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SOME PROPERTIES OF ORTHOGONAL POLYNOMIALS RELATED TO HERMITE POLYNOMIALS

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Some generating relations for certain orthogonal polynomials related to Hermite polynomials (introduced by Thakare and Karande 1973) are obtained. {See also the various remarks and comments surrounding the obvious connections (2.1) and (2.2) of the so-called "generalized" Hermite polynomials with the classical Laguerre polynomials.} In deriving these results here the use of certain operational techniques is indicated.

§1. Recently, Thakare and Karande (1973) studied the class of orthogonal polynomials related to Hermite polynomials by considering the following self-adjoint differential equations:

$$D\,[e^{-x^{2k}}Dy] + \lambda x^{2k-2}\, e^{-x^{2k}}\, y = 0 \qquad \ldots(1.1)$$

$$D \equiv d/dx;\; k = 1, 2, 3, \ldots .$$

They denoted the polynomial solutions of (1.1) by $H_{2pk}(x; k)$ and $H_{2pk+1}(x; k)$ for even and odd polynomials, respectively.

In course of this discussion they obtained various properties of the aforesaid polynomials such as orthogonality relation, Rodrigues formulae, hypergeometric forms, recurrence relations, linear generating relations, integrals, expansion and some operational formulae. The interesting feature regarding these polynomials is that there are two families of differential equations, recurrence relations and hypergeometric representations corresponding to even and odd polynomials. The object of the present note is to mention some more properties such as linear, bilinear generating relations and integral representations for the aforesaid polynomials.

§2. The so-called "generalized" even and odd Hermite polynomials considered by Thakare and Karande (1973) are constant multiples of the classical Laguerre polynomials. Specifically, we have the following connecting relationships:

$$H_{2pk}(z, \beta) = (-1)^p\, 2^{2p}(\tfrac{1}{2})_p\, \{(1 + \beta)_p\}^{-1}\, L_p^{(\beta)}(z) \qquad \ldots(2.1)$$

and

$$H_{2pk+1}(z, \beta) = (-1)^p\, 2^{2p+1}(\tfrac{3}{2})_p\, \{(1 - \beta)_p\}^{-1}\, z^{-\beta}\, L_p^{(-\beta)}(z) \qquad \ldots(2.2)$$

where $\beta = -\dfrac{1}{2k}$ and $z = x^{2k}$.

In view of (2.1) and (2.2), the results of the paper by Thakare and Karande (1973) are trivial consequences of the corresponding well-known results involving Laguerre polynomials. Moreover, the connections (2.1) and (2.2) show their usefulness in trivially deriving generating relations and other results from the corresponding well-known results involving Laguerre polynomials. Consequently, we mention the following results:

$$\sum_{p=0}^{\infty} \frac{(1+\beta)_p}{(\frac{1}{2})_p} H_{2pk}(x;k) \frac{t^p}{p!} = (1+4t)^{-\beta-1} \exp\left(\frac{4x^{2k}t}{1+4t}\right) \qquad \text{...(2.3)}$$

$$\sum_{p=0}^{\infty} \frac{(1-\beta)_p}{(\frac{3}{2})_p} H_{2pk+1}(x;k) \frac{t^p}{p!} = 2(x^{2k})^{-\beta} (1+4t)^{\beta-1} \,.\, \exp\left(\frac{4x^{2k}t}{1+4t}\right) \qquad \text{...(2.4)}$$

$$\sum_{p=0}^{\infty} \frac{(1+\beta+q)_p}{(\frac{1}{2}+q)_p} H_{2pk+2qk}(x;k) \frac{t^p}{p!} = (1+4t)^{-\beta-q-1} \times \exp\left(\frac{4x^{2k}t}{1+4t}\right) H_{2qk}\left[\frac{x}{(1+4t)^{1/2k}};k\right] \qquad \text{...(2.5)}$$

$$\sum_{p=0}^{\infty} \frac{(1-\beta+q)_p}{(\frac{3}{2}+q)_p} H_{2pk+2qk+1}(x;k) \frac{t^p}{p!} = (1+4t)^{-q-1} \times \exp\left(\frac{4x^{2k}t}{1+4t}\right) H_{2qk+1}\left[\frac{x}{(1+4t)^{1/2k}};k\right]. \qquad \text{...(2.6)}$$

We remark that the results (2.3) to (2.6) can also be established directly by making use of the operational formulae (Thakare and Karande 1973, §10, p. 68) and the following properties of operator $\delta \equiv xD_x$:

$$a^{\delta} f(x) = f(ax)$$

$$(1+t)^{-\delta-\alpha} f(x) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (\delta+\alpha)_n f(x)$$

and

$$(1+t)^{-\delta-\alpha} = (1+t)^{-\delta} (1+t)^{-\alpha}.$$

§3. Lastly, we mention the following integral representation for Laguerre polynomials in terms of $H_{2pk}(z)$:

$$L_p^{(\alpha)}(x) = \frac{\Gamma(p + \alpha + 1)}{\Gamma(p + \beta + 1)\,\Gamma(\alpha - \beta)}\, x^{-\alpha} \int_0^x (x - z)^{\alpha-\beta-1} z^{\beta}$$

$$\times \frac{(1 + \beta)_p\,(-1)^p}{p!(p + 1)_p}\, H_{2pk}(z)\, dz. \qquad ...(3.1)$$

From Kogbetliantz (1932, p. 156), we have

$$L_p^{(\alpha)}(x) = \frac{\Gamma(p + \alpha + 1)}{\Gamma(p + \nu + 1)\,\Gamma(\alpha - \nu)}\, x^{-\alpha} \int_0^x (x - u)^{\alpha-\nu-1} u^{\nu}\, L_p^{(\nu)}(u)\, du.$$

Taking $\nu = \beta$ and $u = z$, in the above relation, we get

$$L_p^{(\alpha)}(x) = \frac{\Gamma(p + \alpha + 1)\, x^{-\alpha}}{\Gamma(p + \beta + 1)\,\Gamma(\alpha - \beta)} \int_0^x (x - z)^{\alpha-\beta-1} z^{\beta}\, L_p^{(\beta)}(z)\, dz. \qquad ...(3.2)$$

Now in view of (2.1) the result (3.1) follows from (3.2).

If we take $\beta = -1/2k$, $z = x^{2k}$ and $k = 1$, we get an unnoticed result for even Hermite polynomials. Similar result for the odd polynomials can also be established.

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