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# IDENTITIES FOR A CLASS OF SUMS INVOLVING HORADAM'S GENERALIZED NUMBERS $\{W_n\}$

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## 1. INTRODUCTION

Following Horadam [8], we consider the sequence $\{W_n = W_n(W_0, W_1; a, b)\}_{n=0}^{\infty}$ generated by the recurrence relation

$$W_{n+2} = aW_{n+1} - bW_n; \tag{1}$$

$W_0 = W_{n=0}$ and $W_1 = W_{n=1}$ are initial values. This sequence can be extended to negative subscripts and $W_n$ has the Binet representation $(n = 0, \pm 1, \pm 2, \ldots)$

$$W_n = A\alpha^n + B\beta^n, \tag{2}$$

where $A$ and $B$ are constants and

$$\alpha, \beta = (a \pm \sqrt{a^2 - 4b})/2; \;\; \alpha\beta = b. \tag{3}$$

Expressions for sums involving Fibonacci numbers or Pell numbers have been given by many authors (e.g., [2], [3], [6], [10]-[12], [14]-[18]). The interested reader should consult these references as well as Bicknell's "primer" on Pell numbers [1]for further details and references.

The purpose of this article is to obtain a general identity for the following sum:

$$S(u; m; q, s, n) \equiv \sum_{r=0}^{n-1} r^m u^r W_{qr+s}. \tag{4a}$$

This identity provides a means of evaluating $S$, where $n \geq 1$, $m \geq 0$, $q$ and $s$ are integers, and $u$ is an arbitrary parameter. This general identity, which consists of a sum of $(m+1)(m+4)$ terms, regardless of the value of $n$, collapses into a sum containing only 2 $(m+1)$ terms for certain values of $u$. This simpler identity applies to the following:

$$S_1(m, p, q, s; n) \equiv \sum_{r=0}^{n-1} r^m (U_p / U_{p+q})^r W_{qr+s}, \tag{4b}$$

for $p \neq 0, q \neq 0, q \neq -p$, as will be shown. Here

$$U_n \equiv (\alpha^n - \beta^n)/(\alpha - \beta); \;\; U_{-n} = -U_n / b^n. \tag{5}$$

The collapsing of the sum, from $(m+1)(m+4)$ terms to 2 $(m+1)$ terms, rests on the following identity:

$$U_p^r W_{qr+s} = \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{\ell} b^{p(r-\ell)} U_q^\ell U_{q-p}^{r-\ell} W_{p\ell+s}, \tag{6}$$

where $r \geq 0$, $s$, $p\ (\neq 0)$, and $q\ (\neq 0, p)$ are arbitrary integers. This type of identity is usually referred to as a Fibonacci-binomial identity when it applies specifically to generalized Fibonacci or Lucas numbers. The reader may find an interesting approach to such identities in [4]. Layman

[13] was among the first to consider identities of this type, for Fibonacci numbers, but his results were only partial. Carlitz [3] subsequently gave a generalization in the form of (6), but again for Fibonacci numbers only. Haukkanen [7] recently extended the result of Carlitz by making use of exponential generating functions. His generalization applies to Lucas, Pell, and Pell-Lucas numbers. By contrast, the present approach is more general in some aspects to be clarified later, and it covers the results obtained by Carlitz and by Haukkanen as four special cases.

Several of the identities obtained in the references quoted earlier may be obtained as special cases of (4). Specific cases will be discussed in the closing section of this article.

## 2. ASSOCIATED GEOMETRIC POLYNOMIALS

Let $x \neq 1$ be a real variable and define the following polynomial for integers $n \geq 1$ and $m \geq 0$:

$$P_{n-1}^{m}(x) \equiv \sum_{r=0}^{n-1} r^m x^r. \tag{7}$$

$P_{n-1}^{m}(x)$ can be obtained from the geometric polynomial of degree $n-1$, $P_{n-1}(x) = \Sigma_{r=0}^{n-1} x^r$, by repeated use of the differential operator $D = xd/dx$:

$$P_{n-1}^{m} = D^m \sum_{r=0}^{n-1} x^r = \sum_{r=0}^{n-1} r^m x^r. \tag{8}$$

The convention that $r^0 = 1$ for all $r$, including $r = 0$, will be used throughout. We consider $P_{n-1}^{m}(x)$ briefly in what follows.

