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Certain Theorems on Bilateral Generating Functions Involving Hermite, Laguerre, and Gegenbauer Polynomials

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The object of this paper is to present simpler proofs of the various generalizations of some interesting results on bilateral generating functions which were derived recently by group-theoretic methods. It is also shown how one of our main theorems on generating functions would apply not only to the Bessel function $J_\nu(x)$, but indeed also to the Konhauser biorthogonal polynomials $Y_n^\alpha(x; s)$ whose special case when $s=2$ was encountered in certain analytical calculations involving the penetration of gamma rays through matter.

1. INTRODUCTION AND PRELIMINARIES

An interesting generalization of the classical Hermite polynomials is due to Gould and Hopper [3] who introduced the polynomials

$$H_n^\gamma(x, \alpha, \beta) = (-1)^n x^{-\alpha} \exp(\beta x^\gamma) D_x^n\{x^\alpha \exp(-\beta x^\gamma)\}, \tag{1.1}$$

where $D_x = d/dx$, and the parameters α, β, and γ are unrestricted, in general. In fact, in terms of the classical Hermite polynomials, it is easily seen that

$$H_n^2(x, 0, 1) = H_n(x) \qquad (n = 0, 1, 2, \ldots). \tag{1.2}$$

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Making use of group-theoretic methods (see, e.g., [6, Chaps. 2 and 3; 15, Chap. 6]), Shrivastava and Kaur [9] established two theorems on bilateral generating functions involving the (Gould–Hopper) generalized Hermite polynomials (1.1), the Laguerre polynomials $L_n^{(\alpha)}(x)$, and the Gegenbauer (or ultraspherical) polynomials $C_n^\nu(x)$ (cf. [8, 18]). In our attempt to give relatively simpler proofs of their results, without using group-theoretic methods, we noticed an error in one of their main assertions appearing throughout their paper [9]; more importantly, we were led in this manner to the various generalizations of each of their theorems, which we shall present in Sections 2 and 3. We shall also give several interesting applications of our main classes of bilateral generating functions.

For the sake of ready reference, we choose to recall here the main results of Shrivastava and Kaur [9] in the following (modified, corrected, and substantially improved) forms:

THEOREM A. *In terms of the Gould–Hopper polynomials defined by* (1.1), *let*

$$F_N(x, y, t) = \sum_{n=0}^{\infty} a_n H_n^\gamma(x, \alpha, \beta)\, L_N^{(\lambda+n)}(y)\, t^n \qquad (a_n \neq 0), \tag{1.3}$$

where α, β, γ, *and* λ *are arbitrary* (*real or complex*) *parameters. Also let*

$$f_{m,n}(z, w, x) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k z^{m-k} w^{n-k}}{k!\,(n-k)!}\, a_{m-k} H_{m+n-2k}^\gamma(x, \alpha, \beta). \tag{1.4}$$

Then

$$\sum_{m,n=0}^{\infty} f_{m,n}(z, w, x)\, L_N^{(\lambda+m)}(y)\, t^m$$
$$= x^{-\alpha}(x-w)^\alpha \exp(\beta\{x^\gamma - (x-w)^\gamma\} - t)\, F_N(x-w, y+t, zt), \tag{1.5}$$

provided that each member exists.

THEOREM B. *Suppose that there exists a bilateral generating function in the form*:

$$G_{M,N}(x, y, t) = \sum_{n=0}^{\infty} a_n C_M^{\nu+n}(x)\, L_N^{(\alpha+n)}(y)\, t^n \qquad (a_n \neq 0), \tag{1.6}$$

where ν and α are arbitrary (real or complex) parameters. Also let

$$g_{m,n}(z, w, x) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k z^{m-k} w^{n-k}}{k!\,(n-k)!} \cdot a_{m-k}(\nu+m-k)_{n-k}\, C_M^{\nu+m+n-2k}(x), \tag{1.7}$$

where, as usual, $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$.
Then

$$\sum_{m,n=0}^{\infty} g_{m,n}(z, w, x)\, L_N^{(\alpha+m)}(y)\, t^m = (1-w)^{-\nu-M/2} \exp(-t)\, G_{M,N}\left[\frac{x}{\sqrt{1-w}}, y+t, \frac{zt}{1-w}\right], \tag{1.8}$$

provided that each member exists.

It may be of interest to remark in passing that the parameters M and N in the above theorems need not necessarily be constrained to take on non-negative integer values. Thus, in general, Theorems A and B hold true for the Laguerre and Gegenbauer (or ultraspherical) functions.

