

Special functions and the Mellin transforms of Laguerre and Hermite functions

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Abstract

We present explicit expressions for the Mellin transforms of Laguerre and Hermite functions in terms of a variety of special functions. We show that many of the properties of the resulting functions, including functional equations and reciprocity laws, are direct consequences of transformation formulae of hypergeometric functions. Interest in these results is reinforced by the fact that polynomial or other factors of the Mellin transforms have zeros only on the critical line $\text{Re } s = 1/2$. We additionally present a simple-zero Proposition for the Mellin transform of the wavefunction of the D -dimensional hydrogenic atom. These results are of interest to several areas including quantum mechanics and analytic number theory.

Key words and phrases

Mellin transformation, Hermite polynomial, associated Laguerre polynomial, hypergeometric series, transformation formulas, generating function, reciprocity law, recursion relation, functional equation

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Introduction

The work of Bump et al. [5, 6, 17] on a property of the zeros of the Mellin transforms of Hermite and associated Laguerre functions has created significant interest. Since the zeros of these functions lie on the critical line $\text{Re } s = 1/2$, suggestions of connections with the Riemann hypothesis have been made. The Mellin transforms of orthogonal polynomials have additional properties and some of these have been pointed out [5, 6].

In this paper a complementary point of view is developed through the theory of special functions. We obtain explicit results with the aid of, and in terms of, a variety of special functions. In particular, when the Mellin transforms are written in terms of the Gauss hypergeometric function, well known transformation formulae yield functional equations, reciprocity laws, and other properties.

Although many of our final results recover those found in Refs. [5] and [6], our approach may be useful in finding clues to other algebraic connections, since, for instance, the harmonic oscillator Hamiltonian may be given a group theoretic interpretation via the Weil representation of $\text{SL}_2(\mathbb{R})$ [17]. Our work is related to the development of extended theta function representations of the Riemann zeta function ζ . We have very recently shown how to correspondingly generalize the important Riemann-Siegel integral formula [9]. Bump and Ng [5] and Keating [14] have shown how to generalize Riemann's second proof of the functional equation of $\zeta(s)$ by using Mellin and Fourier transforms of Hermite polynomials. These types of results extend to L -functions and automorphic forms [6, 17]. As we encounter the classical Whit-

taker function, there are analogs in a more general automorphic context. Much of this appears to have an interpretation in terms of group representations.

We recall that the eigensolution of the fundamental quantum mechanical problems of the harmonic oscillator and hydrogenic atoms contain Hermite or associated Laguerre polynomials, depending upon the coordinate system used and the spatial dimension (e.g., [21, 8]). Two quantum numbers appear for indexing the energy levels and the angular momentum. These wavefunctions have a wide variety of applicability, including to image processing and the combinatorics of zero-dimensional quantum field theory [8]. Very recently additional analytic properties of these "quantum shapelets" have been expounded [8]. The one-dimensional Coulomb problem has recently reappeared as a model in quantum computing with electrons on liquid helium films [23, 22]. In addition, given the self-reciprocal Fourier transform property of Hermite polynomials, there are several applications in Fourier optics [7, 13]. Hermite and Laguerre polynomials are also important in random matrix theory.

The simple harmonic oscillator and Coulomb problems may be transformed to one another. For example, the 4-dimensional (8-dimensional) harmonic oscillator may be transformed to a 3-dimensional (5-dimensional) Coulomb problem [16, 18]. Many more mappings are realizable, especially for the corresponding radial problems [16]. Therefore, the eigensolutions are closely related for these two problems with central potentials.

We specifically call attention to Ref. [3] wherein Mellin transforms were calculated for hypergeometric orthogonal polynomials and relations within the Askey scheme [15]

were discussed. However, that reference did not consider the location of the complex zeros of the transforms or connections to the integral representations of zeta functions. Moreover, given a family of orthogonal polynomials $p_n(x, \{\alpha_i\})$ where α_i are real parameters, orthogonal with respect to a weight function $w(x, \{\alpha_i\})$, we are most interested in the Mellin transform of the associated functions $\sqrt{w(x, \{\alpha_i\})}p_n(x, \{\alpha_i\})$. This manner of including the weight function is very useful in identifying the orthogonality and other properties of the Mellin transforms, and was often not the case in Ref. [3].

