5

# GENERATING FUNCTION OF LAGUERRE POLYNOMIALS

ABDULLAH ÖNER

Rodrigez formula for Laguerre polynomials is defined as

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t})$$

Evaluating the first three derivatives of $ws^n = e^{-t}t^n$ we get the following results:

$$\frac{d}{dt}(t^n e^{-t}) = nt^{n-1}e^{-t} - t^n e^{-t}$$

$$\frac{d^2}{dt^2}(t^n e^{-t}) = n(n-1)t^{n-2}e^{-t} - 2nt^{n-1}e^{-t} + t^n e^{-t}$$

$$\frac{d^3}{dt^3}(t^n e^{-t}) = n(n-1)(n-2)t^{n-3}e^{-t} - 3n(n-1)t^{n-2}e^{-t} + 3nt^{n-1}e^{-t} - t^n e^{-t}$$

Then

$$\frac{d^n}{dt^n}(t^n e^{-t}) = \sum_{k=0}^{n} C(n,k)(-1)^k t^k \frac{n!}{k!}e^{-t}.$$

Then we find

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}) = \sum_{k=0}^{n} C(n,k)(-1)^k \frac{t^k}{k!}.$$

Hence we evaluate the generating function as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n(t)w^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} C(n,k)(-1)^k \frac{t^k}{k!}w^n \\
&= (-1)^0 C(0,0)\frac{t^0}{0!}w^0 + \sum_{k=0}^{1} C(1,k)(-1)^k \frac{t^k}{k!}w + \dots + \sum_{k=0}^{n} C(n,k)(-1)^k \frac{t^k}{k!}w^n + \dots \\
&= (-1)^0 \frac{t^0}{0!}[C(0,0)w^0 + C(1,0)w + \dots + C(n,0)w^n + \dots] \\
&+ (-1)^1 \frac{t}{1!}[C(1,1)w + C(2,1)w^2 + \dots + C(n,1)w^n + \dots] \\
&+ (-1)^2 \frac{t^2}{2!}[C(2,2)w^2 + C(3,2)w^3 + \dots + C(n,2)w^n + \dots] \\
&+ \dots \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{t^k}{k!}\sum_{n=k}^{\infty} C(n,k)w^n.
\end{aligned}$$

Let us observe the following identity

$$\begin{aligned}\sum_{n=k}^{\infty} C(n,k)w^n &= \sum_{n=0}^{\infty} C(n+k,k)w^{n+k}\\ &= w^k \sum_{n=0}^{\infty} C(n+k,k)w^n\\ &= w^k \sum_{n=0}^{\infty} C(n+k,n)w^n\\ &= w^k[C(k,o)w^0 + C(k+1,1)w + ... + C(k+n,n)w^n + ...].\end{aligned}$$

Consider the function $f(w) = \dfrac{1}{(1-w)^{k+1}} = (1-w)^{-k-1}$. The first three derivatives of $f$ at 0 are $\frac{df}{dw}(0) = k+1$, $\frac{d^2f}{dw^2}(0) = (k+1)(k+2)$, $\frac{d^3f}{dw^3}(0) = (k+1)(k+2)(k+3)$. Then the Maclaruen series of $f$ is

$$\begin{aligned}f(w) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} w^n\\ &= 1 + (k+1)w + \frac{(k+1)(k+2)}{2!} w^2 + ... + \frac{\prod_{m=1}^{n}(k+m)}{n!} w^n + ...\\ &= \sum_{n=0}^{\infty} C(n+k,n)w^n.\end{aligned}$$

Then we conclude that the generating function of Laguerre polynomials for $\nu = 0$ is

$$\begin{aligned}\sum_{n=0}^{\infty} L_n(t)w^n &= \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \frac{w^k}{(1-w)^{k+1}}\\ &= \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{tw}{(1-w)}\right)^k}{k!} \frac{1}{(1-w)}\\ &= e^{-\frac{tw}{1-w}} \frac{1}{1-w}.\end{aligned}$$

## References

[1] Junghanns Peter, *Orthogonal Polynomials, Lecture Notes*, Summer Term 2012

5/5

# GENERATING FUNCTION FOR HERMITE POLYNOMIALS

BEKIR DANIS

## Generating function for hermite polynomials

We know that the rodriguez type formula for hermite poylnomials is $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ by taking $s(x) = 1$ and $w(x) = e^{-x^2}$ ( $x \in [-\infty, \infty]$). This can be found in the book [1].