Let

$$f_\nu(x) \equiv \frac{x^\nu}{(1-x)}, \tag{9}$$

where $\nu$ is arbitrary, and observe that (7) may be written as

$$\sum_{r=0}^{n-1} r^m x^r = D^m f_0(x) - D^m f_n(x). \tag{10}$$

As a result, a study of the function

$$g_m^\nu(x) = D^m f_\nu(x) \tag{11}$$

will provide the means to calculate $\Sigma r^m x^r$. In general, one can show that [5]

$$g_m^\nu(x) = \sum_{\ell=0}^{m} a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}}, \tag{12}$$

where the set of coefficients $\{a_\ell^m(\nu); \ell = 0, 1, \ldots, m\}$ is simply found by acting $m$ times on $f_\nu(x)$ with the operator $D$. Here is a short list, for easy reference:

$m = 0$: $a_0^0(\nu) = 1$;
$m = 1$: $a_0^1(\nu) = \nu, a_1^1(\nu) = 1$;
$m = 2$: $a_0^2(\nu) = \nu^2, a_1^2(\nu) = 2\nu + 1, a_2^2(\nu) = 2$;
$m = 3$: $a_0^3(\nu) = \nu^3, a_1^3(\nu) = 3\nu^2 + 3\nu + 1, a_2^3(\nu) = 6\nu + 6, a_3^3(\nu) = 6$;
$m = 4$: $a_0^4(\nu) = \nu^4, a_1^4(\nu) = 4\nu^3 + 6\nu^2 + 4\nu + 1$;
$a_2^4(\nu) = 12\nu^2 + 24\nu + 14, a_3^4(\nu) = 24\nu + 36, a_4^4(\nu) = 24$;

... and so on. A hierarchy of equations is easily found for the set of coefficients $\{a_\ell^{m+1}(\nu);\ \ell = 0, 1, 2, \ldots, m+1\}$ in terms of the set $\{a_\ell^m(\nu)\}$. Indeed, from (11) and (12), with $m$ replaced by $m+1$,

$$g_{m+1}^{\nu}(x) \equiv D^{m+1} f_\nu(x) = \sum_{\ell=0}^{m+1} a_\ell^{m+1}(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}}. \tag{13}$$

But one can also write

$$g_{m+1}^{\nu}(x) \equiv D g_m^{\nu}(x) = D \sum_{\ell=0}^{m} a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}} \tag{14}$$

when using (12) for $g_m^\nu(x)$. Operating on the sum with $D$ then gives

$$g_{m+1}^{\nu}(x) = \sum_{\ell=0}^{m} (\nu+\ell) a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}} + \sum_{\ell=0}^{m} (\ell+1) a_\ell^m(\nu) \frac{x^{\nu+\ell+1}}{(1-x)^{\ell+2}}. \tag{15}$$

To cast this result in the form of a sum over $\ell$ from 0 to $m+1$ [see (13)], first define $a_{m+1}^m(\nu) \equiv 0$ and $a_{-1}^m(\nu) \equiv 0$ to obtain, from (15),

$$g_{m+1}^{\nu}(x) = \sum_{\ell=0}^{m+1} (\nu+\ell) a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}} + \sum_{\ell=0}^{m+1} \ell a_{\ell-1}^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}}. \tag{16}$$

This is achieved by extending the upper limit of the first sum in (15) to $m+1$ and by shifting $\ell$ to $\ell-1$ in the second sum; the value $\ell = 0$ can be included in the latter with the help of $a_{-1}^m(\nu) = 0$. A comparison of (13) and (16) then gives the desired recurrence for the unknown coefficients,

$$a_\ell^{m+1}(\nu) = (\nu+\ell) a_\ell^m(\nu) + \ell a_{\ell-1}^m(\nu); \tag{17}$$

here, $m = 0, 1, 2, \ldots$ and $\ell = 0, 1, 2, \ldots, m+1$. This equation was used to generate the list of coefficients $\{a_\ell^m(\nu);\ 0 \le \ell \le m;\ 0 \le m \le 4\}$ presented earlier in this section.

In what follows, the set of coefficients $\{a_\ell^m(\nu)\}$ is assumed to be known and will be used to obtain identities for (4a) and (4b).

