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A note on generalized k-Horadam sequence

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ABSTRACT

In this paper, we define generalized k-Horadam sequence $\{H_{k,n}\}_{n\in\mathbb{N}}$. After that, we study the properties of the generalized k-Horadam sequence and prove some of these properties by means of determinant. Also, we obtain a generating function for the generalized k-Horadam sequence.

1. Introduction

There are so many studies in the literature that concern about the special second order sequences such as generalized k-Fibonacci and k-Lucas, k-Fibonacci, k-Lucas, Generalized Fibonacci, Horadam, Fibonacci, Lucas, Pell, Jacobsthal and Jacobsthal–Lucas sequences (see, for instance, [1–13]). For rich applications of these numbers in science and nature, one can see the citations in [14–20]. For instance, the ratio of two consecutive Fibonacci numbers converges to the Golden section $\alpha = \frac{1+\sqrt{5}}{2}$. The applications of the Golden ratio appear in many research areas, particularly in Physics, Engineering, Architecture, Nature and Art. Physicists Naschie and Marek-Crnjac gave some examples of the Golden ratio in Theoretical Physics and Physics of High Energy Particles.

In this paper, we define a generalization $\{H_{k,n}\}_{n\in\mathbb{N}}$ of the special second order sequences such as generalized k-Fibonacci and k-Lucas, k-Fibonacci, k-Lucas, Horadam, Fibonacci, Lucas, Pell, Jacobsthal and Jacobsthal–Lucas sequences. For these numbers, we obtain generalized Binet formula. In addition to this definition, we investigate the some new algebraic properties via a determinant for the generalized k-Horadam sequence.

2. Main results

In this section, we define a generalization $\{H_{k,n}\}_{n\in\mathbb{N}}$ of the special second order sequences. Also, we obtain some equalities related with this generalization. Now, we note that most of the following preliminary material is actually defined the first time.

Definition 1. Let k be any positive real number and $f(k)$, $g(k)$ are scaler-value polynomials. For $n \geq 0$ and $f^2(k) + 4g(k) > 0$, the generalized k-Horadam sequence $\{H_{k,n}\}_{n\in\mathbb{N}}$ is defined by

$$H_{k,n+2} = f(k)\, H_{k,n+1} + g(k)\, H_{k,n} \tag{1}$$

with initial conditions $H_{k,0} = a$, $H_{k,1} = b$.

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The equation in (1) is the second order linear difference equation and its characteristic equation is follows

$$\lambda^2 = f(k)\lambda + g(k). \tag{2}$$

This equation has two real roots as $r_1 = \frac{f(k)+\sqrt{f^2(k)+4g(k)}}{2}$ and $r_2 = \frac{f(k)-\sqrt{f^2(k)+4g(k)}}{2}$ $(r_1 > r_2)$. It means that the following relations hold for the numbers r_1, r_2:

$$r_1 + r_2 = f(k), \qquad r_1 - r_2 = \sqrt{f^2(k) + 4g(k)}, \qquad r_1r_2 = -g(k). \tag{3}$$