We define a function $f$ as $e^{-x^2}$. This is just for convenience. $\frac{d^n}{dx^n} e^{-x^2} = f^n(x) = (-1)^n e^{-x^2} H_n(x)$. We know $f$ is entire function so we can use the taylor series expansion of $f$ at 0.

We write the taylor expansion of $f(x+t)$ for any $t$.

$$f(x+t) = \sum_0^\infty \frac{t^n}{n!} f^{(n)}(x) = \sum_0^\infty \frac{t^n}{n!} (-1)^n e^{-x^2} H_n(x)$$

By taking $-t$ instead of $t$, we can get the following equation;

$$f(x-t) = \sum_0^\infty \frac{t^n}{n!} e^{-x^2} H_n(x)$$

Multiplying both sides by $e^{x^2}$, we get the following;

$$f(x-t)e^{x^2} = \sum_0^\infty \frac{t^n}{n!} H_n(x)$$

Remember that $f(x) = e^{-x^2}$. Thus, we have

$$e^{-(x-t)^2} e^{x^2} = e^{2xt-t^2} = \sum_0^\infty \frac{t^n}{n!} H_n(x)$$

In last equation, $H_n(x)$ correponds $\frac{\delta^n}{\delta t^n} e^{2xt-t^2}$ at $t = 0$ ( From Taylor series expansion). Hence, we can get all $H_n(x)$ from this function $e^{2xt-t^2}$.

As a result, $e^{2xt-t^2}$ is the generating function for hermite polynomials $H_n(x)$.

## References

[1] Logan, J.David, *Applied Mathematics*, John Willey and Sons, Inc., New York 1997 (Second Edition)

# MATH 543 BONUS PROJECT 1

CAN TURKUN

ABSTRACT. In this paper, we will obtain a generating function for Tchebychev Polynomials of the second kind, denoted $U_n(x)$, by the recursion formula they satisfy. The recursion formula will also be obtained.

## 1. INTRODUCTION

We know from the lecture that if $\omega(x)$=$\sqrt{1-x^2}$ and $s(x)$=$1-x^2$ then the Rodriguez formula is

$$\frac{1}{w(x)}\frac{d^n}{dx^n}(\omega(x)s^n(x)) = U_n(x) \text{ for n} = 0, 1, 2, ... \tag{1.1}$$

and

$$\|U_n(x)\|^2 = \int_{-1}^{1}\omega(x)U_n^2(x)dx = \frac{\pi}{2} \text{ hence } \|U_n(x)\| = \sqrt{\frac{\pi}{2}} \tag{1.2}$$

Computing some of the polynomials by the Rodriguez formula we get

$U_0(x) = 1$
$U_1(x) = 2x$
$U_2(x) = 4x^2 - 1$
$U_3(x) = 8x^3 - 4x$
$U_4(x) = 16x^4 - 12x^2 + 1$
$U_5(x) = 32x^5 - 32x^3 + 6x$

Thus, one can easily show that (by induction) the leading coefficient of $U_n(x)$ is $2^n$ and the coefficient of $x^{n-1}$ is 0. That is, if $U_n(x) = k_nx^n + k_n^{'}x^{n-1} + ...$ then $k_n = 2^n$ and $k_n^{'} = 0$.

## 2. FINDING GENERATING FUNCTION

Let $h_n = \|U_n(x)\|^2 = \int_{-1}^{1}\omega(x)U_n^2(x)dx$ then $h_n = \frac{\pi}{2}$ for n = 0, 1, 2, ... also let $k_n$ and $k_n^{'}$ is the same as above. Then we know from the lecture that $U_n(x)$ satisfy the following recursion relation:

$$U_{n+1}(x) - \frac{k_{n+1}}{k_n}xU_n(x) = a_n^nU_n(x) + a_{n-1}^nU_{n-1}(x) \tag{2.1}$$

where

$$a_{n-1}^{n} = -\frac{k_{n-1}k_{n+1}}{k_n^2}\frac{h_n}{h_{n-1}} \text{ and } a_n^n = \frac{k'_{n+1}}{k_n} - \frac{k_{n+1}}{k_n^2}k'_n$$

Thus we have

$$a_{n-1}^{n} = -\frac{2^{n-1}2^{n+1}}{2^{2n}}\frac{\frac{\pi}{2}}{\frac{\pi}{2}} = -1 \text{ and } a_n^n = \frac{0}{2^n} - \frac{2^{n+1}}{2^{2n}}0 = 0$$