### 3. OBTAINING EXPRESSIONS FOR $S(u; m, q, s, n)$ AND $S_1(m, p, q, s; n)$

To evaluate (4a) and (4b) explicitly, first multiply both sides of (10), (11), and (12) by $A\alpha^s$, where $s$ is an arbitrary integer and get

$$\sum_r A\alpha^s r^m x^r = \sum_{\nu,\ell} \xi_\nu a_\ell^m(\nu) \frac{x^{\nu+\ell}}{(1-x)^{\ell+1}} A\alpha^s, \tag{18}$$

where $\nu = 0$ and $n$, with $\xi_0 = 1$, $\xi_n = -1$. The limits on the sum over $\ell$ are from 0 to $m$, as before. To simplify the notation, the limits on all sums will be omitted, as they always remain the same in what follows. Next, replace $x$ by $y$, and $A\alpha^s$ by $B\beta^s$ in (18), add the resulting expression to (18), and set $x = u\alpha^q$, $y = u\beta^q$ to get

$$\sum_r r^m u^r W_{qr+s} = \sum_{\nu,\ell} \xi_\nu a_\ell^m(\nu) u^{\nu+\ell} \left[ A \frac{\alpha^{q(\nu+\ell)+s}}{(1-u\alpha^q)^{\ell+1}} + B \frac{\beta^{q(\nu+\ell)+s}}{(1-u\beta^q)^{\ell+1}} \right]; \tag{19}$$

$q$ is an arbitrary integer and $u$ an arbitrary parameter such that $u\alpha^q \neq 1$ and $u\beta^q \neq 1$. We shall examine two cases, in turn: 1°) the denominators will be inverted and a binomial expansion made; 2°) the denominators will be made proportional to pure powers in $\alpha$ and $\beta$.

To invert the denominators without invoking infinite series, consider $(1-u\alpha^q)^{-1}$ first:

$$\begin{aligned}(1-u\alpha^q)^{-(\ell+1)} &= [(1-u\alpha^q)(1-u\beta^q)]^{-(\ell+1)}(1-u\beta^q)^{\ell+1}\\ &= N_q^{\ell+1}(u)(1-ub^q\alpha^{-q})^{\ell+1}\\ &= N_q^{\ell+1}(u)\sum_{j=0}^{\ell+1}(-1)^j\binom{\ell+1}{j}b^{qj}u^j\alpha^{-qj},\end{aligned} \tag{20}$$

where

$$N_q(u) \equiv [1-uV_q+u^2b^q]^{-1};\; V_q \equiv \alpha^q+\beta^q;\; V_{-q}=V_q/b^q. \tag{21}$$

A similar approach for the other denominator finally gives the desired identity for (4a):

$$\begin{aligned}S(u;m,q,s,n) &\equiv \sum_r r^m u^r W_{qr+s}\\ &= \sum_{\nu,\ell,j}(-1)^j\binom{\ell+1}{j}\xi_\nu a_\ell^m(\nu)N_q^{\ell+1}b^{qj}u^{\nu+\ell+j}W_{q(\nu+\ell-j)+s}.\end{aligned} \tag{22}$$

This general identity allows an evaluation of the left-hand sum in a closed form. Consider the case in which $m=0$, for example: $a_\ell^0(\nu)=1$, $\ell=0$; $a_\ell^0(\nu)=0$, $\ell\neq 0$. Then, $j=0,1$ and $\nu=0,n$, so that the right-hand expression reduces to only four terms:

$$\sum_{r=0}^{n-1}u^rW_{qr+s} = N_q[W_s-ub^qW_{-q+s}-u^nW_{qn+s}+u^{n+1}b^qW_{q(n-1)+s}]. \tag{23}$$

Now let us turn to the other situation in which the denominators in (19) are proportional to pure powers of $\alpha$ and $\beta$, say, when

$$1-u\alpha^q=\gamma\alpha^p;\; 1-u\beta^q=\gamma\beta^p, \tag{24}$$

with $p$ and $q$ integers; the parameters $u=u(p,q)$, $\gamma=\gamma(p,q)$ are to be determined. To do so, multiply the first equation of (24) by $\alpha^{-p}$, the second by $\beta^{-p}$, and equate the results to get