Particular cases of the previous definition are

- If $f(k) = k$ and $g(k) = 1$, the generalized k-Fibonacci and k-Lucas sequence is obtained
$$G_{k,n+2} = kG_{k,n+1} + G_{k,n}, \qquad G_{k,0} = a, \qquad G_{k,1} = b.$$
- If $f(k) = k$, $g(k) = 1$, $a = 0$ and $b = 1$, the k-Fibonacci sequence is obtained
$$F_{k,n+2} = kF_{k,n+1} + F_{k,n}, \qquad F_{k,0} = 0, \qquad F_{k,1} = 1.$$
- If $f(k) = k$, $g(k) = 1$, $a = 2$ and $b = k$, the k-Lucas sequence is obtained
$$L_{k,n+2} = kL_{k,n+1} + L_{k,n}, \qquad L_{k,0} = 0, \qquad L_{k,1} = k.$$
- If $f(k) = p$ and $g(k) = q$, the Horadam sequence is obtained
$$H_{n+2} = pH_{n+1} + qH_n, \qquad H_0 = a, \qquad H_1 = b.$$
- If $f(k) = 1$, $g(k) = 1$, $a = 0$ and $b = 1$, the Fibonacci sequence is obtained
$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \qquad F_1 = 1.$$
- If $f(k) = 1$, $g(k) = 1$, $a = 2$ and $b = 1$, the Lucas sequence is obtained
$$L_{n+2} = L_{n+1} + L_n, \qquad L_0 = 2, \qquad L_1 = 1.$$
- If $f(k) = 2$, $g(k) = 1$, $a = 0$ and $b = 1$, the Pell sequence is obtained
$$P_{n+2} = 2P_{n+1} + P_n, \qquad P_0 = 0, \qquad P_1 = 1.$$
- If $f(k) = 1$, $g(k) = 2$, $a = 0$ and $b = 1$, the Jacobsthal sequence is obtained
$$J_{n+2} = J_{n+1} + 2J_n, \qquad J_0 = 0, \qquad J_1 = 1.$$
- If $f(k) = 1$, $g(k) = 2$, $a = 2$ and $b = 1$, the Jacobsthal Lucas sequence is obtained
$$j_{n+2} = j_{n+1} + 2j_n, \qquad j_0 = 2, \qquad j_1 = 1.$$

We can find the more information associated with these sequences in [16,2–4,19].

Now, we give the Binet formula for the generalized k-Horadam sequence. Firstly, let us first consider the following Lemma which will be needed for the Binet Formula.

Theorem 2. *For every $n \in \mathbb{N}$, we can write the Binet formula*

$$H_{k,n} = \frac{Xr_1^n - Yr_2^n}{r_1 - r_2},$$

where $X = b - ar_2$ and $Y = b - ar_1$.

The following lemma will be used to prove the above theorem.

Lemma 3. *Let r_1 and r_2 be roots of Eq. (2). Then, we have*

$$H_{k,n} = r_1H_{k,n-1} + \left(H_{k,1} - r_1H_{k,0}\right) r_2^{n-1}. \tag{4}$$

Proof. By using (3), we write to equation in (1) as follows:

$$H_{k,n} = (r_1 + r_2)\, H_{k,n-1} - (r_1r_2)H_{k,n-2},$$
$$H_{k,n} - r_1H_{k,n-1} = r_2(H_{k,n-1} - r_1H_{k,n-2}). \tag{5}$$

Similarly, we can write

$$H_{k,n-1} = (r_1 + r_2)\, H_{k,n-2} - (r_1r_2)H_{k,n-3},$$
$$H_{k,n-1} = r_1H_{k,n-2} + r_2H_{k,n-2} - (r_1r_2)H_{k,n-3}. \tag{6}$$

By substituting Eq. (6) into (5), we get

$$H_{k,n} - r_1H_{k,n-1} = r_2^2(H_{k,n-2} - r_1H_{k,n-3}).$$

After that, by continuing this reduction procedure, we obtain

$$H_{k,n} - r_1 H_{k,n-1} = r_2^{n-1}(H_{k,1} - r_1 H_{k,0}),$$

as required. □

Proof of Theorem 2. In the above Lemma, by dividing by r_2^n both sides of (4), we have

$$\frac{H_{k,n}}{r_2^n} = \frac{r_1}{r_2}\frac{H_{k,n-1}}{r_2^{n-1}} + \frac{H_{k,1} - r_1 H_{k,0}}{r_2}.$$

Now, let us take $\frac{H_{k,n}}{r_2^n} = v_n$. Then we obtain the first order linear difference equation as follows:

$$v_n = \frac{r_1}{r_2} v_{n-1} + \frac{H_{k,1} - r_1 H_{k,0}}{r_2}.$$

The solution of this equation is given by

$$\begin{aligned} v_n &= H_{k,0}\left(\frac{r_1}{r_2}\right)^n + \frac{H_{k,1} - r_1 H_{k,0}}{r_2}\frac{\left(\frac{r_1}{r_2}\right)^n - 1}{\left(\frac{r_1}{r_2}\right) - 1} \\ &= \frac{1}{r_2^n}\left(r_1^n H_{k,0} + \frac{H_{k,1} - r_1 H_{k,0}}{r_2}\left(r_1^n - r_2^n\right)\right). \end{aligned}$$

