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SOME RELATIONS ON HERMITE MATRIX POLYNOMIALS

Levent Kargin and Veli Kurt

Department of Mathematics, Faculty Science, University of Akdeniz TR-07058, Antalya, Turkey.
lkargin@akdeniz.edu.tr , vkurt@akdeniz.edu.tr

Abstract- In this study we give addition theorem, multiplication theorem and summation formula for Hermite matrix polynomials. We write Hermite matrix polynomials as hypergeometric matrix functions. We also obtain a new generating function for Hermite matrix polynomials and using this function, we prove some new results and relations.

Key Words- Hypergeometric matrix functions, Hermite matrix polynomials, generating matrix functions.

1. INTRODUCTION

In the recent two decades, orthogonal matrix polynomials comprise an emerging field of study, with important results in both theory and applications continuing to appear in the literature. Hermite matrix polynomials are introduced by Jodar and Company in [9]. Moreover, some properties of the Hermite matrix polynomials are given in [3, 14, 15] and a generalized form of the Hermite matrix polynomials have been introduced and studied in [16, 17, 19, 20, 22, 24]. Other classical orthogonal polynomials as Laguerre, Gegenbauer, Chebyshev and Jacobi polynomials have been extended to orthogonal matrix polynomials, and some results have been investigated, see for example [4, 5, 8, 21, 23]. From the connection with orthogonal matrix polynomials, special matrix functions have been introduced and studied by some mathematicians. Gamma matrix function is introduced and studied in [7, 10] for matrices in $\mathbb{C}^{r\times r}$ whose eigenvalues are all in the right open half-plane. Apart from the close relationships with the well-known beta and gamma matrix functions, the emerging theory of orthogonal matrix polynomials and its operational calculus suggest the study of hypergeometric matrix function. Hypergeometric matrix function $F(-;A;z)$ has been recently introduced in [13]. Explicit closed form for general solutions of the hypergeometric matrix differential equations is given in [12]. The paper is organized as follows. In the next section we deal with important properties of the Hermite matrix polynomials such as addition, multiplication theorems and summation formula. We obtain a generating function for Hermite matrix polynomials and write these polynomials as hypergeometric matrix functions. We obtain some results which follow from this generating function.

Throughout this paper, if A is a matrix in $\mathbb{C}^{r\times r}$, its spectrum (A) will denotes the set of all the eigenvalues of A. Its 2-norm will be denoted by $\|A\|$ and defined by

$$\|A\| = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2},$$

where for a y in $\mathbb{C}^r$, $\|y\|_2 = (y^T,y)^{1/2}$ is the Euclidean norm of y. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathbb{C}^{r\times r}$ such that $\sigma(A)\subset\Omega$, then from the properties of matrix functional calculus [6, page 558], it follows that

$$f(A)g(A)=\ g(A)f(A).$$

If D_0 is the complex plane cut along the negative real axis and $\log z$ denotes the principle logarithm of z, then $z^{1/2}$ represents $exp\left(\frac{1}{2}\log z\right)$. If A is a matrix $\mathbb{C}^{r\times r}$ in which $\sigma(A)\subset D_0$ then $A^{1/2}=\sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A.

Let A be a matrix in $\mathbb{C}^{r\times r}$ such that

$$Re(z)>0\ for\ every\ eigenvalues\ z\in\sigma(A), \tag{1}$$

Then the Hermite matrix polynomials $H_n(x,A)$ are defined by [9]

$$H_n(x,A)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k n!\left(x\sqrt{2A}\right)^{n-2k}}{k!\,(n-2k)!},\qquad n\geq 0 \tag{2}$$

and satisfying the three-terms recurrence relation

$$H_n(x,A)=\ x\sqrt{2A}H_{n-1}(x,A)-2(n-1)H_{n-2}(x,A),\qquad n\geq 1.$$

$$H_{-1}(x,A)=0,\qquad H_1(x,A)=I,$$

where I is the unit matrix in $\mathbb{C}^{r\times r}$. According to [9], we have

$$\sum_{n=0}^{\infty}\frac{H_n(x,A)}{n!}t^n=exp\left(xt\sqrt{2A}-t^2\right). \tag{3}$$

