

OPERATIONAL FORMULAE FOR THE COMPLEX HERMITE POLYNOMIALS $H_{p,q}(z, \bar{z})$

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ABSTRACT. We give operational formulae of Burchnell type involving complex Hermite polynomials. Short proofs of some known formulae are given and new results involving these polynomials, including Nielsen's identities and Runge addition formula, are derived.

Keywords: Operational formula; Complex Hermite polynomials; Runge's addition formula; Generating function; Nielsen's identity

1. INTRODUCTION

Operational formulae, involving special orthogonal polynomials, appear as useful tools to obtain some new results or to give short proofs of known formulae, see Burchnell [4, 5], Carlitz [6], Al-Salam [2], Gould-Hopper [12], Chatterjea [7], Singh [24], Karande and Thakare [18], Al-Salam and Ismail [3], Patil and Thakare [21], Chatterjea and Srivastava [8], Purohit and Raina [22], Kaanoglu [16] among others. As basic example, one cites the classical real Hermite polynomials ([13, 20, 26, 27]):

$$H_m(x) = (-1)^m e^{x^2} D^m(e^{-x^2}); \quad D := \frac{d}{dx},$$

which are extensively studied and have found wide application in various branches of mathematics, physics and technology. They possess a complete and rich list of remarkably interesting properties and some of them can be obtained using the Burchnell's operational formula ([4]), to wit

$$(-D + 2x)^m (f) = m! \sum_{k=0}^m \frac{(-1)^k H_{m-k}(x)}{k! (m-k)!} D^k(f). \quad (1.1)$$

Indeed, it can be employed to give a direct and simple proof of the Runge addition formula ([23], [17]):

$$H_m(x+y) = \left(\frac{1}{2}\right)^{m/2} m! \sum_{k=0}^n \frac{H_k(\sqrt{2}x) H_{m-k}(\sqrt{2}y)}{k! (m-k)!}. \quad (1.2)$$

as well as of the quadratic recurrence formula (Nielsen's identity [19]):

$$H_{m+n}(x) = m!n! \sum_{k=0}^{\min(m,n)} \frac{(-2)^k H_{m-k}(x) H_{n-k}(x)}{k! (m-n)! (n-k)!}. \quad (1.3)$$

An extension of (1.1) is given by Gould and Hopper in [12] for the generalized Hermite polynomials

$$H_m^\gamma(x, \alpha, p) := (-1)^m x^{-\alpha} e^{px^\gamma} D^m(x^\alpha e^{-px^\gamma}).$$

As an interesting extension of the real Hermite polynomials $H_m(x)$ are the complex Hermite polynomials $H_{p,q}(z, \bar{z})$, for $z = x + iy \in \mathbb{C}$; $x, y \in \mathbb{R}$. They were considered by Ito in [15] in the context of complex Markov process,

$$H_{p,q}(z, \bar{z}) = (-1)^{p+q} e^{z\bar{z}} \partial_{\bar{z}}^p \partial_z^q \left(e^{-z\bar{z}} \right),$$

where ∂_z and $\partial_{\bar{z}}$ stand for

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \partial_{\bar{z}} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right); \quad i = \sqrt{-1}.$$

For the unity of the formulation, we define trivially $H_{p,q}(z, \bar{z}) = 0$ whenever $p < 0$ or $q < 0$. This class of polynomials is connected to $H_m(x)$ by

$$H_{p,q}(z, \bar{z}) = p!q! \left(\frac{i}{2} \right)^{(p+q)/2} \sum_{j=0}^p \sum_{k=0}^q \frac{i^{j+k} H_{j+k}(x) H_{p+q-j-k}(y)}{j!k! (p-j)!(q-k)!}$$

and constitutes an orthogonal basis of the Hilbert space $L^2(\mathbb{C}; e^{-z\bar{z}} dx dy)$ (see [14, 11]):

$$\int_{\mathbb{C}} H_{p,q}(z, \bar{z}) \overline{H_{m,n}(z, \bar{z})} e^{-z\bar{z}} dx dy = \delta_{pm} \delta_{qn} p!q! \pi.$$

Moreover, they appear naturally when investigating spectral properties of some second order differential operators of Laplacian type ([25, 10, 28, 30]) and can be connected to some classes of generalized Bargmann spaces. Several interesting features of $H_{p,q}(z, \bar{z})$ in connection of coherent states theory have been studied recently (see for examples [1, 9, 29]).

