A NOTE ON HORADAM'S SEQUENCE

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1. INTRODUCTION

Horadam's sequence $\{w_n(a,b;p,q)\}$, or briefly $\{w_n\}$, is defined by the recurrence relation $w_0=a$, $w_1=b$, $w_n=pw_{n-1}-qw_{n-2}$ $(n\geq 2)$ (see, e.g., [1], [2], [3]). The sequence $\{u_n(p,q)\}$, or briefly $\{u_n\}$, is defined as $u_n=w_n(1,p;p,q)$, and the sequence $\{v_n(p,q)\}$, or briefly $\{v_n\}$, is defined as $v_n=w_n(2,p;p,q)$. The sequences $\{u_n\}$ and $\{v_n\}$, respectively, are generalized Fibonacci and Lucas sequences.

In this note we study linear combinations of Horadam's sequences and the generating function of the ordinary product of two of Horadam's sequences. Similar results for the generalized Fibonacci sequence $\{u_n\}$ are given by McCarthy and Sivaramakrishnan [4]. In [1], Horadam studied the generating function of powers of $\{w_n\}$. The main results are in Sections 2 and 3 below

2. LINEAR COMBINATIONS

Let $p_1,p_2,\ldots,p_k$ be distinct complex numbers and let $w_n^{(j)}=w_n(a,b;p_j,q)$ and $u_n^{(j)}=u_n(p_j,q)$ for $j=1,2,\ldots,k$. McCarthy and Sivaramakrishnan show that the sequences $\{u_n^{(j)}\}$ are linearly independent and that if $u_n=c_1u_n^{(1)}+c_2u_n^{(2)}+\cdots+c_ku_n^{(k)}$ for all $n\geq 0$ then, for some h with $1\leq h\leq k$, we have $c_h=1$, $c_j=0$ for $j\neq h$ and $p=p_h$ (see Theorems 3 and 4 in [4]).

In this section we show that these results hold for the more general sequences $\{w_n\}$ and $\{w_n^{(j)}\}$. In the proofs, we need the identities

$$w_n=a\sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}p^{n-2k}q^k+(b-pa)\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n-1-k}{k}p^{n-1-2k}q^k \tag{2.1}$$

(see eq. (1.7) in [3]) and

$$w_n=bu_{n-1}-qau_{n-2} \tag{2.2}$$

(see eq. (2.14) in [2]).

We now give the generalizations.

Theorem 2.1: Let $a\neq 0$. The sequences $\{w_n^{(j)}\}$ are linearly independent.

Proof: Suppose that

$$\sum_{j=1}^{k}c_jw_n^{(j)}=0 \text{ for } n\geq 0.$$

Then the first k of these equations form a system of k linear equations with $c_1,c_2,\ldots,c_k$ as unknowns. The matrix of coefficients is $[w_i^{(j)}]$, $i=0,1,\ldots,k-1$; $j=1,2,\ldots,k$. The row with $i=0$ is $a,a,\ldots,a$. The other rows can be obtained by equation (2.1). Thus, if $i\geq 1$, then by adding appropriate multiples of the rows $0,1,\ldots,i-1$ to row i, the matrix can be transformed into one having $2ap_1^i,2ap_2^i,\ldots,2ap_k^i$ as its row i. Thus,

$$\det[w_i^{(j)}] = a(2a)^{k-1} \prod_{1\le i<j\le k} (p_j - p_i) \neq 0,$$

and hence $c_j = 0,\ j = 1, 2, \ldots, k$. □

Theorem 2.2: Let $a, b \neq 0$. If, for complex numbers $c_1, c_2, \ldots, c_k$,

$$\sum_{j=1}^{k} c_j w_n^{(j)} = w_n, \quad n \ge 0, \tag{2.3}$$

then for some h with $1 \le h \le k$ we have $c_h = 1$, $c_j = 0$ for $j \neq h$ and $p = p_h$.