We first consider the Mellin transformation of Laguerre functions, outlining multiple proofs of a result in terms of the hypergeometric function ${}_2F_1$. We then briefly present similar results for the Mellin transforms of Hermite functions. Corollaries of our Propositions include the functional equations and reciprocity laws at negative integers of polynomial or other factors of the transforms. Other results include generating functions and recursion relations of the Mellin transforms. We also strengthen a zero result of Ref. [6]. We give a simple-zero result for the Mellin transform of the wavefunction for D -dimensional hydrogenic atoms and then supply some concluding remarks.

Mellin transform of associated Laguerre functions

We put for the Laguerre functions $\mathcal{L}_n^\alpha(x) = x^{\alpha/2}e^{-x/2}L_n^\alpha(x)$ for $\alpha > -1$, where L_n^α is the associated Laguerre polynomial [2, 12, 19]. We also put for the Mellin transform

$$M_n^\alpha(s) \equiv \int_0^\infty \mathcal{L}_n^\alpha(x)x^{s-1}dx = 2^{s+\alpha/2}\Gamma(s + \alpha/2)P_n^\alpha(s). \quad (1)$$

This transform exists for $\text{Re}(s + \alpha/2) > 0$.

Proposition 1 We have

$$P_n^\alpha(s) = \frac{(1 + \alpha)_n}{n!} {}_2F_1(-n, s + \alpha/2; \alpha + 1; 2), \quad (2)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol and Γ is the Gamma function.

Corollary 1 The polynomials $P_n^\alpha(s)$ satisfy the functional equation

$$P_n^\alpha(s) = (-1)^n P_n^\alpha(1 - s). \quad (3)$$

Corollary 2 The polynomials $P_n^\alpha(s)$ satisfy the reciprocity law

$$\frac{(1 + \alpha)_m}{m!} P_n^\alpha(-m - \alpha/2) = \frac{(1 + \alpha)_n}{n!} P_m^\alpha(-n - \alpha/2). \quad (4)$$

Corollary 3 The polynomials $P_n^\alpha(s)$ have derivative

$$\frac{d}{ds} P_n^\alpha(s) = \frac{(1 + \alpha)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{(1 + \alpha)_k} (s + \alpha)_k \frac{2^k}{k!} \sum_{j=0}^{k-1} \frac{1}{s + \alpha/2 + j}. \quad (5)$$

Corollary 4 The polynomials $P_n^\alpha(s)$ may be written as

$$P_n^\alpha(s) = \frac{(1 + \alpha)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{(1 + \alpha)_k} \frac{2^k}{k!} \sum_{j=0}^k (-1)^{k+j} s(k, j) (s + \alpha/2)^j, \quad (6)$$

where $s(k, j)$ are Stirling numbers of the first kind.

Equation (3) has the form $\xi(s) = a\xi(1 - s)$ where $|a| = 1$. Such a functional equation is standard in the theory of completed zeta functions. Corollary 1 follows from Proposition 1 upon the use of the transformation formula [12]

$${}_2F_1(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha} {}_2F_1\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z - 1}\right). \quad (7)$$

Corollary 2, that has a combinatorial interpretation from counting lattice points [6], follows from Proposition 1 from the obvious symmetry ${}_2F_1(-n, -m; \alpha + 1; 2) = {}_2F_1(-m, -n; \alpha + 1; 2)$. Corollary 3 follows from Proposition 1 by using the series definition of ${}_2F_1$ and the derivative of the Pochhammer symbol in terms of the digamma function $\psi \equiv \Gamma'/\Gamma$. We have

$$\frac{d}{ds} P_n^\alpha(s) = \frac{(1 + \alpha)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{(1 + \alpha)_k} (s + \alpha)_k \frac{2^k}{k!} [\psi(s + \alpha/2 + k) - \psi(s + \alpha/2)]. \quad (8)$$

Applying the functional equation of the digamma function [12] then gives Eq. (5). The same method may be used to obtain higher order derivatives of $P_n^\alpha(s)$. The series representation (6) for $P_n^\alpha(s)$ follows from Proposition 1 by the series definition of ${}_2F_1$ and the expression for a Pochhammer symbol in terms of $s(k, j)$ [1].