In course of our investigation, we establish new operational formulae involving $H_{p,q}(z, \bar{z})$ analogous to the Burchnell one (1.1). In addition, we will use them in a simple way to obtain new properties satisfied by these polynomials. Mainly, we are interested in Nielsen's identities, generating functions and Runge addition formula.

2. BURCHNALL'S OPERATIONAL FORMULA FOR $H_{p,q}(z, \bar{z})$

In order to obtain some operational formulae for $H_{p,q}(z, \bar{z})$ of Burchnell type, we begin by noting first that the complex Hermite polynomials

$$H_{p,q}(z, \bar{z}) = (-1)^{p+q} e^{z\bar{z}} \partial_{\bar{z}}^p \partial_z^q \left(e^{-z\bar{z}} \right) \quad (2.1)$$

$$= (-1)^p e^{z\bar{z}} \partial_{\bar{z}}^p \left(\bar{z}^q e^{-z\bar{z}} \right) \quad (2.2)$$

$$= (-1)^q e^{z\bar{z}} \partial_z^q \left(z^p e^{-z\bar{z}} \right), \quad (2.3)$$

can be rewritten in the following equivalent form

$$H_{p,q}(z, \bar{z}) = (-\partial_{\bar{z}} + z)^p (\bar{z}^q) \quad (2.4)$$

$$= (-\partial_z + \bar{z})^q (z^p), \quad (2.5)$$

which appears as special cases of the following

Lemma 2.1. *For every sufficiently differentiable function f , we have*

$$(-\partial_{\bar{z}} + z)^p (f) = (-1)^p e^{z\bar{z}} \partial_{\bar{z}}^p \left(e^{-z\bar{z}} f \right) \quad (2.6)$$

$$(-\partial_z + \bar{z})^q (f) = (-1)^q e^{z\bar{z}} \partial_z^q \left(e^{-z\bar{z}} f \right). \quad (2.7)$$

Proof. We note first that (2.6) can be handled in a similar way as down below for (2.7). Indeed, direct computation yields $(-\partial_z + \bar{z})(e^{z\bar{z}}g) = e^{z\bar{z}}(-\partial_z g)$. Next, by induction, we get

$$(-\partial_z + \bar{z})^q (e^{z\bar{z}}g) = (-1)^q e^{z\bar{z}} \partial_z^q(g)$$

which infers (2.7) for $g = e^{-z\bar{z}}f$. \square

Consequently, it follows

Proposition 2.2. *For given positive integers p, q and every sufficiently differentiable function f , we have*

$$a) \quad (-\partial_z + \bar{z})^q (f) = q! \sum_{k=0}^q \frac{(-1)^k}{k!} \frac{\bar{z}^{q-k}}{(q-k)!} \partial_z^k(f) \quad (2.8)$$

$$b) \quad (-\partial_z + \bar{z})^q (f) = q! \sum_{k=0}^q \frac{(-1)^k}{k!} \frac{H_{p,q-k}(z, \bar{z})}{(q-k)!} \partial_z^k(z^{-p}f). \quad (2.9)$$

Proof. Application of Leibnitz formula to the product function $e^{-z\bar{z}}f$ in (2.7) gives

$$\begin{aligned} (-\partial_z + \bar{z})^q (f) &= (-1)^q e^{z\bar{z}} \sum_{k=0}^q \binom{q}{k} \partial_z^{q-k} (e^{-z\bar{z}}) \partial_z^k(f) \\ &= q! \sum_{k=0}^q \frac{(-1)^k}{k!} \frac{\bar{z}^{q-k}}{(q-k)!} \partial_z^k(f). \end{aligned}$$

Now by writing f as $f = z^p(z^{-p}f)$ in (2.7) and next applying Leibnitz formula, we get

$$\begin{aligned} (-\partial_z + \bar{z})^q (f) &= (-1)^q e^{z\bar{z}} \sum_{k=0}^q \binom{q}{k} \partial_z^{q-k} (z^p e^{-z\bar{z}}) \partial_z^k(z^{-p}f) \\ &\stackrel{(2.3)}{=} q! \sum_{k=0}^q \frac{(-1)^k}{k!} \frac{H_{p,q-k}(z, \bar{z})}{(q-k)!} \partial_z^k(z^{-p}f). \end{aligned}$$

