined on the vector space $\{p(x) \in \mathbb{C}[x] \,:\, \deg(p) \leq (\beta-2)/2\}$ by

$$\mathcal{L}(x^k) = k! \cdot (\beta - 2 - k)! \quad \text{for } 0 \leq k \leq \beta - 2.$$

Then the transformed Jacobi polynomials $\{\widehat{P}_n^{(0,-\beta)}(x) \,:\, 0 \leq n \leq (\beta-2)/2\}$ form an orthogonal basis in the with respect to inner product $\langle f, g \rangle := \mathcal{L}(f \cdot g)$. For odd β we may extend $\mathcal{L}$ and the induced inner product to polynomials of degree at most $(\beta-1)/2$ by making $\mathcal{L}(x^{\beta-1})$ large enough to make the determinant of the $(\beta+1)/2 \times (\beta+1)/2$ matrix

$$\begin{pmatrix} \mathcal{L}(x^0) & \mathcal{L}(x^1) & \cdots & \mathcal{L}(x^{(\beta-1)/2}) \\ \mathcal{L}(x^1) & \mathcal{L}(x^2) & \cdots & \mathcal{L}(x^{(\beta-1)/2+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}(x^{(\beta-1)/2}) & \mathcal{L}(x^{(\beta-1)/2+1}) & \cdots & \mathcal{L}(x^{\beta-1}) \end{pmatrix}$$

positive. The polynomial $\widehat{P}_{(\beta-1)/2}^{(0,-\beta)}(x)$ may then be added to the orthogonal basis.

Elements of the proof

For $0 \leq k \leq \beta - 2$ we have:

$$\mathcal{L}(x^k) = (\beta - 1)! B(k + 1, \beta - 1 - k).$$

Here $B(z, w)$ is the *beta function*

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)}.$$

$$\mathcal{L}(x^k) = (\beta - 1)! \int_0^1 \left(\frac{t}{1 - t}\right)^k (1 - t)^{\beta - 2}\ dt.$$

$$\langle f, g \rangle = (\beta - 1)! \int_0^1 f\left(\frac{t}{1 - t}\right) \cdot g\left(\frac{t}{1 - t}\right) \cdot (1 - t)^{\beta - 2}\ dt$$

Thus we have an inner product for polynomials of degree at most $(\beta - 1)/2$.

Orthogonality:

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{m+j}{m}\binom{\beta-2-m-j}{n-m-1}=0.$$

Total weight of all (X, A, B) where

(i) $X \subseteq \{1, 2, \ldots, n\}$;

(ii) $A = \{a_1, \ldots, a_m\}$ is an m-element multiset such that each a_i belongs to $X \cup \{0\}$;

(iii) $B = \{b_1, \ldots, b_{n-m-1}\}$ is an $(n-m-1)$-element multiset such that each b_j belongs to $\{1, \ldots, \beta - n\} \setminus X$.

The weight of (X, A, B) is $(-1)^{|X|}$. Since $|A| + |B| = n - 1$, there is $c \in \{1, \ldots, n\}$ that does not appear in A, nor in B. For each $X \subset \{1, \ldots, n\} \setminus \{c\}$, the weight of (X, A, B) and of $(X \cup \{c\}, A, B)$ cancel.

Extending to degree $(\beta - 1)/2$ for odd β:

Only need to make sure entire matrix has positive determinant, all other principal minors have. The determinant is a linear function of $\mathcal{L}(x^\beta)$ whose coefficient is positive.

Weighted Schröder numbers

Schröder path from $(0,0)$ *to* (n,n)*:* a Delannoy path not going above the line $y = x$.

weighted Schröder numbers $s_n^{u,v,w}$*:* the total weight of all Schröder paths from $(0,0)$ to (n,n), where each east step $(0,1)$ has weight u, each north step has weight v, and each northeast step has weight w.

Schröder polynomials: $S_n(x) := s_n^{1,x,-1}$.

$$S_n(x) = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{j+1} \binom{2j}{j} \binom{n+j}{n-j} x^j \quad \text{for } n \geq 1.$$

$$S_n(x) = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j+1} \binom{n+j}{n} x^j$$

For $n \geq 1$ we also have

$$(x-1)\widetilde{P}_n^{(1,-1)}(x) = \sum_{k=0}^{n+1} (-1)^{n+1-k} x^k \binom{n+1}{k} \binom{n-1+k}{n}$$

Therefore,

$$S_n(x) = \frac{x-1}{(n+1)x} \widetilde{P}_n^{(1,-1)}(x).$$

Facts about $s_n^{u,v,w}$ and $S_n(x)$

$$s_n^{u,v,w} = (-w)^n S_n\left(-\frac{uv}{w}\right)$$

$$s_n^{u,v,w} = \frac{(-w)^n}{n+1}\left(1+\frac{w}{uv}\right)\widetilde{P}_n^{(1,-1)}\left(-\frac{uv}{w}\right).$$

$$s_n := s_n^{1,1,1} = \frac{(-1)^n 2}{n+1}\widetilde{P}_n^{(1,-1)}(-1) = \frac{(-1)^n 2}{n+1}P_n^{(1,-1)}(-3)$$