The Pochhammer symbol or shifted factorial is defined by [11]

$$(A)_n=(A)(A+I)\cdots(A+(n-1)I),\quad n\geq 1, \tag{4}$$

with $(A)_0=I$. By using (4) it is easy to show that

$$(A)_{2n}=2^{2n}\left(\frac{A}{2}\right)_n\left(\frac{A+I}{2}\right)_n. \tag{5}$$

The hypergeometric matrix function $F(A,B;C;z)$ has been given in [11]

$$F(A,B;C;z)=\sum_{n=0}^{\infty}\frac{(A)_n(B)_n[(C)_n]^{-1}}{n!}z^n,\quad |z|<1,$$

where A,B,C are matrices in $\mathbb{C}^{r\times r}$ such that

$$C+nI\ \text{is invertible for all integers}\ n\geq 0.$$

Note that by (4) if $A=-iI$ where i is a natural number then $(A)_{i+j}=0$ for $j\geq 1$ and $F(A,B;C;z)$ becomes a matrix polynomial of degree i.

Lemma 1: ([18]) Let $\|.\|$ denotes any matrix norm for which $\|I\|=1$. If $\|M\|<1$ for a matrix M in $\mathbb{C}^{r\times r}$, then $(I+M)^{-c}$ exists and given by

$$(I-M)^{-c}=\sum_{n=0}^{\infty}\frac{(c)_n M^n}{n!},$$

where c is a positive integer.

We conclude this section recalling a result related to the rearrangement of the terms in iterated series. If $\mathrm{A}(k,n)$ and $\mathrm{B}(k,n)$ are matrices in $\mathbb{C}^{r\times r}$ for $n\geq 0$, $k\geq 0$, then in an analogous way to the proof of Lemma 11 of [22], it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{A}(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \mathrm{A}(k,n-2k), \tag{6}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{B}(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathrm{B}(k,n-k). \tag{7}$$

2. SOME RELATIONS ON HERMITE MATRIX POLYNOMIALS

Proposition 2: Hermite matrix polynomials satisfy the multiplication and addition formula as follows:

$$H_n(\mu x, A) = \mu^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{k!\,(n-2k)!}\left(1-\frac{1}{\mu^2}\right)^k H_{n-2k}(x,A), \tag{8}$$

$$\left(\mu^2+\lambda^2\right)^{n/2} H_n\left(\frac{\lambda z_1+\mu z_2}{\left(\mu^2+\lambda^2\right)^{1/2}}, A\right) = \sum_{k=0}^{n}\binom{n}{k}\lambda^k \mu^{n-k} H_k(z_1,A)H_{n-k}(z_2,A) \tag{9}$$

where μ and λ are constants.

Proof: Taking μx for x and $\frac{t}{\mu}$ for t in (3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(\mu x, A)\frac{t^n}{\mu^n n!} &= exp\left(xt\sqrt{2A}-\left(\frac{t}{\mu}\right)^2\right)\\ &= exp\left(xt\sqrt{2A}-t^2+t^2-\left(\frac{t}{\mu}\right)^2\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} H_n(x,A)\left(1-\frac{1}{\mu^2}\right)^k \frac{t^{n+2k}}{n!\,k!}.\end{aligned}$$

By using (6) and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get (8). We can similarly prove the equation (9).