\square

As immediate consequence of (2.8) when taking $f = z^p$, we get the explicit expansion of the complex Hermite polynomial,

$$H_{p,q}(z, \bar{z}) = p!q! \sum_{k=0}^{\min(p,q)} \frac{(-1)^k}{k!} \frac{\bar{z}^{q-k}}{(q-k)!} \frac{z^{p-k}}{(p-k)!}. \quad (2.10)$$

The expression (2.9) for the complex Hermite polynomial can be considered as an analogue of (1.1). However, below we obtain a more appropriate Burchnell's operational formula involving $H_{p,q}(z, \bar{z})$.

Proposition 2.3. *For given positive integers p, q and every sufficiently differentiable function f , we have*

$$e^{z\bar{z}} \partial_z^p \partial_z^q (e^{-z\bar{z}}f) = (-1)^{p+q} p!q! \sum_{j=0}^p \sum_{k=0}^q \frac{(-1)^{j+k}}{j!k!} \frac{H_{p-j,q-k}(z, \bar{z})}{(p-j)!(q-k)!} \partial_z^j \partial_z^k(f). \quad (2.11)$$

Proof. By applying repetitively the Leibnitz formula, it follows

$$e^{z\bar{z}} \partial_{\bar{z}}^p \partial_z^q (e^{-z\bar{z}} f) = e^{z\bar{z}} \sum_{j=0}^p \sum_{k=0}^q (-1)^{q-k} \binom{p}{j} \binom{q}{k} \partial_{\bar{z}}^{p-j} (\bar{z}^{q-k} e^{-z\bar{z}}) \partial_{\bar{z}}^j \partial_z^k (f).$$

But from (2.2), i.e., $e^{z\bar{z}} \partial_{\bar{z}}^m (\bar{z}^n e^{-z\bar{z}}) = (-1)^m H_{m,n}(z, \bar{z})$, we deduce

$$e^{z\bar{z}} \partial_{\bar{z}}^p \partial_z^q (e^{-z\bar{z}} f) = \sum_{j=0}^p \sum_{k=0}^q (-1)^{j+k} \binom{p}{j} \binom{q}{k} H_{p-j, q-k}(z, \bar{z}) \partial_{\bar{z}}^j \partial_z^k (f).$$

This completes the proof. \square

Remark 2.4. For $f(z) = e^{-\xi z/2}$ with $\xi \in \mathbb{C}$, the right hand side of (2.11) leads to the generalized complex polynomials of Hermite type studied in [11] and suggested by a special magnetic Schrödinger operator.

Remark 2.5. Using (2.7) together with the established fact $(-\partial_{\bar{z}} + z)^p (e^{z\bar{z}} g) = (-1)^p e^{z\bar{z}} (\partial_{\bar{z}}^p g)$, we get

$$(-\partial_{\bar{z}} + z)^p (-\partial_z + \bar{z})^q (f) = (-1)^{p+q} e^{z\bar{z}} \partial_{\bar{z}}^p \partial_z^q (e^{-z\bar{z}} f).$$

Therefore, the operational formula (2.11) can be reworded as

$$(-\partial_{\bar{z}} + z)^p (-\partial_z + \bar{z})^q (f) = p!q! \sum_{j=0}^p \sum_{k=0}^q \frac{(-1)^{j+k}}{j!k!} \frac{H_{p-j, q-k}(z, \bar{z})}{(p-j)!(q-k)!} \partial_{\bar{z}}^j \partial_z^k (f). \quad (2.12)$$

Moreover, it is interesting to observe that $H_{p,q}(z, \bar{z})$ can be realized also as

$$H_{p,q}(z, \bar{z}) = (-\partial_{\bar{z}} + z)^p (-\partial_z + \bar{z})^q \cdot (1). \quad (2.13)$$

In the next section, we are concerned with some properties, for the complex Hermite polynomials, that follow directly from the previous operational representations, including (2.5), (2.9) and its variant (2.8) as well as (2.12).