In its special case when

$$a_n = 1\ (n=0, 1, 2, \ldots); \qquad w=t; \qquad \lambda=0; \qquad N=0, 1, 2, \ldots, \tag{1.9}$$

Theorem A corresponds essentially to Theorem I of Shrivastava and Kaur [9]. On the other hand, Theorem B with

$$a_n = 1\ (n=0, 1, 2, \ldots); \qquad w=2t; \qquad \nu=\alpha=0; \qquad M,N=0, 1, 2, \ldots, \tag{1.10}$$

and with t in (1.8) replaced by ω, would yield in the corrected version of Theorem II of Shrivastava and Kaur [9].

Our direct proof of Theorem A, without using group-theoretic methods employed by the earlier authors [9] in the special case when the conditions listed in (1.9) are satisfied, is based upon the known generating function [3, p. 57, Eq. (5.3)]:

$$\sum_{n=0}^{\infty} H_{m+n}^{\gamma}(x, \alpha, \beta) \frac{t^n}{n!} = (1-t/x)^{\alpha} \exp(\beta x^{\gamma}\{1-(1-t/x)^{\gamma}\})\, H_m^{\gamma}(x-t, \alpha, \beta) \qquad (|t| < |x|), \tag{1.11}$$

and the well-known identity (cf. [1, p. 142, Eq. (18)]; see also [15, p. 172, Problem 22(ii)]):

$$e^{-t} L_N^{(\alpha)}(x+t) = \sum_{n=0}^{\infty} L_N^{(\alpha+n)}(x) \frac{(-t)^n}{n!}, \tag{1.12}$$

which follows immediately from Taylor's expansion, since

$$D_x^n \{e^{-x} L_N^{(\alpha)}(x)\} = (-1)^n e^{-x} L_N^{(\alpha+n)}(x). \tag{1.13}$$

Indeed, if we substitute for the polynomials $f_{m,n}(z, w, x)$ from (1.4) into the left-hand side of (1.5), and invert the order of the summations involved, we obtain

$$\begin{aligned} &\sum_{m,n=0}^{\infty} f_{m,n}(z, w, x) \, L_N^{(\lambda+m)}(y) \, t^m \\ &\quad = \sum_{m=0}^{\infty} a_m \, (zt)^m \sum_{n=0}^{\infty} H_{m+n}^{\gamma}(x, \alpha, \beta) \frac{w^n}{n!} \\ &\quad \cdot \sum_{k=0}^{\infty} L_N^{(\lambda+m+k)}(y) \frac{(-t)^k}{k!}. \end{aligned}$$

Now apply (1.11) and (1.12) to sum the inner series, and then interpret the resulting series by means of (1.3). We are thus led easily to the right-hand side of the assertion (1.5) of Theorem A.

Theorem B can be proven in a similar manner by appealing to the identity (1.12) and the elementary result:

$$\sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} C_N^{\nu+n}(x) \, t^n = (1-t)^{-\nu-N/2} \, C_N^{\nu}\left(\frac{x}{\sqrt{1-t}}\right) \qquad (|t|<1), \tag{1.14}$$

which is a rather straightforward consequence of the hypergeometric representation [2, p. 176, Eq. 3.15(8)]:

$$C_N^{\nu}(x) = \frac{(\nu)_N}{N!} (2x)^N \, {}_2F_1\left[-\frac{1}{2}N, -\frac{1}{2}N + \frac{1}{2}; 1-\nu-N; x^{-2}\right]. \tag{1.15}$$

2. Further Generalizations of Theorem A

A closer examination of our proof of Theorem A, using the Gould–Hopper result (1.11), would suggest the existence of much more general classes of bilateral generating functions of the type (1.5). As an illustration, consider a class of functions $\{S_n(x) \mid n=0, 1, 2, ...\}$ generated by [10, p. 755, Eq. (1)]

$$\sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x)\, t^n = f(x, t)\{g(x, t)\}^{-m} S_m(h(x, t)), \tag{2.1}$$

where m is a nonnegative integer, the coefficients $A_{m,n}$ are constants (real or complex), and f, g, h are suitable functions of x and t. The sequence $\{S_n(x)\}$, generated by (2.1), can indeed be specialized to yield a fairly wide variety of special functions (and polynomials) including, for example, the Gould–Hopper polynomials (1.1) for which

$$A_{m,n} = 1/n!, \qquad f(x, t) = (1-t/x)^{\alpha} \exp(\beta x^{\gamma}\{1-(1-t/x)^{\gamma}\}),$$
$$g(x, t) = 1, \qquad h(x, t) = x - t, \qquad \text{and} \qquad S_n(x) = H_n^{\gamma}(x, \alpha, \beta),$$

by virtue of (1.11).