Finally, we get

$$H_{k,n} = \left(\frac{H_{k,1} - r_2 H_{k,0}}{r_1 - r_2}\right) r_1^n - \left(\frac{H_{k,1} - r_1 H_{k,0}}{r_1 - r_2}\right) r_2^n,$$

as required. □

Theorem 4. *For $q > p \geq 0$, we have*

$$\sum_{i=0}^{n} H_{k,pi+q} = \frac{(-g(k))^p (H_{k,pn+q} - H_{k,q-p}) - H_{k,pn+p+q} + H_{k,q}}{(-g(k))^p - r_1^p - r_2^p + 1}.$$

Proof. We will prove the above result using the Binet formula for the generalized k-Horadam sequence. Then

$$\begin{aligned} \sum_{i=0}^{n} H_{k,pi+q} &= \sum_{i=0}^{n} \frac{X r_1^{pi+q} - Y r_2^{pi+q}}{r_1 - r_2} \\ &= \frac{X r_1^q}{r_1 - r_2}\sum_{i=0}^{n} r_1^{pi} - \frac{Y r_2^q}{r_1 - r_2}\sum_{i=0}^{n} r_2^{pi}. \end{aligned}$$

From the sum of the geometric sequence, we get

$$\sum_{i=0}^{n} H_{k,pi+q} = \frac{X r_1^q}{r_1 - r_2}\left(\frac{r_1^{pn+p} - 1}{r_1 - 1}\right) - \frac{Y r_2^q}{r_1 - r_2}\left(\frac{r_2^{pn+p} - 1}{r_2 - 1}\right).$$

By considering (3) and Theorem 2, we obtain

$$\sum_{i=0}^{n} H_{k,pi+q} = \frac{(-g(k))^p (H_{k,pn+q} - H_{k,q-p}) - H_{k,pn+p+q} + H_{k,q}}{(-g(k))^p - r_1^p - r_2^p + 1}. \quad \square$$

The following theorem gives us Cassini's identity for the generalized k-Horadam sequence.

Theorem 5. *Let the entries of each matrix $X_n = \begin{pmatrix} H_{k,n-1} & H_{k,n} \\ H_{k,n} & H_{k,n+1} \end{pmatrix}$ be the generalized k-Horadam numbers. For $n \geq 1$, we get*

$$|X_n| = (-g(k))^{n-1}\left(a^2 g(k) + abf(k) - b^2\right)$$

Proof. Let us use the principle of mathematical induction on m. For $m = 1$,

$$|X_1| = \begin{vmatrix} H_{k,0} & H_{k,1} \\ H_{k,1} & H_{k,2} \end{vmatrix} = (-g(k))^0\left(a^2 g(k) + abf(k) - b^2\right).$$

It is easy to see that, for $m = 2$, we have

$$|X_2| = \begin{vmatrix} H_{k,1} & H_{k,2} \\ H_{k,2} & H_{k,3} \end{vmatrix} = (-g(k)) \left(a^2 g(k) + abf(k) - b^2\right).$$

As the usual next step of inductions, let us assume that it is true for all positive integers m. That is,

$$|X_m| = \begin{vmatrix} H_{k,m-1} & H_{k,m} \\ H_{k,m} & H_{k,m+1} \end{vmatrix} = (-g(k))^{m-1} \left(a^2 g(k) + abf(k) - b^2\right). \tag{7}$$

Therefore, we have to show that it is true for $m + 1$. In other words, we need to check

$$|X_{m+1}| = (-g(k))^m \left(a^2 g(k) + abf(k) - b^2\right). \tag{8}$$

By considering elementary matrix row operations in (7), there are three steps for getting from (7) to (8). At first, the first row is multiplied by $g(k)$, then we multiply the second row by $f(k)$ so that we add the product to first row. Finally, two rows are swapped. At the first step the determinant is multiplied by $g(k)$, for the second step does not affect the determinant, and the last step changes only the sign which is desired. □

When $f(k) = g(k) = 1$, $a = 0$ and $b = 1$, the above result reduces to a known Cassini's identity of Fibonacci numbers.