3. RELATED PROPERTIES

3.1. Recurrence formulae. Note for instance that, since ∂_z and $(-\partial_z + \bar{z})^q$ commute, we get

$$\partial_z H_{p,q}(z, \bar{z}) = \partial_z (-\partial_z + \bar{z})^q (z^p) = (-\partial_z + \bar{z})^q \partial_z (z^p) = p H_{p-1,q}(z, \bar{z}). \quad (3.1)$$

Similarly, we have

$$\partial_{\bar{z}} H_{p,q}(z, \bar{z}) = \partial_{\bar{z}} (-\partial_z + z)^p (\bar{z}^q) = (-\partial_z + z)^p \partial_{\bar{z}} (\bar{z}^q) = q H_{p,q-1}(z, \bar{z}). \quad (3.2)$$

Thus, we see

$$(-\partial_z + \bar{z}) \partial_{\bar{z}} H_{p,q} = q H_{p,q}, \quad (3.3)$$

that is $H_{p,q}$ are eigenfunctions of the second order differential operator of Laplacian type $\Delta := (-\partial_z + \bar{z}) \partial_{\bar{z}}$. Furthermore, the polynomials $H_{p,q}$ obey the recursion relations

$$a) \quad H_{p+1,q} = z H_{p,q} - \partial_{\bar{z}} H_{p,q}. \quad (3.4)$$

$$b) \quad H_{p+1,q} = z H_{p,q} - q H_{p,q-1}. \quad (3.5)$$

$$a') \quad H_{p,q+1} = \bar{z} H_{p,q} - \partial_z H_{p,q}. \quad (3.6)$$

$$b') \quad H_{p,q+1} = \bar{z} H_{p,q} - p H_{p-1,q}. \quad (3.7)$$

Indeed, using (2.5), it follows

$$H_{p+1,q}(z, \bar{z}) = (-\partial_{\bar{z}} + z)^{1+p} (\bar{z}^q) = (-\partial_{\bar{z}} + z) H_{p,q}(z, \bar{z}) = -\partial_{\bar{z}} H_{p,q} + z H_{p,q}.$$

Substitution of (3.2) in the previous equality gives rise to (3.5). Similarly (2.4) yields (3.6) and so (3.7). Recurrence formula (3.6) (resp. (3.7)) can also be obtained from (3.4) (resp. (3.5)) by conjugation since the polynomials $\overline{H_{p,q}(z, \bar{z})}$ satisfy the following symmetry relation with respect to index permutation $\overline{H_{p,q}(z, \bar{z})} = H_{q,p}(z, \bar{z})$.

3.2. Quadratic recurrence formulae. By writing (2.8) for the special case of $f = H_{p,s}$, we get the following first variant of Nielsen's identity

$$H_{p,q+s}(z, \bar{z}) = p!q! \sum_{k=0}^{\min(p,q)} \frac{(-1)^k}{k!} \frac{\bar{z}^{q-k}}{(q-k)!} \frac{H_{p-k,s}(z, \bar{z})}{(p-k)!}. \quad (3.8)$$

This follows using $H_{p,q+s} = (-\partial_z + \bar{z})^q (H_{p,s})$ combined with the fact that $\partial_z^k (H_{p,s}) = \frac{p!}{(p-k)!} H_{p-k,s}$ for $k \leq p$. Furthermore, by considering the particular case $f = z^{p+s}$ in (2.9), we check that

$$(-\partial_z + \bar{z})^q (z^{p+s}) = q! \sum_{k=0}^q \frac{(-1)^k}{k!} \frac{H_{p,q-k}(z, \bar{z})}{(q-k)!} \partial_z^k (z^s).$$

Whence, the second variant of the Nielsen identity reads

$$H_{p+s,q}(z, \bar{z}) = q!s! \sum_{k=0}^{\min(q,s)} \frac{(-1)^k}{k!} \frac{z^{s-k}}{(s-k)!} \frac{H_{p,q-k}(z, \bar{z})}{(q-k)!}. \quad (3.9)$$

A third variant of Nielsen's identity can be established. More precisely, we have the following proposition.