In fact the polynomials of Eq. (2) are closely connected with the symmetric Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x, \pi/2)$ [3, 15]. The latter polynomials are a special case of $P_n^{(\lambda)}(x, \phi)$ where $\lambda > 0$ and $0 < \phi < \pi$. For the polynomials of Eq. (2) we identify $P_n^\alpha(s) = (-i)^n P_n^{((1+\alpha)/2)}(i/2 - is, \pi/2)$.

For Proposition 1 we describe six separate proofs. These are in addition to the equivalent series result of Ref. [6] presumably based upon the power series of the associated Laguerre polynomials [12, 19] and termwise integration. We believe that the alternative proofs may contain illuminating intermediate results that may have other applications. Additional proofs are possible using, for example, other integral representations of L_n^α . As this work was being finished, we found that Eqs. (1) and (2) are effectively given in Ref. [20] (p. 245). Our alternative proofs illustrate a variety of analytic techniques and exhibit connections with other special functions of

mathematical physics, in line with the theme of this paper.

First proof. We start with the definition of M_n^α and make a change of variable:

$$M_n^\alpha(s) = 2^{\alpha/2+s} \int_0^\infty e^{-y} y^{\alpha/2+s-1} L_n^\alpha(2y) dy. \quad (9)$$

We then apply the property [10]

$$L_m^\beta(\tau x) = \sum_{n=0}^m \binom{\beta+m}{m-n} \tau^n (1-\tau)^{m-n} L_n^\beta(x) \quad (10)$$

at $\tau = 2$, giving

$$M_n^\alpha(s) = 2^{\alpha/2+s} (-1)^n \sum_{j=0}^n (-1)^j 2^j \binom{\alpha+n}{n-j} \int_0^\infty e^{-y} y^{\alpha/2+s-1} L_j^\alpha(y) dy. \quad (11)$$

The integral on the right side of this equation exists for $\text{Re}(s + \alpha/2) > 0$ and is given in terms of [12] (p. 844) ${}_2F_1(-j, \alpha/2 + s; \alpha + 1; 1)$. In turn, this function value may be reduced to the ratio of Gamma functions [12] (p. 1042) by Chu-Vandermonde summation. Making these substitutions, transforming the Gamma factors, and writing them in terms of Pochhammer symbols, gives Eq. (1) with $P_n^\alpha(s)$ as stated in Eq. (2).

Second proof. Here we start with an integral representation of the associated Laguerre polynomials in terms of Bessel functions of the first kind J_α and develop an intermediate integral representation of $M_n(s)$ in terms of a Whittaker function $M_{\mu,\nu}$. We have [2] (p. 286)

$$\mathcal{L}_n^\alpha(x) = \frac{e^{x/2}}{n!} \int_0^\infty t^{n+\alpha/2} J_\alpha(2\sqrt{xt}) e^{-t} dt, \quad (12)$$

so that the interchange of integrations gives

$$M_n^\alpha(s) = \frac{1}{n!} \int_0^\infty t^{n+\alpha/2} e^{-t} dt \int_0^\infty x^{s-1} e^{x/2} J_\alpha(2\sqrt{xt}) dx. \quad (13)$$

Since [12] (p. 720)

$$\int_0^\infty x^{s-1} e^{x/2} J_\alpha(2\sqrt{xt}) dx = \frac{\Gamma(s + \alpha/2)}{\sqrt{t}\Gamma(\alpha + 1)} e^t \left(-\frac{1}{2}\right)^{1/2-s} M_{s-1/2, \alpha/2}(-2t), \quad (14)$$

with a change of variable we have

$$M_n^\alpha(s) = \frac{\Gamma(s + \alpha/2)}{n!\Gamma(\alpha + 1)} \int_0^\infty y^{n+\alpha/2-1/2} M_{s-1/2, \alpha/2}(y) dy \quad (15)$$

$$= \frac{\Gamma(s + \alpha/2)}{n!\Gamma(\alpha + 1)} (-1)^{n+\alpha/2-s+1} 2^{\alpha/2+s} \Gamma(n+\alpha+1) {}_2F_1(n+\alpha+1, 1-s+\alpha/2; \alpha+1; 2). \quad (16)$$