Hence the recursion relation (2.1) turns into this recursion formula

$$U_{n+1}(x) - 2xU_n(x) = -U_{n-1}(x) \tag{2.2}$$

Multiplying the recursion formula (2.2) by $t^{n+1}$ and summing over $n = 1$ to $\infty$ we get

$$\sum_{n=1}^{\infty} t^{n+1}U_{n+1}(x) - 2xt\sum_{n=1}^{\infty} t^n U_n(x) = -t^2\sum_{n=1}^{\infty} t^{n-1}U_{n-1}(x) \tag{2.3}$$

By shifting the indices $n$ to $n+1$ in (2.3) we get

$$\sum_{n=0}^{\infty} t^{n+2}U_{n+2}(x) - 2xt\sum_{n=0}^{\infty} t^{n+1}U_{n+1}(x) = -t^2\sum_{n=0}^{\infty} t^n U_n(x) \tag{2.4}$$

Since the generating function is $g(x,t) = \sum_{n=0}^{\infty} t^n U_n(x)$ note that

$g(x,t) = U_0(x) + \sum_{n=1}^{\infty} t^n U_n(x) = 1 + \sum_{n=0}^{\infty} t^{n+1}U_{n+1}(x)$ by shifting the indices and using $U_0(x) = 1$. Also,

$g(x,t) = U_0(x) + tU_1(x) + \sum_{n=2}^{\infty} t^n U_n(x) = 1 + 2xt + \sum_{n=0}^{\infty} t^{n+2}U_{n+2}(x)$ by shifting the indices and using $U_1(x) = 2x$.

So we find $\sum_{n=0}^{\infty} t^{n+1}U_{n+1}(x) = g(x,t) - 1$ and $\sum_{n=0}^{\infty} t^{n+2}U_{n+2}(x) = g(x,t) - 2xt - 1$

putting these into (2.4) we get

$$g(x,t) - 1 - 2xt - 2xt(g(x,t) - 1) = -t^2 g(x,t) \tag{2.5}$$

solving this equation for $g(x,t)$ we get

$$g(x,t) - 1 - 2xt - 2xtg(x,t) + 2xt = -t^2 g(x,t)$$

$$g(x,t) - 1 - 2xtg(x,t) = -t^2 g(x,t)$$

$$g(x,t) - 2xtg(x,t) + t^2 g(x,t) = 1$$

$$g(x,t)(1 - 2xt + t^2) = 1$$

$$g(x,t) = \frac{1}{1 - 2xt + t^2} \tag{2.6}$$

Consequently, we find the generating function for Tchebychev Polynomials of the second kind,$U_n(x)$, which is shown as in (2.6)

$$g(x,t) = \sum_{n=0}^{\infty} t^n U_n(x) = \frac{1}{1 - 2xt + t^2}$$

# GENERATING FUNCTION OF FIRST KIND CHEBYSHEV POLYNOMIALS

BURAK HATİNOĞLU

In this project we will find generating function of first kind Chebyshev polynomials by using the identity $T_n(x) = \cos n\theta$, where $x = \cos\theta$, $\theta \in [0, \pi]$ *and* $n \in \mathbb{N}_0 := \{0, 1, 2...\}$. So let us first prove this identity by induction. The Rodrigues formula of first kind Chebyshev polynomials is as follows:

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n}[(1-x^2)^{n-\frac{1}{2}}]$$

First let us show the case $n = 0$.

$$T_0(x) = (1-x^2)^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}} = 1 = \cos 0$$

Assume $T_n(x) = \cos n\theta$ for some fixed $n \in \mathbb{N}$. Note that $\frac{dx}{d\theta} = -\sin\theta$. Let us show for $n+1$.

$$\begin{aligned} T_{n+1}(x) &= \frac{(-1)^{n+1} 2^{n+1} (n+1)!}{(2n+2)!}(1-x^2)^{\frac{1}{2}} \frac{d^{n+1}}{dx^{n+1}}[(1-x^2)^{n+\frac{1}{2}}] \\ &= \frac{(-1)^{n+1} 2^n (n)!}{(2n+1)!}(1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n}[(n+\frac{1}{2})(1-x^2)^{n-\frac{1}{2}}(-2x)] \\ &= \frac{(-1)^n 2^n (n)!}{(2n)!}(1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n}[x(1-x^2)^{n-\frac{1}{2}}] \end{aligned}$$