Proposition 3.1. *We have*

$$H_{p+s,q}(z, \bar{z}) = \sqrt{2}^{p+s-q} q! \sum_{k=0}^q \frac{H_{s,k}\left(\frac{z}{\sqrt{2}}, \frac{\bar{z}}{\sqrt{2}}\right)}{k!} \frac{H_{p,q-k}\left(\frac{z}{\sqrt{2}}, \frac{\bar{z}}{\sqrt{2}}\right)}{(q-k)!}. \quad (3.10)$$

Proof. The proof of (3.10) can be handled by writing $e^{z\bar{z}} (-\partial_z + \bar{z})^q (z^{p+s} e^{-z\bar{z}})$ in two manners. Indeed, the first one follows by taking $f = z^{p+s} e^{-z\bar{z}}$ in (2.7). In fact,

$$\begin{aligned} e^{z\bar{z}} (-\partial_z + \bar{z})^q (z^{p+s} e^{-z\bar{z}}) &= (-1)^q e^{2z\bar{z}} \partial_z^q (z^{p+s} e^{-2z\bar{z}}) \\ &= \sqrt{2}^{q-p-s} H_{p+s,q}(\sqrt{2}z, \sqrt{2}\bar{z}) \end{aligned}$$

upon an appropriate change of variable. The second one follows from (2.9), to wit

$$\begin{aligned} e^{z\bar{z}} (-\partial_z + \bar{z})^q (z^{p+s} e^{-z\bar{z}}) &= q! e^{z\bar{z}} \sum_{k=0}^q \frac{(-1)^k}{k!} \frac{H_{p,q-k}(z, \bar{z})}{(q-k)!} \partial_z^k (z^s e^{-z\bar{z}}) \\ &\stackrel{(2.3)}{=} q! \sum_{k=0}^q \frac{H_{s,k}(z, \bar{z})}{k!} \frac{H_{p,q-k}(z, \bar{z})}{(q-k)!}. \end{aligned}$$

Thence, we have proved that

$$H_{p+s,q}(\sqrt{2}z, \sqrt{2}\bar{z}) = \sqrt{2}^{p+s-q} q! \sum_{k=0}^q \frac{H_{s,k}(z, \bar{z})}{k!} \frac{H_{p,q-k}(z, \bar{z})}{(q-k)!}$$

which gives rise to (3.10) by replacing z by $z/\sqrt{2}$. \square

We have called here (3.8) (resp. (3.9)) by the first (resp. second) variant of Nielsen identity, since we can write it as a weighted sum of a product of the same polynomials, according to the fact that $\bar{z}^{q-k} = H_{0,q-k}(z, \bar{z})$ (resp. $z^{s-k} = H_{s-k,0}(z, \bar{z})$). We reserve the appellation Nielsen's identity to the following

Proposition 3.2. *We have the following quadratic recurrence formula*

$$H_{p+m,q+n}(z, \bar{z}) = p!q!m!n! \sum_{j=0}^{\min(p,n)} \sum_{k=0}^{\min(q,m)} \frac{(-1)^{j+k}}{j!k!} \frac{H_{p-j,q-k}(z, \bar{z})}{(p-j)!(q-k)!} \frac{H_{m-k,n-j}(z, \bar{z})}{(m-k)!(n-j)!}. \quad (3.11)$$

Proof. By taking $f = H_{m,n}(z, \bar{z})$ in (2.12), we obtain

$$(-\partial_{\bar{z}} + z)^p (-\partial_z + \bar{z})^q (H_{m,n}(z, \bar{z})) = p!q! \sum_{j=0}^p \sum_{k=0}^q \frac{(-1)^{j+k}}{j!k!} \frac{H_{p-j,q-k}(z, \bar{z})}{(p-j)!(q-k)!} \partial_{\bar{z}}^j \partial_z^k (H_{m,n}(z, \bar{z})).$$

Therefore the result (3.11) follows upon making use of

$$(-\partial_{\bar{z}} + z)^p (-\partial_z + \bar{z})^q (H_{m,n}(z, \bar{z})) = H_{p+m,q+n}(z, \bar{z})$$

together with the fact that

$$\partial_{\bar{z}}^j \partial_z^k (H_{m,n}(z, \bar{z})) = m!n! \frac{H_{m-k,n-j}(z, \bar{z})}{(m-k)!(n-j)!}$$

for $k \leq m$, $j \leq n$ and $\partial_{\bar{z}}^j \partial_z^k (H_{m,n}(z, \bar{z})) = 0$ otherwise. \square