Making use of the generating function (2.1), instead of its special case (1.11), it is not difficult to prove a generalization of Theorem A contained in

THEOREM 1. *Corresponding to the functions $S_n(x)$, generated by* (2.1), *let*

$$\Theta_N[x, y, t] = \sum_{n=0}^{\infty} a_n S_n(x)\, L_N^{(\lambda+n)}(y)\, t^n \qquad (a_n \neq 0), \tag{2.2}$$

where λ is an arbitrary (real or complex) parameter. Suppose also that

$$\theta_{m,n}(z, w, x) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k}{k!} a_{m-k} A_{m-k,n-k} z^{m-k} w^{n-k} S_{m+n-2k}(x). \tag{2.3}$$

Then

$$\sum_{m,n=0}^{\infty} \theta_{m,n}(z, w, x)\, L_N^{(\lambda+m)}(y)\, t^m$$
$$= \exp(-t) f(x, w)\, \Theta_N[h(x, w), y+t, zt/g(x, w)], \tag{2.4}$$

provided that each member exists.

It should be pointed out that, by setting

$$a_n = b_n T_n(z_1, ..., z_s) \qquad (b_n \neq 0; n = 0, 1, 2, ...), \tag{2.5}$$

where $T_n(z_1, ..., z_s)$ is a nonvanishing function of s variables $z_1, ..., z_s$ $(s \geqslant 1)$, Theorem 1 (and each of its consequences considered below) can be applied to derive various classes of mixed multilateral generating functions analogous to those considered in the literature (cf. [12; 15, Sections 8.4 and 8.5]). Furthermore, as already observed by one of us elsewhere [12, p. 222], the definition (2.1) can easily be transformed to include cases when m is an arbitrary complex parameter. Thus, for example, Theorem 1 applies also to the familiar Bessel function generated by [18, p. 141, Eq. 5.22(5)]

$$\sum_{n=0}^{\infty} J_{\mu+n}(x)\frac{t^n}{n!}=(1-2t/x)^{-\mu/2}J_\mu(\sqrt{x^2-2xt}), \tag{2.6}$$

where μ is an arbitrary complex number. Setting $\mu=\nu+m$ in (2.6), where m is a nonnegative integer, and comparing the resulting equation with the generating function (2.1), we find that

$$A_{m,n}=1/n!, \qquad f(x,t)=(1-2t/x)^{-\nu/2}, \qquad g(x,t)=\sqrt{1-2t/x},$$

$$h(x,t)=\sqrt{x^2-2xt}, \qquad \text{and} \qquad S_n(x)=J_{\nu+n}(x).$$

Consequently, Theorem 1 yields

COROLLARY 1. *If*

$$\Phi_N[x,y,t]=\sum_{n=0}^{\infty} a_n J_{\nu+n}(x)\,L_N^{(\lambda+n)}(y)\,t^n \qquad (a_n\neq 0) \tag{2.7}$$

and

$$\varphi_{m,n}(z,w,x)=\sum_{k=0}^{\min(m,n)}\frac{(-1)^k z^{m-k}\,w^{n-k}}{k!\,(n-k)!}\,a_{m-k}J_{\nu+m+n-2k}(x), \tag{2.8}$$

where ν and λ are arbitrary complex parameters, then

$$\sum_{m,n=0}^{\infty}\varphi_{m,n}(z,w,x)\,L_N^{(\lambda+m)}(y)\,t^m$$

$$=\exp(-t)(1-2w/x)^{-\nu}\,\Phi_N[\sqrt{x^2-2xw},\,y+t,\,zt(1-2w/x)^{-1/2}], \tag{2.9}$$

provided that each member exists.