Here we should observe the follwing identity.

$$\begin{aligned} \frac{d^n}{dx^n}[x(1-x^2)^{n-\frac{1}{2}}] &= \frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}] + \frac{d^{n-1}}{dx^{n-1}} x \frac{d}{dx}[(1-x^2)^{n-\frac{1}{2}}] \\ &= 2\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}] + \frac{d^{n-2}}{dx^{n-2}} x \frac{d^2}{dx^2}[(1-x^2)^{n-\frac{1}{2}}] \\ &= 3\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}] + \frac{d^{n-3}}{dx^{n-3}} x \frac{d^3}{dx^3}[(1-x^2)^{n-\frac{1}{2}}] \\ &= \cdots \\ &= n\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}] + x\frac{d^n}{dx^n}[(1-x^2)^{n-\frac{1}{2}}] \end{aligned}$$

Hence

$$T_{n+1}(x) = \frac{(-1)^n 2^n (n)!}{(2n)!}(1-x^2)^{\frac{1}{2}}\{n\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}] + x\frac{d^n}{dx^n}[(1-x^2)^{n-\frac{1}{2}}]\}$$
$$= \sin\theta\frac{(-1)^n 2^n (n)!}{(2n)!}n\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}] + \cos\theta\cos n\theta,$$

by our assumption and $x = \cos\theta$. Let us also observe the following identity:

$$\int \cos n\theta d\theta = \int \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{\frac{1}{2}}\frac{d^n}{dx^n}[(1-x^2)^{n-\frac{1}{2}}]d\theta$$
$$= \frac{(-1)^n 2^n n!}{(2n)!}\int \sin\theta\frac{d}{-\sin\theta d\theta}[\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}]]d\theta$$
$$= -\frac{(-1)^n 2^n n!}{(2n)!}\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}]$$

Therefore we get the following identity:

$$-\sin n\theta = \frac{(-1)^n 2^n n!}{(2n)!}n\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}]$$

This identity gives us the result.

$$T_{n+1}(x) = \sin\theta\frac{(-1)^n 2^n (n)!}{(2n)!}n\frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}] + \cos\theta\cos n\theta$$
$$= \cos\theta\cos n\theta - \sin\theta\sin n\theta$$
$$= \cos(n+1)\theta$$

This completes proof, so $T_n(x) = \cos n\theta$, where $x = \cos\theta$, $\theta \in [0,\pi]$ *and* $n \in \mathbb{N}_0 := \{0, 1, 2...\}$. Hence for any $n$ we have

$$Re(t^n e^{in\theta}) = Re(t^n \cos n\theta + it^n \sin n\theta) = t^n T_n(x).$$

It is natural to ask is this equality also valid for infinite sum, i.e. is the following equality valid:

$$Re(\sum_{n=0}^{\infty} t^n e^{in\theta}) = \sum_{n=0}^{\infty} t^n T_n(x)$$

where $x = \cos\theta$.

Let us find the value of the infinite sum on the left hand side.

$$\sum_{n=0}^{\infty} t^n e^{in\theta} = \sum_{n=0}^{\infty}(te^{i\theta})^n = \lim_n \frac{1-(te^{i\theta})^n}{1-te^{i\theta}}$$

where $n$ goes to plus infinity, but this limit is finite only if $-1<t<1$. Hence take $-1<t<1$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} t^n e^{in\theta} &= \frac{1}{1-te^{i\theta}} \\ &= \frac{1}{1-t\cos\theta - it\sin\theta} \\ &= \frac{1-t\cos\theta + it\sin\theta}{(1-t\cos\theta - it\sin\theta)(1-t\cos\theta + it\sin\theta)} \\ &= \frac{1-t\cos\theta + it\sin\theta}{1-2t\cos\theta + t^2}\end{aligned}$$

where $x = \cos\theta$. Hence for $-1<t<1$ we get the following result and find generating function of Chebyshev polynomials of first kind.

$$\sum_{n=0}^{\infty} t^n T_n(x) = Re(\sum_{n=0}^{\infty} (te^{i\theta})^n) = \frac{1-tx}{1-2tx+t^2} = g(x,t).$$

## References

[1] Cheney E.W., *Introduction to Approximation Theory*, Chelsea Publishing Company, 1982.
[2] Dennery P., Krzywicki A., *Mathematics for Physicists*, Dover Publications, 1995.

Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey
*E-mail address*: burak.hatinoglu@bilkent.edu.tr