As a special case of (3.11) when taking $m = q$ and $n = p$ (keeping in mind the fact that $\overline{H_{m,n}(z, \bar{z})} = H_{n,m}(z, \bar{z})$), we state the following

Corollary 3.3. *We have*

$$H_{p+q,p+q}(z, \bar{z}) = (p!q!)^2 \sum_{j=0}^p \sum_{k=0}^q \frac{(-1)^{j+k}}{j!k!} \frac{|H_{p-j,q-k}(z, \bar{z})|^2}{((p-j)!(q-k)!)^2}. \quad (3.12)$$

3.3. Generating functions.

Proposition 3.4. *We have the following generating functions:*

$$a) \quad \sum_{p=0}^{+\infty} \frac{u^p}{p!} H_{p,q}(z, \bar{z}) = (\bar{z} - u)^q e^{uz}. \quad (3.13)$$

$$a') \quad \sum_{q=0}^{+\infty} \frac{v^q}{q!} H_{p,q}(z, \bar{z}) = (z - v)^p e^{v\bar{z}}. \quad (3.14)$$

$$b) \quad \sum_{p,q=0}^{+\infty} \frac{u^p v^q}{p! q!} H_{p,q}(z, \bar{z}) = e^{uz+v\bar{z}-uv}. \quad (3.15)$$

These equalities are valid for all u, v, z complex.

Proof. The generating function $a)$ is in fact the conjugate counterpart of $a')$. To prove $a')$, we begin by writing

$$\sum_{q=0}^{+\infty} \frac{v^q}{q!} (-\partial_z + \bar{z})^q (f) = e^{-v\partial_z + v\bar{z}}(f) = e^{v\bar{z}} e^{-v\partial_z}(f).$$

Now, since $e^{-v\partial_z}(z^p) = \sum_{k=0}^p \binom{p}{k} (-v)^k z^{p-k}$, it follows

$$\sum_{q=0}^{+\infty} \frac{v^q}{q!} H_{p,q}(z, \bar{z}) = e^{v\bar{z}} \sum_{k=0}^p \binom{p}{k} (-v)^k z^{p-k} = (z-v)^p e^{v\bar{z}}.$$

This is exactly (3.14) from which we deduce

$$\sum_{p,q=0}^{+\infty} \frac{u^p}{p!} \frac{v^q}{q!} H_{p,q}(z, \bar{z}) = e^{v\bar{z}} \sum_{p,q=0}^{+\infty} \frac{u^p}{p!} (z-v)^p = e^{uz + v\bar{z} - uv}.$$

□

Corollary 3.5. *We have*

$$\sum_{p,q=0}^{+\infty} \frac{\bar{z}^p z^q}{p! q!} H_{p,q}(z, \bar{z}) = e^{z\bar{z}}.$$

Furthermore, for every fixed integers $m, n \geq 0$, we have the identities

$$\sum_{p=0}^{+\infty} \frac{\bar{z}^p}{p!} H_{p,n}(z, \bar{z}) = 0 \quad \text{and} \quad \sum_{q=0}^{+\infty} \frac{z^q}{q!} H_{m,q}(z, \bar{z}) = 0.$$

According to the referee suggestion, it would be more interesting, in obtaining reproducing kernels using coherent states, if one can provide the sum

$$\sum_{q=0}^{+\infty} \frac{H_{p,q}(z, \bar{z}) \overline{H_{p,q}(w, \bar{w})}}{p! q!}.$$

The following result gives a closed form of such sum.

Proposition 3.6. *For every $z, w \in \mathbb{C}$ and fixed positive integers p, m , we have*

$$\sum_{q=0}^{+\infty} \frac{H_{p,q}(z, \bar{z}) \overline{H_{m,q}(w, \bar{w})}}{m! q!} = \frac{(-1)^m}{m!} H_{p,m}(z-w, \overline{z-w}) e^{w\bar{z}}. \quad (3.16)$$

Proof. Multiplication of both sides of (3.14) by $e^{-w\bar{w}}$ and next differentating the obtained formula m -times with respect to w infers

$$\sum_{q=0}^{+\infty} \frac{1}{q!} H_{p,q}(z, \bar{z}) \partial_z^m (w^q e^{-w\bar{w}}) = \partial_z^m \left((z-w)^p e^{w(\bar{z}-\bar{w})} \right).$$