Proof: We prove that

$$\sum_{j=1}^{k} c_j u_n^{(j)} = u_n, \quad n \ge 0. \tag{2.4}$$

Assume that $n = 0$. Then $\sum_{j=1}^{k} c_j w_0^{(j)} = w_0$ or $\sum_{j=1}^{k} c_j a = a$. Since $a \neq 0$, we have $\sum_{j=1}^{k} c_j = 1$ or $\sum_{j=1}^{k} c_j u_0^{(j)} = u_0$. Thus, (2.4) holds for $n = 0$. Assume that (2.4) holds for $n < m$. Then, with the aid of (2.2) and the induction assumption,

$$\sum_{j=1}^{k} c_j w_{m+1}^{(j)} = b\sum_{j=1}^{k} c_j u_m^{(j)} - qa\sum_{j=1}^{k} c_j u_{m-1}^{(j)} = b\sum_{j=1}^{k} c_j u_m^{(j)} - qau_{m-1}.$$

On the other hand, with the aid of (2.3) and (2.2),

$$\sum_{j=1}^{k} c_j w_{m+1}^{(j)} = w_{m+1} = bu_m - qau_{m-1}.$$

Since $b \neq 0$, we see that (2.4) holds for $n = m$. This completes the proof of (2.4). Now, applying the result of McCarthy and Sivaramakrishnan [4] referred to at the beginning of this section to (2.4), we have $c_h = 1$, $c_j = 0$ for $j \neq h$ and $p = p_h$ for some h with $1 \le h \le k$. This completes the proof of Theorem 2.2. □

Remark 2.1: Considering the sequence $\{w_n / a\}$, $a \neq 0$, and using the results of [4], it can be shown that Theorems 2.1 and 2.2 hold for the sequences $\{w_n(a, p; p, q)\}$ and $\{w_n(a, p_j; p_j, q)\}$. Thus, Theorems 2.1 and 2.2 hold for the generalized Lucas sequence $\{v_n\}$.

3. GENERATING FUNCTIONS

Let the sequence $\{w_n'(a', b'; p', q')\}$, or briefly $\{w_n'\}$, be defined by the recurrence relation $w_0' = a'$, $w_1' = b'$, $w_n' = p'w_{n-1}' - q'w_{n-2}'$ $(n \ge 2)$. In this section we evaluate the generating function of the ordinary product $\{w_n w_n'\}$ of two of Horadam's sequences. For the sake of brevity, we denote $W_2(x) = \sum_{n=0}^{\infty} w_n w_n' x^n$. We thus evaluate $W_2(x)$. As special cases, we obtain the generating functions of $\{u_n u_n'\}$ and $\{w_n^2\}$ evaluated by McCarthy and Sivaramakrishnan [4] and by Horadam [1], respectively.

Lemma 3.1:

$$\sum_{n=0}^{\infty} w_{n+1} w_n' x^n = \frac{a'b - paa' + (p'qaa' - qab')x + (p - p'qx)W_2(x)}{1 - qq'x^2}.$$

Proof: It is clear that

$$(1-qq'x^2)\sum_{n=0}^{\infty} w_{n+1}w'_n x^n = \sum_{n=0}^{\infty} w_{n+1}w'_n x^n - qq'\sum_{n=0}^{\infty} w_{n+1}w'_n x^{n+2}$$

$$= a'b + \sum_{n=1}^{\infty}(pw_n - qw_{n-1})w'_n x^n + q\sum_{n=0}^{\infty} w_{n+1}(w'_{n+2} - p'w'_{n+1})x^{n+2}$$

$$= a'b + pW_2(x) - paa' - q\sum_{n=1}^{\infty} w_{n-1}w'_n x^n + q\sum_{n=2}^{\infty} w_{n-1}w'_n x^n - p'qx(W_2(x) - aa')$$

$$= a'b - paa' - qab'x + p'qaa'x + pW_2(x) - p'qxW_2(x).$$

The proof of Lemma 3.1 is complete. □

Lemma 3.2:

$$\sum_{n=0}^{\infty} w_n w'_{n+1} x^n = \frac{ab' - p'aa' + (pq'aa' - q'a'b)x + (p' - pq'x)W_2(x)}{1 - qq'x^2}.$$

Lemma 3.2 follows from Lemma 3.1 by replacing a, b, p, and q with a', b', p', and q', respectively.