Numerous further applications of Theorem 1 would involve the familiar classes of polynomials considered in the earlier works stemming from the definition (2.1) (see, e.g., [10; 14; 12; 15, Sect. 8.2 *et seq.*]). For instance, the polynomials $G_n^{(\alpha)}(x,\gamma,\beta,\zeta)$ defined by [16, p. 75, Eq. (1.3)]

$$G_n^{(\alpha)}(x,\gamma,\beta,\zeta)=\frac{x^{-\alpha-\zeta n}}{n!}\exp(\beta x^\gamma)(x^{\zeta+1}D_x)^n\,\{x^\alpha\exp(-\beta x^\gamma)\} \tag{2.10}$$

were introduced by Srivastava and Singhal [16] in an attempt to present a unified theory of the various known generalizations of the classical Hermite and Laguerre polynomials, including (for example) the Gould–Hopper polynomials (1.1). These polynomials are known to possess the generating functions (cf. [16; 12, p. 239]):

$$\sum_{n=0}^{\infty} \binom{m+n}{n} G_{m+n}^{(\alpha)}(x, \gamma, \beta, \zeta)\, t^n$$
$$= (1-\zeta t)^{-m-\alpha/\zeta} \exp(\beta x^{\gamma}\{1-(1-\zeta t)^{-\gamma/\zeta}\})$$
$$\cdot G_m^{(\alpha)}(x(1-\zeta t)^{-1/\zeta}, \gamma, \beta, \zeta), \qquad (\zeta \neq 0; m=0, 1, 2, ...) \quad (2.11)$$

and

$$\sum_{n=0}^{\infty} \binom{m+n}{n} G_{m+n}^{(\alpha-\zeta n)}(x, \gamma, \beta, \zeta)\, t^n$$
$$= (1+\zeta t)^{-1+\alpha/\zeta} \exp(\beta x^{\gamma}\{1-(1+\zeta t)^{\gamma/\zeta}\})$$
$$\cdot G_m^{(\alpha)}(x(1+\zeta t)^{1/\zeta}, \gamma, \beta, \zeta), \qquad (\zeta \neq 0; m=0, 1, 2, ...). \quad (2.12)$$

Making use of (2.11) and (2.12) in conjunction with Theorem 1, we obtain the following results on bilateral generating functions for the Srivastava–Singhal polynomials defined by (2.10).

COROLLARY 2. *If*

$$\Psi_N[x, y, t] = \sum_{n=0}^{\infty} a_n G_n^{(\alpha)}(x, \gamma, \beta, \zeta)\, L_N^{(\lambda+n)}(y)\, t^n \qquad (a_n \neq 0) \quad (2.13)$$

and

$$\psi_{m,n}(z, w, x) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k}{k!} \binom{m+n-2k}{n-k} a_{m-k}$$
$$\cdot z^{m-k}\, w^{n-k} G_{m+n-2k}^{(\alpha)}(x, \gamma, \beta, \zeta), \quad (2.14)$$

then

$$\sum_{m,n=0}^{\infty} \psi_{m,n}(z, w, x)\, L_N^{(\lambda+m)}(y)\, t^m$$
$$= (1-\zeta w)^{-\alpha/\zeta} \exp(\beta x^{\gamma}\{1-(1-\zeta w)^{-\gamma/\zeta}\} - t)$$
$$\cdot \Psi_N[x(1-\zeta w)^{-1/\zeta}, y+t, zt/(1-\zeta w)] \qquad (\zeta \neq 0), \quad (2.15)$$

provided that each member exists.

COROLLARY 3. *If*

$$\Xi_N[x, y, t] = \sum_{n=0}^{\infty} a_n G_n^{(\alpha-\zeta n)}(x, \gamma, \beta, \zeta)\, L_N^{(\lambda+n)}(y)\, t^n \qquad (a_n \neq 0) \tag{2.16}$$

and

$$\xi_{m,n}(z, w, x) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k}{k!} \binom{m+n-2k}{n-k} a_{m-k} z^{m-k} w^{n-k} \cdot G_{m+n-2k}^{(\alpha-(m+n-2k)\zeta)}(x, \gamma, \beta, \zeta), \tag{2.17}$$

then

$$\sum_{m,n=0}^{\infty} \xi_{m,n}(z, w, x)\, L_N^{(\lambda+m)}(y)\, t^m = (1+\zeta w)^{-1+\alpha/\zeta} \exp(\beta x^\gamma \{1-(1+\zeta w)^{\gamma/\zeta}\} - t) \cdot \Xi_N[x(1+\zeta w)^{1/\zeta}, y+t, zt/(1+\zeta w)] \qquad (\zeta \neq 0), \tag{2.18}$$

provided that each member exists.