Now, in view of (2.3), i.e., $\partial_z^m (w^q e^{-w\bar{w}}) = (-1)^m H_{q,m}(w, \bar{w}) e^{-w\bar{w}}$, we obtain

$$\sum_{q=0}^{+\infty} \frac{1}{q! m!} H_{p,q}(z, \bar{z}) H_{q,m}(w, \bar{w}) = \frac{(-1)^m}{m!} e^{w\bar{w}} \partial_z^m \left((z-w)^p e^{w(\bar{z}-\bar{w})} \right). \quad (3.17)$$

The right hand side of the previous equation can be rewritten as

$$\begin{aligned}
\frac{(-1)^m}{m!} e^{w\bar{w}} \partial_z^m \left((z-w)^p e^{w(\bar{z}-\bar{w})} \right) &= \frac{(-1)^m}{m!} e^{w\bar{z}} \left[e^{(z-w)(\bar{z}-\bar{w})} \partial_z^m \left((z-w)^p e^{-(z-w)(\bar{z}-\bar{w})} \right) \right] \\
&= \frac{(-1)^m}{m!} e^{w\bar{z}} \left[(-1)^m e^{\xi\bar{\xi}} \partial_z^m \left(\xi^p e^{-\xi\bar{\xi}} \right) \Big|_{\xi=z-w} \right] \\
&\stackrel{(2.3)}{=} \frac{(-1)^m}{m!} H_{p,m}(z-w, \bar{z}-\bar{w}) e^{w\bar{z}}.
\end{aligned}$$

Replacing this in (3.17) leads to the desired result. \square

Remark 3.7. As a particular case of (3.16), when taking $p = m$, we get

$$\sum_{q=0}^{+\infty} \frac{H_{p,q}(z, \bar{z}) \overline{H_{p,q}(w, \bar{w})}}{p!q!} = {}_1F_1(-p; 1; |z-w|^2) e^{w\bar{z}}, \quad (3.18)$$

since in this case $H_{p,p}(\xi, \bar{\xi}) = (-1)^p p! {}_1F_1(-p; 1; |\xi|^2)$, see [14], where ${}_1F_1(a; b; x)$ stands for the confluent hypergeometric function. Furthermore, by rewriting the sum in the left hand side of (3.18) as $\sum_{q=0}^{+\infty} = \sum_{q=0}^{p-1} + \sum_{q=p}^{+\infty}$, and next taking $z = w$ and $n = q - p$ for $q \geq p$, one deduces the following

$$\sum_{n=0}^{+\infty} \frac{|H_{p+n,p}(z, \bar{z})|^2}{(p+n)!p!} = \frac{(-1)^p}{p!} e^{z\bar{z}} - \sum_{q=0}^{p-1} \frac{|H_{p,q}(z, \bar{z})|^2}{p!q!}. \quad (3.19)$$

Proposition 3.8. We have the following summation formula:

$$\sum_{p,q,r,s=0}^{+\infty} \frac{H_{p,q}(z, \bar{z}) H_{p,s}(u, \bar{u}) H_{q,r}(v, \bar{v})}{p!q!r!s!} = e^{z(u+1)} e^{\bar{z}(v+1)} e^{-(u+v+uv+1)}. \quad (3.20)$$

Proof. We make use of (2.5) to write the left hand side of (3.20) as

$$\begin{aligned}
&\sum_{p,q,r,s=0}^{+\infty} \frac{1}{p!q!r!s!} H_{p,q}(z, \bar{z}) H_{p,s}(u, \bar{u}) H_{q,r}(v, \bar{v}) \\
&= \sum_{r,s=0}^{+\infty} \frac{1}{r!s!} (-\partial_v + \bar{v})^r (-\partial_u + \bar{u})^s \left(\sum_{p,q=0}^{+\infty} \frac{u^p v^q}{p! q!} H_{p,q}(z, \bar{z}) \right)
\end{aligned}$$