$$\alpha^{-p}-u\alpha^{q-p}=\gamma=\beta^{-p}-u\beta^{q-p}. \tag{25}$$

For $p\neq 0$, $q\neq 0$, and $q\neq p$, this gives

$$u=U_{-p}/U_{q-p}=-U_p/(b^pU_{q-p}), \tag{26}$$

where $U_n$ is defined in (5). Similarly, multiplying the first equation by $\alpha^{-q}$ and the second by $\beta^{-q}$, one finds, for $p\neq 0$, $q\neq 0$, and $q\neq p$:

$$\gamma=U_{-q}/U_{p-q}=U_q/(b^pU_{q-p}). \tag{27}$$

This set of values, $(u,\gamma)$, satisfies (24), and insertion in (19) gives, for $\gamma\neq 0$:

$$\sum_r r^m u^r W_{qr+s}=\sum_{\nu,\ell}\xi_\nu a_\ell^m(\nu)\frac{u^{\nu+\ell}}{\gamma^{\ell+1}}W_{q(\nu+\ell)-p(\ell+1)+s}, \tag{28}$$

i.e., we get the identity ($p \neq 0, q \neq 0, q \neq -p$) referred to in (4b), namely,

$$\begin{aligned} S_1(m,p,q,s;n) &\equiv \sum_r r^m (U_p / U_{p+q})^r W_{qr+s} \\ &= \sum_{\nu,\ell} \xi_\nu a_\ell^m(\nu) b^{-p(\ell+1)} U_p^{\nu+\ell} U_q^{-(\ell+1)} U_{p+q}^{1-\nu} W_{q(\nu+\ell)+p(\ell+1)+s}, \end{aligned} \tag{29}$$

after replacing $p$ by $-p$ and making use of the second form for $\gamma$ in (27). As mentioned in the introduction, the right-hand expression in (29) only contains 2 $(m+1)$ terms, whereas that in (22) contains $(m+1)$ $(m+4)$ such terms. As a result, (29) represents a very special class of identities. We now use (24) to establish a generalization of the Carlitz theorem [3].

From (24), consider the following,

$$(u\alpha^q)^r = (1-\gamma\alpha^p)^r \tag{30}$$

for $r = 0, 1, 2, \ldots$, where $u$ and $\gamma$ are as given in (26) and (27). On using the binomial expansion, we readily obtain the identity

$$u^r \alpha^{qr} = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \gamma^j \alpha^{pj}; \tag{31}$$

a similar result also holds for $u^r \beta^{qr}$. Now multiply (31) by $A\alpha^s$, then substitute $B$ for $A$, $\beta$ for $\alpha$, and add the two resulting equations to get

$$u^r W_{qr+s} = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \gamma^j W_{pj+s}. \tag{32}$$

Inserting $(u, \gamma)$ from (26), (27) then gives (6); this is, as stated earlier, a generalization of the "if" part of a theorem, due to Carlitz [3], for Fibonacci numbers, and recently extended by Haukkanen [7] to Lucas, Pell, and Pell-Lucas numbers. The present proof is simpler and more general. The present approach does not establish the "only if" part of those theorems, however.

The solutions $u$ and $\gamma$ can also be used to generate interesting associated identities similar to (6). Indeed, from

$$1^r = (u\alpha^q + \gamma\alpha^p)^r, \tag{33}$$

one obtains, for $p \neq 0, q \neq 0, q \neq -p, r = 0, 1, 2, \ldots$, and $s$ an arbitrary integer,

$$b^{pr} U_q^r W_s = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} U_p^\ell U_{q+p}^{r-\ell} W_{q\ell+pr+s}. \tag{34}$$

The proof is similar to that of (6); thus, we omit the details. Similarly, starting with

$$(u\gamma\beta^{p+q})^{-r} = (\gamma^{-1}\beta^{-p} + u^{-1}\beta^{-q})^r, \tag{35}$$

one finds, again for $p \neq 0, q \neq 0, q \neq -p, r = 0, 1, 2, \ldots$, and $s$ an arbitrary integer,

$$U_{2p+q}^r W_{qr+s} = \sum_{\ell=0}^{r} \binom{r}{\ell} b^{(p+q)(r-\ell)} U_p^{r-\ell} U_{p+q}^\ell W_{(2p+q)\ell-pr+s}. \tag{36}$$