Theorem 6. *Let the entries of each matrix* $Y_r = \begin{pmatrix} H_{k,n+r} & H_{k,n} \\ H_{k,n+r+1} & H_{k,n+1} \end{pmatrix}$ *be the generalized k-Horadam numbers. For* $r \geq 0$, *the following properties hold:*

(i) $|Y_{r+2}| = f(k)\,|Y_{r+1}| + g(k)\,|Y_r|$,
(ii) $|Y_r| = (-g(k))^n\,(bH_{k,r} - aH_{k,r+1})$.

Proof. Firstly, let us show that the equality in (i) is satisfied.

(i) Let $A = f(k)\,|Y_{r+1}| + g(k)\,|Y_r|$ be the right hand side of equation (i), for $r \geq 0$, we write

$$\begin{aligned} A &= f(k) \begin{vmatrix} H_{k,n+r+1} & H_{k,n} \\ H_{k,n+r+2} & H_{k,n+1} \end{vmatrix} + g(k) \begin{vmatrix} H_{k,n+r} & H_{k,n} \\ H_{k,n+r+1} & H_{k,n+1} \end{vmatrix} \\ &= f(k) \left(H_{k,n+1}H_{k,n+r+1} - H_{k,n}H_{k,n+r+2}\right) + g(k) \left(H_{k,n+1}H_{k,n+r} - H_{k,n}H_{k,n+r+1}\right) \\ &= H_{k,n+1}(f(k)H_{k,n+r+1} + g(k)H_{k,n+r}) - H_{k,n}(f(k)H_{k,n+r+2} + g(k)H_{k,n+r+1}). \end{aligned}$$

By using (1), if we rewrite this last equality, then we get

$$\begin{aligned} f(k)\,|Y_{r+1}| + g(k)\,|Y_r| &= H_{k,n+1}H_{k,n+r+2} - H_{k,n}H_{k,n+r+3} \\ &= |Y_{r+2}|, \end{aligned}$$

as required.

(ii) We need the follow induction steps on r. For $r = 0$, it is easy to see that $|Y_0| = 0$. For $r = 1$, by using Theorem 5, we can write

$$\begin{aligned} |Y_1| &= (-g(k))^n\,(b^2 - a^2 g(k) - abf(k)) \\ &= (-g(k))^n\,(bH_{k,1} - aH_{k,2}). \end{aligned}$$

As the usual next step of inductions, let us assume that it is true for all positive integers r. That is,

$$|Y_r| = (-g(k))^n\,(bH_{k,r} - aH_{k,r+1}). \tag{9}$$

Therefore, we have to show that is true for $r + 1$. In other words,

$$|Y_{r+1}| = (-g(k))^n\,(bH_{k,r+1} - aH_{k,r+2}).$$

By considering equation (i) and (9), we write

$$\begin{aligned} |Y_{r+1}| &= f(k)\,|Y_r| + g(k)\,|Y_{r-1}| \\ &= f(k)\,(-g(k))^n\,(bH_{k,r} - aH_{k,r+1}) + g(k)\,(-g(k))^n\,(bH_{k,r-1} - aH_{k,r}) \\ &= (-g(k))^n \left[b\left(f(k)H_{k,r} + g(k)H_{k,r-1}\right) - a\left(f(k)H_{k,r+1} + g(k)H_{k,r}\right)\right] \\ &= (-g(k))^n\,(bH_{k,r+1} - aH_{k,r+2}), \end{aligned}$$

which ends up the induction and the proof. □

It is notable that, taking m instead of $n + r$ in the above theorem, we get the d'Ocagne identity for the generalized k-Horadam sequences as

$$H_{k,m}H_{k,n+1} - H_{k,m+1}H_{k,n} = (-g(k))^n\,(bH_{k,m-n} - aH_{k,m-n+1}).$$

Also, for special values of $f(k)$, $g(k)$, a and b, we obtain the d'Ocagne identity for all special second order sequences. For instance, taking $f(k) = g(k) = 1$, $a = 0$ and $b = 1$, we get the d'Ocagne identity for the Fibonacci sequence as $F_mF_{n+1} - F_{m+1}F_n = (-1)^nF_{m-n}$.