In the last step we used Ref. [12] (p. 859). Examining the asymptotic form of $M_{\mu, \nu}(z)$ for large $|z|$ shows that formula 7.621.1 there is valid for $\text{Re } s > -1/2$, not just for $\text{Re } s > 1/2$. The reduction of Eq. (16) to Eq. (1) with (2) is accomplished by applying the transformation formula [12]

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; z). \quad (17)$$

Third proof. From the power series of the Laguerre polynomials we have

$$M_n^\alpha(s) = (1+\alpha)_n \sum_{k=0}^n \frac{(-1)^k}{k!(n-k)!(1+\alpha)_k} \int_0^\infty x^{\alpha/2+s-1+k} e^{-x/2} dx. \quad (18)$$

We evaluate the integral in terms of the Gamma function. We then manipulate the series, using $1/(n-k)! = (-n)_k (-1)^k / n!$, and Eq. (2) follows.

Fourth proof. We substitute into the definition of $M_n^\alpha(s)$ the expansion [19] (p. 89)

$$x^{s-\alpha/2-1} = \Gamma(s + \alpha/2)\Gamma(s - \alpha/2) \sum_{j=0}^\infty \frac{(-1)^j L_j^\alpha(x)}{\Gamma(j + \alpha + 1)\Gamma(s - \alpha/2 - j)}, \quad (19)$$

make a change of variable, and twice apply property (10). We then use the orthogonality of the associated Laguerre polynomials with respect to the measure $x^\alpha e^{-x} dx$.

Fifth proof. We make use of the connection with the confluent hypergeometric function ${}_1F_1$ (e.g., [19], p. 273),

$$L_n^\alpha(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n; \alpha; x). \quad (20)$$

We then use a tabulated integral [12] (p. 860), giving Eq. (2). Thus, this method is among the most expedient in obtaining the desired Mellin transform.

Sixth proof. By using the exponential generating function of L_n^α [2] (p. 288), it is readily verified that [6]

$$\sum_{n=0}^{\infty} M_n^\alpha(s) t^n = 2^{s+\alpha/2} \Gamma(s+\alpha/2) (1-t)^{s-\alpha/2-1} (1+t)^{-s-\alpha/2}, \quad |t| < 1. \quad (21)$$

By then using the generalized binomial expansion

$$(1-t)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} t^n = {}_2F_1(a, b; b; t), \quad (22)$$

we have

$$\sum_{n=0}^{\infty} P_n^\alpha(s) t^n = {}_1F_0(\alpha/2 - s + 1; ; t) {}_1F_0(\alpha/2 + s; ; -t) \quad (23a)$$

$$= \sum_{n=0}^{\infty} \sum_{\ell=0}^n \frac{(\alpha/2 - s + 1)_\ell (\alpha/2 + s)_{n-\ell}}{\ell! (n-\ell)!} (-1)^{n-\ell} t^n. \quad (23b)$$

By comparing powers of t on both sides of Eq. (23b) we have

$$P_n^\alpha(s) = \sum_{\ell=0}^n \frac{(\alpha/2 - s + 1)_\ell (\alpha/2 + s)_{n-\ell}}{\ell! (n-\ell)!} (-1)^{n-\ell} \quad (24a)$$

$$= \frac{(-1)^n \Gamma(\alpha/2 + n + s)}{n! \Gamma(s + \alpha/2)} {}_2F_1(-n, 1 + \alpha/2 - s; 1 - \alpha/2 - n - s; -1). \quad (24b)$$

We then apply the linear transformation [20] (p. 48)

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z^{-a} {}_2F_1\left(a, a-c+1; a+b-c+1; 1-\frac{1}{z}\right)$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} {}_2F_1\left(c-a, 1-a; c-a-b+1; 1-\frac{1}{z}\right). \quad (25)$$

The appearance of the prefactor $1/\Gamma(-n)$ before the second resulting ${}_2F_1$ and the existence of simple poles of the Gamma function at nonpositive integers annuls the second term. We then simplify the first term using $\Gamma(-\alpha)/\Gamma(-\alpha-n) = (1+\alpha)_n$ and Eq. (2) follows.