Now we recall the relationship [14, p. 315, Eq. (83)]:

$$Y_n^\alpha(x; s) = s^{-n} G_n^{(\alpha+1)}(x, 1, 1, s) \qquad (\alpha > -1;\, s = 1, 2, 3, ...), \tag{2.19}$$

where $Y_n^\alpha(x; s)$ denotes the Konhauser biorthogonal polynomials (cf. [4, 5, 7, 13]). In particular,

$$Y_n^\alpha(x; 1) = L_n^{(\alpha)}(x) \qquad (\alpha > -1;\, n = 0, 1, 2, ...), \tag{2.20}$$

and the polynomials $Y_n^\alpha(x; 2)$ were encountered earlier by Spencer and Fano [11] in certain analytical calculations involving the penetration of gamma rays through matter.

In view of the relationship (2.19), Corollaries 2 and 3 can readily be applied to derive the corresponding bilateral generating functions for these biorthogonal polynomials. Furthermore, since [16, p. 76, Eq. (1.6)]

$$G_n^{(\alpha)}(x, \gamma, \beta, -1) = G_n^{(\alpha-n+1)}(x, \gamma, \beta, 1) = \frac{(-x)^n}{n!} H_n^\gamma(x, \alpha, \beta) \tag{2.21}$$

in terms of the Gould–Hopper polynomials (1.1), Theorem A is an obvious further special case of Corollary 2.

Numerous other special cases of Corollaries 2 and 3 can be derived by appealing to the various known relationships (cf. [16, p. 76]).

3. Further Generalizations of Theorem B

In terms of a suitably bounded sequence $\{\Omega_n\}_{n=0}^{\infty}$ of complex numbers, let us define

$$\omega_N^{\nu}(x) = (\nu)_N \sum_{k=0}^{\infty} \frac{\Omega_k x^{N-2k}}{(1-\nu-N)_k}, \tag{3.1}$$

where the parameters ν and N are unrestricted, in general. Then it is easily verified that

$$\sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \omega_N^{\nu+n}(x)\, t^n = (1-t)^{-\nu-N/2}\, \omega_N^{\nu}\left(\frac{x}{\sqrt{1-t}}\right). \tag{3.2}$$

Setting

$$\Omega_n = \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{n!\,(\beta_1)_n \cdots (\beta_q)_n} \qquad (n = 0, 1, 2, \ldots),$$

the generating function (3.2) assumes the hypergeometric form:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_pF_{q+1}\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ 1-\lambda-n, \beta_1, \ldots, \beta_q; \end{matrix}\; z\right] t^n$$

$$= (1-t)^{-\lambda}\, {}_pF_{q+1}\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ 1-\lambda, \beta_1, \ldots, \beta_q; \end{matrix}\; z(1-t)\right], \tag{3.3}$$

which can indeed be proven directly.

Each of the formulas (3.2) and (3.3) provides a generalization of the generating function (1.14) involving the Gegenbauer polynomial (or function). In fact, by suitably choosing the coefficients Ω_n in (3.1) or the various parameters occurring in (3.3), the generating function (3.2) can be applied to numerous other hypergeometric polynomials (including, for example, the Jacobi polynomials [8, Chap. 16]).

Applying the generating function (1.12), in conjunction with (3.2) instead of (1.14), we can prove a further generalization of Theorem B given by

THEOREM 2. *Suppose that there exists a bilateral generating function in the form:*

$$H_{M,N}(x, y, t) = \sum_{n=0}^{\infty} a_n \omega_M^{\nu+n}(x)\, L_N^{(\alpha+n)}(y)\, t^n \qquad (a_n \neq 0), \tag{3.4}$$

where ν and α are arbitrary (real or complex) parameters. Also let

$$h_{m,n}(z, w, x) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k z^{m-k} w^{n-k}}{k!\,(n-k)!} \cdot a_{m-k}\,(\nu+m-k)_{n-k}\,\omega_M^{\nu+m+n-2k}(x). \tag{3.5}$$

Then

$$\sum_{m,n=0}^{\infty} h_{m,n}(z, w, x)\, L_N^{(\alpha+m)}(y)\, t^m = (1-w)^{-\nu-M/2} \exp(-t)\, H_{M,N}\left[\frac{x}{\sqrt{1-w}}, y+t, \frac{zt}{1-w}\right], \tag{3.6}$$

provided that each member exists.

Theorem 2 can be shown to contain several classes of bilateral generating functions analogous to those given by Theorem B. Indeed the choice of the coefficients a_n given by (2.5) continues to remain valid for Theorem 2 and for each of its consequences including Theorem B.

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