Theorem 7. *Let the entries of each matrix* $Z_s = \begin{pmatrix} H_{k,n} & H_{k,n-r} \\ H_{k,n+s} & H_{k,n-r+s} \end{pmatrix}$ *be generalized k-Horadam numbers. For* $s \geq 0$, *the following properties hold:*

(i) $|Z_{s+2}| = f(k)\,|Z_{s+1}| + g(k)\,|Z_s|$,
(ii) $|Z_s| = (-g(k))^{n-r}\,\frac{(bH_{k,r}-aH_{k,r+1})(bH_{k,s}-aH_{k,s+1})}{b^2-a^2g(k)-abf(k)}$.

Proof. Proof of this theorem can be seen easily in a similar manner with Theorem 6. □

It is notable that, taking $m - n + r$ instead of s in the above theorem, we obtain a generalization of some equalities such as d'Ocagne's and Catalan's equalities for the generalized k-Horadam sequences as

$$B = \frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,m-n+r} - aH_{k,m-n+r+1}\right)}{b^2 - a^2g(k) - abf(k)}, \tag{10}$$

where $B = H_{k,m}H_{k,n} - H_{k,m+r}H_{k,n-r}$.

Also, as applications of (10) for the generalized k-Horadam sequence, we have the following results:

- Taking $n + 1$ instead of n and $r = 1$ in (10), we obtain d'Ocagne identity in Theorem 7.
- Taking n instead of m in (10), we obtain Catalan's identity.

Theorem 8 (*Generating Function of* $\{H_{k,n}\}_{n\in\mathbb{N}}$). *The generating function of this sequence is given by*

$$\sum_{i=0}^{\infty} H_{k,i}x^i = \frac{H_{k,0} + x\left(H_{k,1} - f(k)H_{k,0}\right)}{1 - f(k)x - g(k)x^2}.$$

Proof. Let $H(x)$ be a generating function for the $\{H_{k,n}\}_{n\in\mathbb{N}}$ sequence. Then we write

$$H(x) = H_{k,n} = H_{k,0} + xH_{k,1} + \cdots + x^nH_{k,n} + \cdots. \tag{11}$$

If it is multiplying Eq. (11) with $f(k)x$ and $g(k)x^2$, respectively, then we have

$$f(k)xH_{k,n} = f(k)xH_{k,0} + f(k)x^2H_{k,1} + \cdots + f(k)x^{n+1}H_{k,n} + \cdots \tag{12}$$

$$g(k)x^2H_{k,n} = g(k)x^2H_{k,0} + g(k)x^3H_{k,1} + \cdots + g(k)x^{n+2}H_{k,n} + \cdots. \tag{13}$$

Consequently, considering (11)–(13), the following equation is obtained

$$\left(1 - f(k)x - g(k)x^2\right)H_{k,n} = H_{k,0} + x\left(H_{k,1} - f(k)H_{k,0}\right)$$

$$\sum_{i=0}^{\infty} H_{k,i}x^i = \frac{H_{k,0} + x\left(H_{k,1} - f(k)H_{k,0}\right)}{1 - f(k)x - g(k)x^2},$$

as required. □

If we take $H_{k,i+1}$ instead of $H_{k,i}$, $a = 0$ and $b = 1$ in the above theorem, the dynamic behavior of the one-dimensional family of maps $H_{f(k),g(k)}(x) = \frac{1}{1-f(k)x-g(k)x^2}$, for specific values of the control parameters $f(k)$ and $g(k)$ is obtained. Besides, in this study it is observed that, as the parameters vary, the behavior of maps progresses from periodicity through bifurcations to a state of chaos. Period doubling bifurcations and periodic windows are visualized in a manner similar to the logistic map in [19].

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