Corollary 3: The Hermite matrix polynomials have the following relations:

$$2^{n/2} H_n\left(\frac{z_1+z_2}{\sqrt{2}}, A\right) = \sum_{k=0}^{n}\binom{n}{k} H_k(z_1,A)H_{n-k}(z_2,A), \tag{10}$$

$$2^{n/2} H_n\left(\sqrt{2}x, A\right) = \sum_{k=0}^{n}\binom{n}{k} H_k(x,A)H_{n-k}(y,A), \tag{11}$$

$$2^{n/2} H_n(x+y, A) = \sum_{k=0}^{n}\binom{n}{k} H_k\left(\sqrt{2}x,A\right)H_{n-k}\left(\sqrt{2}y,A\right). \tag{12}$$

Proposition 4: The summation formulas for $H_n(x,A)$ are given as follows:

$$H_n(x,A)H_m(x,A) = m!\,n! \sum_{k=0}^{\min(m,n)} \frac{2^k \mathrm{H}_{m+n-2k}(x,A)}{(m-k)!\,(n-k)!\,k!} \tag{13}$$

and for $x \neq y$,

$$\sum_{m=0}^{n} \frac{\sqrt{2A} H_m(x,A)\, H_m(y,A)}{2^{m+1} m!} = \frac{H_{n+1}(y,A)H_n(x,A) - H_{n+1}(x,A)H_n(y,A)}{2^{n+1} n!\,(y-x)}. \tag{14}$$

Proof: Using (3), we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_n(x,A)\, H_m(x,A) \frac{u^n}{n!}\frac{v^m}{m!} = \exp(\sqrt{2}Ax(u+v) - (u+v)^2 + 2uv$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{2^m u^m v^m}{m!} H_n(x,A) \frac{(u+v)^n}{n!}.$$

Making necessary arrangement and comparing the coefficients of $\frac{u^n}{n!}\frac{v^m}{m!}$ completes the proof of (13). Equation (14) can be easily proved by using the three-terms recurrence relation for Hermite matrix polynomials.

Proposition 5: Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying condition (1) and $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$. Then Hermite matrix polynomials have the following generating function:

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x,A)}{n!} t^n = (I - xt\sqrt{2A})^{-c} F\left(\frac{cI}{2}, \frac{(c+1)I}{2}; -; -4t^2 \left(I - xt\sqrt{2A}\right)^{-2}\right),$$

where c is a positive integer and $|t| < 1, |x| < 1$.

Proof: Using (2) and (6), we have

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x,A)}{n!} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\, (c)_n (x\sqrt{2A})^{n-2k}}{k!\,(n-2k)!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^k (c+2kI)_n (xt\sqrt{2A})^n (c)_{2k} t^{2k}}{k!\,n!}.$$

By using Lemma 1, we have

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x,A)}{n!} t^n = \sum_{k=0}^{\infty} \frac{(-1)^k (c)_{2k}}{k!} (I - xt\sqrt{2A})^{-(c+2k)} t^{2k}.$$

Using the relation (5) and making necessary arrangement, completes the proof.

Theorem 6: Let A be a invertible matrix in $\mathbb{C}^{r\times r}$ satisfying condition (1). Then Hermite matrix polynomials $H_n(x,A)$ can be writen as hypergeometric matrix functions:

$$H_n(x,A) = \left(x\sqrt{2A}\right)^n F\left(\frac{-nI}{2}, \frac{(1-n)I}{2}; -; \frac{-2A^{-1}}{x^2}\right). \tag{15}$$

Proof: From (2), we have

$$H_n(x,A) = \left(x\sqrt{2A}\right)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(-nI)_{2k}(2A)^{-k}}{k!\,x^{2k}}.$$

Since $\frac{n!}{(n-2k)!} = (-nI)_{2k}$, using the relation (5) we get (15).

Theorem 7: Let A be a invertible matrix in $\mathbb{C}^{r\times r}$ satisfying condition (1). For k $\in \mathbb{Z}^+$, Hermite matrix polynomials have the following generating function:

$$\sum_{n=0}^{\infty} H_{n+k}(x,A)\frac{t^n}{n!} = \exp(xt\sqrt{2A} - t^2)H_k\left(x - \left(\sqrt{\frac{A}{2}}\right)^{-1} t, A\right). \qquad (16)$$

Proof : By using (7), we have

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} H_{n+k}(x,A)\frac{t^n}{n!}\frac{u^k}{k!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{H_n(x,A)t^{n-k}u^k}{k!\,(n-k)!}\\ &= \sum_{n=0}^{\infty} H_n(x,A)\frac{(t+u)^n}{n!}\\ &= \exp(xt\sqrt{2A} - t^2)\sum_{k=0}^{\infty} H_k\left(x - \left(\sqrt{\frac{A}{2}}\right)^{-1} t, A\right)\frac{u^k}{k!}.\end{aligned}$$

By comparing the coefficients of $\frac{u^k}{k!}$, we obtain (16).