We remark that Eqs. (24) may be obtained as a highly degenerate case of a formula ${}_2F_1(a, b; c; gz) {}_2F_1(\alpha, \beta; \gamma; hz) = \sum_{n=0}^{\infty} c_k z^k$, where the coefficients c_k may be expressed in terms of a generalized hypergeometric function ${}_4F_3$. In our case, $b = c$, $\beta = \gamma$, and $g = -h = 1$.

We next present example recursion relations satisfied by $M_n^\alpha(s)$ and $P_n^\alpha(s)$. We have

Proposition 2.

$$\frac{1}{2}[M_n^\alpha(s) + M_{n+1}^\alpha(s)] = (\alpha/2 + s - 1)[M_{n+1}^\alpha(s-1) - M_n^\alpha(s-1)], \quad (26a)$$

$$\left[1 + \frac{1}{n}\left(\frac{\alpha}{2} + s\right)\right] M_n^\alpha(s) = \frac{1}{2n} M_n^\alpha(s+1) + \left(1 + \frac{\alpha}{n}\right) M_{n-1}^\alpha(s), \quad (26b)$$

$$(s + \alpha/2)[P_n^\alpha(s) + P_{n+1}^\alpha(s)] = (\alpha/2 + s - 1)[P_{n+1}^\alpha(s-1) - P_n^\alpha(s-1)], \quad (27a)$$

and

$$\left[1 + \frac{1}{n}\left(\frac{\alpha}{2} + s\right)\right] P_n^\alpha(s) = \frac{1}{n}(s + \alpha/2 + 1)P_n^\alpha(s+1) + \left(1 + \frac{\alpha}{n}\right) P_{n-1}^\alpha(s). \quad (27b)$$

Proof. Equations (27) follow from Eqs. (26) upon using the definition of $P_n^\alpha(s)$ in Eq. 1. Equation (26a) follows from the use of the relation [12] (p. 1037)

$$L_n^\alpha(x) = \frac{d}{dx}[L_n^\alpha(x) - L_{n+1}^\alpha(x)] \quad (28)$$

in Eq. (1) and integration by parts. Similarly, Eq. (26b) follows from the relation [12] (p. 1037)

$$L_n^\alpha(x) = \frac{x}{n} \frac{d}{dx} L_n^\alpha(x) + \left(1 + \frac{\alpha}{n}\right) L_{n-1}^\alpha(x), \quad (29)$$

and integration by parts.

Bump et al. [6] noted the orthogonality of the polynomials $P_n^\alpha(1/2 + it)$ with respect to the measure $2^{\alpha+1} |\Gamma(1/2 + \alpha/2 + it)|^2 dt$. By applying the Plancherel formula they demonstrated that all the zeros of $P_n^\alpha(s)$ have real part $1/2$ ([6], Theorem 4). By applying classical results [25] (Theorems 3.3.1-3.3.3), we may strengthen their conclusion to

Proposition 3. (a) The zeros of $P_n^\alpha(s)$ are simple and lie on the line $\text{Re } s = 1/2$.
(b) The zeros of $P_n^\alpha(1/2 + it)$ and $P_{n+1}^\alpha(1/2 + it)$ separate each other.

Proposition 3 also follows from Theorems 5.4.1 and 5.4.2 of Ref. [2].

Due to the relations $L_n^{-1/2}(x) = (-1)^n H_{2n}(\sqrt{x})/2^{2n}n!$ and $L_n^{1/2}(x) = (-1)^n H_{2n+1}(\sqrt{x})/\sqrt{x}2^{2n+1}n!$ [19] (p. 81), where H_m are the Hermite polynomials, the theory of the Hermite polynomials may be deduced from that of the associated Laguerre polynomial L_n^α in the cases that $\alpha = \pm 1/2$. Nonetheless, we believe that some separate discussion for Hermite polynomials is in order. We develop the hypergeometric function representation of their Mellin transforms, relate these to other special functions, and consider reciprocity relations for a factor of the Mellin transforms.