Finally, consider

$$\frac{1}{u\beta^q} = \frac{1}{1-\gamma\beta^p} = N(p,q)(1-\gamma\alpha^p), \tag{37}$$

where

$$N(p,q) \equiv [1-\gamma V_p + \gamma^2 b^p]^{-1}. \tag{38}$$

In the right-hand side of (37), replace $1-\gamma\alpha^p$ by $u\alpha^q$ to obtain the identity

$$1 = b^q u^2(p,q) N(p,q), \tag{39}$$

from which we get, for $p \neq 0, q \neq p$:

$$\begin{aligned} N(p,q) &= b^q\, U_{p-q}^2 / U_p^2 \\ &= b^p\, U_{q-p}^2 / [b^p\, U_{q-p}^2 - V_p U_q U_{q-p} + U_q^2]. \end{aligned} \tag{40}$$

The last line follows from (38).

We now examine special cases of some of the identities that have been obtained.

## 4. ADDITIONAL PROPERTIES AND SOME APPLICATIONS

In this final section, we apply formulas (4b), (29), and (6) to specific cases, some of which will help support our earlier claim that certain identities contained in [2], [3], [6], [7], [10]-[12], and [14]-[18] are special cases of the present formulas.

We first establish the following, for $n,\ k = 0, \pm 1, \pm 2, \ldots$,

$$W_{n+k} = f_{k-1} W_{n+1} + f_{k-2} W_n, \tag{41}$$

where the coefficients $\{f_k\}$ are to be determined. (This formula will prove useful in comparing some of the identities presented here to some of the identities given [2], [3], [6], [7], [10]-[12] and [14]-[18], as mentioned in the introduction.) To do so, assume that (41) holds for all $n$, $k$ and replace $k$ by $k-1$:

$$W_{n+k-1} = f_{k-2} W_{n+1} + f_{k-3} W_n. \tag{42}$$

Multiply (41) by $a$, (42) by $-b$, and add the results to get

$$aW_{n+k} - bW_{n+k-1} = (af_{k-1} - bf_{k-2}) W_{n+1} + (af_{k-2} - bf_{k-3}) W_n. \tag{43}$$

Comparing this result to (1) then gives

$$W_{n+k+1} = (af_{k-1} - bf_{k-2}) W_{n+1} + (af_{k-2} - bf_{k-3}) W_n, \tag{44}$$

so if we let

$$f_k = af_{k-1} - bf_{k-2}, \tag{45}$$

then (41) is satisfied for all $n$, $k$. Thus, $f_k$ can be written in the Binet form

$$f_k = A'\alpha^k + B'\beta^k, \tag{46}$$

where $A'$ and $B'$ are constants. The initial conditions on $\{f_k\}$ are obtained by setting $k=2$ in (41), i.e.,

$$\begin{aligned} W_{n+2} &= f_1 W_{n+1} + f_0 W_n \\ &= aW_{n+1} - bW_n, \end{aligned} \tag{47}$$

where the second line follows from (1). Because $n$, $A$, and $B$ are arbitrary, (47) implies

$$f_0 = -b;\ f_1 = a. \tag{48}$$

Use of (48) in (46) gives, after a little algebra,

$$f_k = aU_k + b^2 U_{k-1}. \tag{49}$$

For $a = 1, b = -1, f_k = F_{k+1}$, which is a well-known result.

Now, armed with these preliminaries, we turn to formulas (4b), (29), and (6). $S_1(m, p, q, s; n)$ is as defined in (4b); we also let [see (6)]

$$S_2(p, q, s; r) \equiv U_p^r W_{qr+s}. \tag{50}$$

$S_1$ and $S_2$ contain additional parameters, $a$, $b$, $A$, and $B$, but those are omitted in the notation for the sake of brevity. In this same spirit, the following nomenclature and notation will be used.

**1. General Horadam (GH) Case**

This is the most general case; previous notation will not be altered except that $S_1$ and $S_2$ will be written as $S_1^{\mathrm{GH}}$ and $S_2^{\mathrm{GH}}$.

**2. General Fibonacci (GF) Case**

This is the special case where $a = 1, b = -1$; all previous notation will be used, except for $S_1^{\mathrm{GF}}$, $S_2^{\mathrm{GF}}$, and $W_n = H_n$ (to follow what now appears to be common practice). Also, $U_n = F_n$ and $V_n = L_n$, where $F_n$ and $L_n$ are the $n^{\text{th}}$ Fibonacci and Lucas numbers, respectively.