By applying (3.15), we obtain

$$\begin{aligned}
& \sum_{p,q,r,s=0}^{+\infty} \frac{1}{p!q!r!s!} H_{p,q}(z, \bar{z}) H_{p,s}(u, \bar{u}) H_{q,r}(v, \bar{v}) \\
&= \sum_{r,s=0}^{+\infty} \frac{1}{r!s!} (-\partial_v + \bar{v})^r (-\partial_u + \bar{u})^s (e^{uz+v\bar{z}-uv}) \\
&= \sum_{r,s=0}^{+\infty} \frac{1}{r!} (-\partial_v + \bar{v})^r \left(\frac{(z-v)^s}{s!} e^{uz+v\bar{z}-uv} \right) \\
&= \sum_{r=0}^{+\infty} \frac{1}{r!} (-\partial_v + \bar{v})^r (e^{z-v} e^{uz+v\bar{z}-uv}) \\
&= \sum_{r=0}^{+\infty} \frac{(-1 + \bar{z} - u)^r}{r!} e^{z-v} e^{uz+v\bar{z}-uv} \\
&= e^{z(u+1)} e^{\bar{z}(v+1)} e^{-(u+v+uv+1)}.
\end{aligned}$$

□

Remark 3.9. The special case of $r = s = 0$ (in (3.20)) leads to the generating relation (3.15).

3.4. Runge's addition formula.

Proposition 3.10. We have the following addition formula

$$H_{p,q}(z+w, \bar{z}+\bar{w}) = p!q! \left(\frac{1}{2}\right)^{(p+q)/2} \sum_{j=0}^p \sum_{k=0}^q \frac{H_{j,k}(\sqrt{2}z, \sqrt{2}\bar{z})}{k!j!} \frac{H_{p-j,q-k}(\sqrt{2}w, \sqrt{2}\bar{w})}{(q-k)!(p-j)!}. \quad (3.21)$$

Proof. Set $A_z := -\frac{\partial}{\partial(\sqrt{2}z)} + (\sqrt{2}z)$ and $A_{\bar{z}} := -\frac{\partial}{\partial(\sqrt{2}\bar{z})} + (\sqrt{2}\bar{z})$, and note that

$$A_{\bar{z}}^m A_z^n \cdot (1) = H_{m,n}(\sqrt{2}z, \sqrt{2}\bar{z}). \quad (3.22)$$

Note also that $H_{p,q}(z+w, \bar{z}+\bar{w})$ can be written in terms of A_z and A_w as

$$\begin{aligned}
H_{p,q}(z+w, \bar{z}+\bar{w}) &= \left(-\frac{d}{d(z+w)} + (z+w) \right)^p \left(-\frac{d}{d(\bar{z}+\bar{w})} + (\bar{z}+\bar{w}) \right)^q \cdot (1) \\
&= \left(\frac{1}{2}\right)^{(p+q)/2} \{(A_z + A_w)^p (A_{\bar{z}} + A_{\bar{w}})^q\} \cdot (1).
\end{aligned}$$

Now, since the involved differential operators A_z , $A_{\bar{z}}$, A_w and $A_{\bar{w}}$ commute, the binomial formula yields

$$\begin{aligned}
H_{p,q}(z+w, \bar{z}+\bar{w}) &= \left(\frac{1}{2}\right)^{(p+q)/2} \left(\sum_{j=0}^p \sum_{k=0}^q \binom{p}{j} \binom{q}{k} A_z^j A_w^{p-j} A_{\bar{z}}^k A_{\bar{w}}^{q-k} \right) \cdot (1) \\
&= \left(\frac{1}{2}\right)^{(p+q)/2} \sum_{j=0}^p \sum_{k=0}^q \binom{p}{j} \binom{q}{k} (A_z^j A_{\bar{z}}^k \cdot (1)) (A_w^{p-j} A_{\bar{w}}^{q-k} \cdot (1)).
\end{aligned}$$

Thence, we obtain the asserted result according to (3.22). □

Remark 3.11. *An alternative way to prove (3.21) is to write $H_{p,q}(z+w, \bar{z}+\bar{w})$ in terms of the operator A_z as*

$$H_{p,q}(z+w, \bar{z}+\bar{w}) = \left(\frac{1}{2}\right)^{(p+q)/2} \sum_{j=0}^p \binom{p}{j} (A_z + A_w)^q \cdot ((\sqrt{2}z)^j (\sqrt{2}w)^{p-j})$$

and next applying the binomial formula, keeping in mind that the operators A_z and A_w commute and satisfy

$$A_z^r(z^s) = 2^{-s/2} H_{s,r}(\sqrt{2}z, \sqrt{2}\bar{z}).$$

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