Mellin transform of Hermite functions

Following the normalization of Ref. [5], we put

$$f_n(x) = (8\pi)^{-n/2} H_n(\sqrt{2\pi}x) e^{-\pi x^2}. \quad (30)$$

For the Mellin transform, we set

$$M_n(s) = 2 \int_0^\infty f_n(x) x^{s-1} dx = \pi^{-s/2} \Gamma(s/2) p_{n/2}(s), \quad \text{Re } s > 0. \quad (31)$$

We first consider the case of even degree Hermite polynomials and have

Proposition 4.

$$p_n(s) = (8\pi)^{-n} (-1)^n \frac{(2n)!}{n!} {}_2F_1(-n, s/2; 1/2; 2). \quad (32)$$

Corollary 5 The polynomials $p_n(s)$ satisfy the functional equation

$$p_n(s) = (-1)^n p_n(1-s). \quad (33)$$

Corollary 6 The polynomials $p_n(s)$ satisfy the reciprocity law

$$(8\pi)^{-m} (-1)^m \frac{(2m)!}{m!} p_n(-2m) = (8\pi)^{-n} (-1)^n \frac{(2n)!}{n!} p_m(-2n). \quad (34)$$

Corollary 7 The polynomials $p_n(s)$ may be expressed as

$$p_n(s) = (8\pi)^{-n} \frac{\Gamma(s/2 - n)}{\Gamma(s/2)} (2n)! C_{2n}^{s/2-n}(\sqrt{2}), \quad (35)$$

where C_n^λ is the Gegenbauer (generalized Legendre) polynomial [12, 19, 20].

Corollary 5 again follows from the transformation (7). Corollary 6, that may not have been observed before, follows from the symmetry of ${}_2F_1$ when its two numerator

parameters are interchanged. It would be of interest to know if Eq. (34) has a combinatorial interpretation. Corollary 7 follows from the relation [12] (p. 1030)

$${}_2F_1(-n, s/2; 1/2; 2) = (-1)^n \frac{s}{2} B(s/2 - n, n + 1) C_{2n}^{s/2-n}(\sqrt{2}), \quad (36)$$

where B is the Beta function. In this way, known recursion formulas for the Gegenbauer polynomials may be used to obtain the corresponding ones for the polynomials $p_n(s)$.

A great many methods may be used to obtain Eq. (32). We simply indicate two of them here.

Proof 1. If we use the relation [12] (p. 1033), $H_{2n}(x) = (-1)^n (2n)! {}_1F_1(-n, 1/2; x^2)/n!$, with a change of variable we may use a tabulated integral [12] (p. 860).

Proof 2. If we perform the Mellin transform directly, we have

$$\begin{aligned} M_{2n}(s) &= 2(8\pi)^{-n} (2\pi)^{-s/2} \int_0^\infty y^{s-1} H_{2n}(y) e^{-y^2/2} dy \\ &= 2(8\pi)^{-n} (2\pi)^{-s/2} \frac{(-1)^n}{\sqrt{\pi}} 2^{2n-3/2-(s-1)/2} 4^{s/2} \Gamma(s/2) \Gamma(n+1/2) {}_2F_1(-n, s/2; 1/2; 2), \end{aligned} \quad (37)$$

where we used [12] (p. 838). Equation (32) is obtained after simplification and noting that $\Gamma(n+1/2) = \sqrt{\pi} (2n)! 2^{-2n}/n!$.

The function $p_{n/2}(s)$ is no longer a polynomial when n is an odd integer. We now have

Proposition 5.

$$p_{n+1/2} = 2^{5/2} (8\pi)^{-n-1/2} (-1)^n (n+1/2) \frac{(2n)! \Gamma(s/2 + 1/2)}{n! \Gamma(s/2)} {}_2F_1(-n, s/2 + 1/2; 3/2; 2). \quad (38)$$

In this case, the reciprocity law relates $p_{n+1/2}(-2m-1)$ and $p_{m+1/2}(-2n-1)$. The Mellin transform $M_{2n+1}(s)$ may be performed directly, with the aid of [12] (p. 838). While $p_{n+1/2}(s)$ is not a polynomial in s , the truncating ${}_2F_1$ in Eq. (38) is.