As an example of equation (16), let us derive the following theorem:

Theorem 8: Let A be a invertible matrix in $\mathbb{C}^{r\times r}$ satisfying condition (1). Hermite matrix polynomials satisfy the following relation:

$$\sum_{k=0}^{\infty} H_n(x,A)H_n(y,A)\frac{t^n}{n!} = \left(1 - 4t^2\right)^{-\frac{1}{2}}\exp\left(\frac{2A\,(xyt - (x^2 + y^2)t^2)}{1 - 4t^2}\right), \qquad (17)$$

where $|t| < \frac{1}{2}$.

Proof: By using (6) and (16), we have

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(x,A)H_n(y,A)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\left(x\sqrt{2A}\right)^{n-2k}H_n(y,A)t^n}{k!\,(n-k)!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{H_{n+2k}(y,A)\left(x\sqrt{2A}\right)^n}{n!}\frac{(-1)^k t^{2k}}{k!}\\ &= \sum_{n=0}^{\infty} \frac{\exp(2\,Axyt - 2Ax^2t^2)^k H_{2k}(y - 2xt, A)\,(-1)^k\,t^2}{k!}.\end{aligned}$$

Since

$$H_{2k}(y-2xt,A)=\sum_{s=0}^{k}\frac{(-1)^s(2k)!\,(y-2xt)^{2k-2s}\left(\sqrt{2A}\right)^{2k-2s}}{s!\,(2k-2s)!},$$

and $(2k)!=(1)_{2k}=2^{2k}k!\left(\frac{1}{2}\right)_k$, it follows that

$$\sum_{n=0}^{\infty}H_n(x,A)H_n(y,A)\frac{t^n}{n!}$$
$$=(1-4t^2)^{-\frac{1}{2}}\exp\left(2\mathrm{Axyt}-2Ax^2t^2\right)\exp\left(\frac{-2At^2(y-2xt)^2}{1-4t^2}\right).$$

Combining the exponential factors, we arrive at (17).

Theorem 9: Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying condition (1), $\left\|\sqrt{A}\right\|<\frac{1}{\sqrt{2}}$ and c be a positive integer. Hermite matrix polynomials satisfy the following relation:

$$\sum_{n=0}^{\infty}F\ (nI,c;-;y)H_n(x,A)\frac{t^n}{n!}=\exp(\mathrm{xt}\sqrt{2\mathrm{A}}-t^2)(I+xyt\sqrt{2A}-2yt^2)^{-c}$$
$$\times F\left(\frac{cI}{2},\frac{(c+1)I}{2};-;-4y^2t^2(I+xyt\sqrt{2A}-2yt^2)^{-2}\right).$$

Proof: Applying equation (16) to Proposition 5, we complete the proof.

3. CONCLUSIONS

In this paper, we carry the properties of classical scalar Hermite polynomials to the Hermite matrix polynomials. Equation (12) is the matrix analog of the Runge addition formula of the scalar Hermite polynomials. For the case $A=[2]_{1\times1}$, the expression (13) concides with the formula which was proved by Feldheim for classical scalar Hermite polynomials. Also Proposition 5 is the matrix analogous of the Bateman's generating relation for classical scalar Hermite polynomials given in [1]. Replacing *t* with $\frac{t}{2}$ in (17), we give another proof for the equation (41) in [14]. Theorem 9 is the matrix analog of the Brafman's relation for classical scalar Hermite polynomials in [2].

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