The "swapping rule" yields

$$s_n = \frac{2}{n+1}\cdot P_n^{(-1,1)}(3) \quad \text{for } n \geq 1.$$

$$d_{n,n}^{u,v,w} = 2uv\sum_{k=0}^{n-1} d_{k,k}^{u,v,w} s_{n-k-1}^{u,v,w} + w d_{n-1,n-1}^{u,v,w}.$$

$$\widetilde{P}_n(x) = 2x\sum_{k=0}^{n-1}\widetilde{P}_k(x)S_{n-k-1}(x) - \widetilde{P}_{n-1}(x).$$

$$\widetilde{P}_n(x) = 2\sum_{k=0}^{n-2}\widetilde{P}_k(x)\frac{x-1}{n-k}\widetilde{P}_{n-k-1}^{(1,-1)}(x) + (2x-1)\widetilde{P}_{n-1}(x) \quad \text{and}$$

$$P_n(x) = \sum_{k=0}^{n-2}P_k(x)\frac{x-1}{n-k}P_{n-k-1}^{(1,-1)}(x) + xP_{n-1}(x) \quad \text{for } n \geq 1.$$

A formula for repeated antiderivatives of the shifted Legendre polynomials

$$S_n(x) = \frac{1}{x}\int_0^x \widetilde{P}_{n-1}(t)\ dt \quad \text{holds for } n \geq 1.$$

Let n and α be positive integers. Applying the antiderivative operator

$$f(x) \mapsto \int_0^x f(t)\ dt$$

to $\widetilde{P}_n(x)$ exactly α times yields the polynomial $\frac{1}{(n+\alpha)_\alpha}(x-1)^\alpha \widetilde{P}_n^{(\alpha,-\alpha)}(x)$.

This follows from

$$\frac{d}{dx}\frac{(x-1)^\alpha \widetilde{P}_n^{(\alpha,-\alpha)}(x)}{(n+\alpha)_\alpha} = \frac{(x-1)^{\alpha-1}\widetilde{P}_n^{(\alpha-1,-(\alpha-1))}(x)}{(n+\alpha-1)_{\alpha-1}}$$

for $\alpha \geq 1$.

Favard's theorem

Favard's theorem states that a sequence of monic polynomials $\{p_n(x)\}_{n\geq 0}$ is an orthogonal polynomial sequence, if and only if it satisfies

$$p_n(x) = (x - c_n)p_{n-1}(x) - \lambda_n p_{n-2}(x) \quad n = 1, 2, 3, \ldots$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$, the numbers c_n and λ_n are constants, $\lambda_n \neq 0$ for $n \geq 2$, and λ_1 is arbitrary. The original proof provides only a recursive description of $\mathcal{L}$. Viennot gave a combinatorial proof of Favard's theorem, upon which he has built a general combinatorial theory of orthogonal polynomials. In his theory, the values $\mathcal{L}(x^n)$ are explicitly given as sums of weighted *Motzkin paths*.

Two notes of Favard's theorem and Viennot's model

The polynomials $\{S_n(x)\}_{n\geq 0}$ *almost* form an orthogonal polynomial sequence.

$$p_n(x) := \frac{1}{\binom{2n}{n}} \frac{x-1}{x} \widetilde{P}_n^{(1,-1)}(x)$$

$$p_n(x) = \left(x - \frac{1}{2}\right) p_{n-1}(x) - \frac{n(n-2)}{4(2n-1)(2n-3)} p_{n-2}(x)$$

for $n \geq 2$. Substituting $n = 2$ yields $\lambda_2 = 0$.

The monic variant of the Legendre polynomials is

$$p_n(x) := \frac{2^n P_n(x)}{\binom{2n}{n}}.$$

Favard's recursion formula takes the form

$$p_n(x) = x p_{n-1} x - \frac{(n-1)^2}{(2n-1)(2n-3)} p_{n-2}(x).$$

Challenge: Consider weighted Motzkin paths from $(0,0)$ to $(n,0)$. The horizontal steps have zero weight, the northeast steps $(1,1)$ have weight 1, the southeast steps $(1,-1)$ have weight $k^2/(4k^2-1)$ if they start at a point whose second coordinate k. Using Viennot's model, the total weight if these paths should be $1/(n+1)$ for all even $n \in \mathbb{N}$.

Connection to Riordan arrays

A *Riordan array* is a pair $(d(t), h(t))$ of formal power series in the variable t. These function define the triangle $d_{n,k} = [t^n]d(t)(th(t))^k$.

The weighted Delannoy number $d_{m,n}^{u,v,w}$ is the coefficient of t^n in $(u + wt)^m/(1 - vt)^{m+1}$. An immediate consequence of this observation is that the n-th row k-th column entry in the Riordan array $(1/(1-vt), t(u+wt)/(1-vt))$ is $d_{k,n-k}^{u,v,w}$. The numbers $d_{m,n}^{1,2,-1}$ appear as entry A1016195 in Sloane [16], listing the entries of the Riordan array $(1/(1 - 2t), t(1 - t)/(1 - 2t))$. Our results should allow to write summation formulas for Jacobi polynomials using the theory of Riordan arrays.

References

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