Theorem 3.1:

$$W_2(x) = \frac{A(x)}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2q'^2x^4},$$

where

$$A(x) = aa' + (bb' - aa'pp')x + (aa'p^2q' + aa'p'^2q - aa'qq' - ab'p'q - a'bpq')x^2$$
$$+ (ab'pqq' + a'bp'qq' - aa'pp'qq' - bb'qq')x^3.$$

Proof: We have

$$W_2(x) = \sum_{n=0}^{\infty} w_n w'_n x^n = aa' + bb'x + \sum_{n=2}^{\infty}(pw_{n-1} - qw_{n-2})(p'w'_{n-1} - q'w'_{n-2})x^n$$

$$= aa' + bb'x + pp'\sum_{n=2}^{\infty} w_{n-1}w'_{n-1}x^n - pq'\sum_{n=2}^{\infty} w_{n-1}w'_{n-2}x^n$$

$$- p'q\sum_{n=2}^{\infty} w_{n-2}w'_{n-1}x^n + qq'\sum_{n=2}^{\infty} w_{n-2}w'_{n-2}x^n$$

$$= aa' + bb'x + pp'x\left(\sum_{n=0}^{\infty} w_n w'_n x^n - aa'\right) - pq'x^2\sum_{n=0}^{\infty} w_{n+1}w'_n x^n$$

$$- p'qx^2\sum_{n=0}^{\infty} w_n w'_{n+1}x^n + qq'x^2\sum_{n=0}^{\infty} w_n w'_n x^n.$$

Applying Lemmas 3.1 and 3.2, we obtain

$$W_2(x) = aa' + bb'x - aa'pp'x + pp'xW_2(x) - pq'x^2\frac{a'b - paa' + p'qaa'x - qab'x + (p - p'qx)W_2(x)}{1 - qq'x^2}$$

$$- p'qx^2\frac{ab' - p'aa' + pq'aa'x - q'a'bx + (p' - pq'x)W_2(x)}{1 - qq'x^2} + qq'x^2W_2(x).$$

Solving for $W_2(x)$, we get Theorem 3.1. □

Corollary 3.1 [4]:

$$\sum_{n=0}^{\infty} u_n u'_n x^n = \frac{1 - qq'x^2}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2q'^2x^4}.$$

Corollary 3.2:

$$\sum_{n=0}^{\infty} v_n v'_n x^n = \frac{4 - 3pp'x + (2p^2q' + 2p'^2q - 4qq')x^2 - pp'qq'x^3}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2q'^2x^4}.$$

Corollary 3.3:

$$\sum_{n=0}^{\infty} u_n v'_n x^n = \frac{2 - pp'x + (p'^2q - 2qq')x^2}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2q'^2x^4}.$$

Corollary 3.4 [1]:

$$\sum_{n=0}^{\infty} w_n^2 x^n = \frac{a^2 + [b^2 - a^2(p^2 - q)]x + q(b - pa)^2x^2}{(1 - qx)[1 - (p^2 - 2q)x + q^2x^2]}.$$

Corollary 3.5 [4]:

$$\sum_{n=0}^{\infty} u_n^2 x^n = \frac{1 + qx}{(1 - qx)[1 - (p^2 - 2q)x + q^2x^2]}.$$

Corollary 3.6:

$$\sum_{n=0}^{\infty} v_n^2 x^n = \frac{4 + (4q - 3p^2)x + p^2qx^2}{(1 - qx)[1 - (p^2 - 2q)x + q^2x^2]}.$$

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