**3. Fibonacci (F) and Lucas (L) Cases**

Then $a = 1, b = -1$ for both cases, while $A = -B = (\alpha - \beta)^{-1}$ for the F-case and $A = +B = 1$ for the L-case; we also write $S_1^{\mathrm{F}}, S_1^{\mathrm{L}}, S_2^{\mathrm{F}}, S_2^{\mathrm{L}}, W_n = F_n$, and $W_n = L_n$.

**4. General Pell (GP) Case**

This is the case where $a = 2, b = -1$. The notations $S_1^{\mathrm{GP}}, S_2^{\mathrm{GP}}$, and $W_n = H_n$ will be used (again to follow a very common practice); also, $U_n = P_n, V_n = R_n$, where $P_n$ and $R_n$ are the $n^{\text{th}}$ Pell and Pell-Lucas numbers, respectively.

**5. Pell (P) and Pell-Lucas (PL) Cases**

Then $a = 2, b = -1$ for both cases, while $A = -B = (\alpha - \beta)^{-1}$ for the P-case and $A = +B = 1$ for the PL-case; we also write $S_1^{\mathrm{P}}, S_1^{\mathrm{PL}}, S_2^{\mathrm{P}}, S_2^{\mathrm{PL}}, W_n = H_n$ (for both), and $U_n = P_n, V_n = R_n$.

We now consider $S_1$ and $S_2$ in the special cases mentioned above.

**I. $S_1(m, p, q, s; n)$**

One has

$$S_1(m, -p-q, q, s; n) = \sum_r r^m b^{-qr} (U_{p+q} / U_p)^r W_{qr+s} \tag{51}$$

and

$$S_1(m, -p, -q, s; n) = b^{-s} \sum_r r^m (U_p / U_{p+q})^r \overline{W}_{qr+s} \tag{52}$$

where, for $n = 0, \pm 1, \pm 2, \ldots,$

$$\overline{W}_n \equiv b^n W_{-n} = B\alpha^n + A\beta^n \tag{53}$$

will be called the transpose of $W_n$. The transpose is obtained by interchanging $A$ and $B$ in $W_n$. In general, $\overline{W}_n \neq W_n$. For Pell and Fibonacci numbers, $\overline{W}_n = -W_n$, while for Lucas and Pell-Lucas numbers, $\overline{W}_n = W_n$.

For $m = 0, p \neq 0, q \neq 0, q \neq -p, b \neq 0$, (29) gives

$$\begin{aligned} S_1(0,p,q,s;n) &\equiv \sum_{r=0}^{n-1} (U_p / U_{p+q})^r W_{qr+s} \\ &= [U_{p+q} W_{p+s} - U_p^n U_{p+q}^{1-n} W_{qn+p+s}] / (b^p U_q). \end{aligned} \tag{54}$$

Specializing further, with $W_{-1} = \overline{W}_1 / b$, $U_{-1} = -b^{-1}$, $U_2 = a$:

$$\begin{aligned} S_1(0,-1,3,0;n) &\equiv \sum_{r=0}^{n-1} (-ab)^r W_{3r} \\ &= ab[a\overline{W}_1 / b - a(-ab)^{-n} W_{3n-1}] / (a^2 - b). \end{aligned} \tag{55}$$

Set $a = 1, b = -1, A = -B = 1/(\alpha - \beta)$ in (55) to obtain

$$\sum_{r=0}^{n-1} F_{3r} = \frac{1}{2}(F_{3n-1} - 1); \tag{56}$$

this corresponds to Iyer's equation (6) in [12], since his result transforms into (56) after use of (41) and (49).

For $m = 3$, (29) gives, with $U_1 = 1$ and $U_2 = a$,

$$\begin{aligned} S_1(3,1,1,0;n) &= \sum_{r=0}^{n-1} r^3 a^{-r} W_r \\ &= \sum_{\nu=0,n} \sum_{\ell=0}^{3} \xi_\nu a_\ell^3(\nu) a^{1-\nu} b^{-(\ell+1)} W_{2\ell+\nu+1}. \end{aligned} \tag{57}$$

From the list of coefficients following (12), we get $a_\ell^3(\nu)$, $\ell = 0, 1, 2, 3$; then

$$\begin{aligned} S_1(3,1,1,0;n) &= a\left[\frac{W_3}{b^2} + 6\frac{W_5}{b^3} + 6\frac{W_7}{b^4}\right] \\ &\quad - a^{1-n}\left[n^3 \frac{W_{n+1}}{b} + (3n^2 + 3n + 1)\frac{W_{n+3}}{b^2} + (6n+6)\frac{W_{n+5}}{b^3} + 6\frac{W_{n+7}}{b^4}\right]. \end{aligned} \tag{58}$$

Now, specializing to the Fibonacci case ($a = 1, b = -1, A = -B = 1/(\alpha - \beta), W_n = F_n$):

$$S_1^{\mathrm{F}}(3,1,1,0;n) = 50 + n^3 F_{n+1} - (3n^2 + 3n + 1)F_{n+3} + 6(n+1)F_{n+5} - 6F_{n+7}. \tag{59}$$

After using (41) and (49), this gives the identity

$$\sum_{r=0}^{n-1} r^3 F_r = 50 + (n^3 - 3n^2 + 9n - 19)F_{n+1} - (3n^2 - 15n + 31)F_{n+2}. \tag{60}$$

This result is identical to Harris's equation 17 in [6].

We now list a few other sums that can be evaluated using (29):

$$S_1^{GC}(m,1,1,s;n) = \Sigma r^m a^{-r} W_{r+s};$$
$$S_1^{GF}(m,1,1,s;n) = \Sigma r^m W_{r+s};$$
$$S_1^{GC}(m,-2,3,s;n) = \Sigma r^m (-a/b^2)^r W_{3r+s};$$
$$S_1^{GP}(m,-2,3,s;n) = \Sigma r^m (-2)^r W_{3r+s}.$$

**II. $S_2(p,q,s;r)$**

$$S_2^{GP}(p,q,s;r) = P_p^r H_{qr+s} = \sum_{\ell=0}^{r} (-1)^{(p+1)(r-\ell)} \binom{r}{\ell} P_q^\ell P_{q-p}^{r-\ell} H_{p\ell+s}. \tag{61}$$

If we now set $A = -B = (\alpha-\beta)^{-1}$ (Pell: P) in this result, we obtain

$$S_2^{P}(p,q,s;r) = P_p^r P_{qr+s} = \sum_{\ell=0}^{r} (-1)^{(p+1)(r-\ell)} \binom{r}{\ell} P_q^\ell P_{q-p}^{r-\ell} P_{p\ell+s}. \tag{62}$$

This is the "if" part of Haukkanen's Theorem 3 in [7]. Now, if $A = B = 1$ (Pell-Lucas: PL),

$$S_2^{PL}(p,q,s;r) = P_p^r P_{qr+s} = \sum_{\ell=0}^{r} (-1)^{(p+1)(r-\ell)} \binom{r}{\ell} P_q^\ell P_{q-p}^{r-\ell} R_{p\ell+s}. \tag{63}$$

This is the "if" part of Haukkanen's Theorem 4 in [7].

$$S_2^{GF}(p,q,s;r) = F_p^r H_{qr+s} = \sum_{\ell=0}^{r} (-1)^{(p+1)(r-\ell)} \binom{r}{\ell} F_q^\ell F_{q-p}^{r-\ell} H_{p\ell+s}. \tag{64}$$

If $A = -B = (\alpha-\beta)^{-1}$ (F), then

$$S_2^{F}(p,q,s;r) = F_p^r F_{qr+s} = \sum_{\ell=0}^{r} (-1)^{(p+1)(r-\ell)} \binom{r}{\ell} F_q^\ell F_{q-p}^{r-\ell} F_{p\ell+s}. \tag{65}$$

This is the "if" part of Carlitz's basic theorem [3]. [Note that (4.8) and (4.9) in [3] are in error due to an omission arising after (4.7).]

Finally, if $A = B = 1$ (L), then

$$S_2^{L} = F_p^r L_{qr+s} = \sum_{\ell=0}^{r} (-1)^{(p+1)(r-\ell)} \binom{r}{\ell} F_q^\ell F_{q-p}^{r-\ell} L_{p\ell+s}. \tag{66}$$

This is the "if" part of Haukkanen's Theorems 1 and 2 in [7]. In our treatment, there is no need to consider $s = 0$ and $s \neq 0$ separately as Haukkanen does due to the fact that we only prove the "if" part of that theorem.

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