Bump and Ng [5] (p. 197) gave a recursion relation for $p_k(s)$. Many more recursion formulae exist for $p_k(s)$ and $M_n(s)$ and we mention two such. We have

Proposition 6

$$M_n(s) = \frac{1}{\sqrt{2\pi}} M_{n-1}(s+1) - \frac{(n-1)}{4\pi} M_{n-2}(s), \quad (39)$$

and

$$M_n(s) = \frac{1}{4(n+1)} \left[M_{n+1}(s+1) - \frac{(s-1)}{2\pi} M_{n+1}(s-1) \right]. \quad (40)$$

Equation (39) follows from the use of $H_n(u) = 2uH_{n-1}(u) - 2(n-1)H_{n-2}(u)$ in Eq. (1) while Eq. (40) follows from the use of $H_n(u) = H'_{n+1}(u)/2(n+1)$ [12] (p. 1033).

Finally, we obtain the generating function of $M_n(s)$ and $p_{n/2}(s)$ and describe how the functional equation (33) arises from that of the parabolic cylinder function D_ν . By using the generating function of the Hermite polynomials [19] (p. 60) we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{M_n(s)}{n!} t^n &= 2(8\pi)^{-n/2} e^{-t^2} \int_0^{\infty} e^{-\pi x^2 + 2\sqrt{2\pi}xt} x^{s-1} dx \\ &= 2(8\pi)^{-n/2} e^{(\pi-1)t^2} (2\pi)^{-s/2} \Gamma(s) D_{-s}(-2t), \quad \text{Re } s > 0, \end{aligned} \quad (41)$$

where we used [12] (p. 337) to evaluate the integral. Thereupon, from Eq. (31), we have

$$\sum_{n=0}^{\infty} \frac{p_{n/2}(s)}{n!} t^n = 2(8\pi)^{-n/2} e^{(\pi-1)t^2} \frac{2^{s/2-1}}{\sqrt{\pi}} \Gamma\left(\frac{s+1}{2}\right) D_{-s}(-2t), \quad (42)$$

where the duplication formula of the Gamma function has been used. The important functional equation (33) is recovered from this equation by using that of D_{-s} [20] (p. 325). Another way to see this is to note the connection with the confluent hypergeometric function [20] (p. 324)

$$D_\nu(z) = 2^{\nu/2} e^{-z^2/4} \left[\frac{\Gamma(1/2)}{\Gamma[(1-\nu)/2]} {}_1F_1(-\nu/2; 1/2; z^2/2) + \frac{z}{\sqrt{2}} \frac{\Gamma(-1/2)}{\Gamma(-\nu/2)} {}_1F_1[(1-\nu)/2; 3/2; z^2/2] \right] \quad (43)$$

and that Kummer's first transformation ${}_1F_1(\alpha; \rho; z) = e^z {}_1F_1(\rho - \alpha; \rho; -z)$ applies.

Mellin transform of the solution for hydrogenic atoms in D -dimensions

We now return to some of the quantum mechanical considerations of the Introduction. We let $\psi(x)$ be the wavefunction for the hydrogenic atom (Coulomb problem) in D -dimensions, where $x = (x_1, \dots, x_D)$ and $r = |x|$. We then have

Proposition 7. The Mellin transform

$$\int_{R^D} \psi(x) r^{s-D/2-1} dx \quad (44)$$

has zeros only on the critical line $\text{Re } s = 1/2$ and these zeros are simple.

Proof. For hydrogenic atoms of nuclear charge Ze , the scaled Hamiltonian is given by $H = -\nabla^2 + V(r)$ with potential energy $V(r) = -Ze^2/r$, where e is the electronic charge. The eigensolutions satisfy $H\psi_{n\ell} = E_n\psi_{n\ell}$, where $n \geq 1$ is the principal quantum number, $\ell = 0, 1, \dots, n-1$ the angular momentum quantum number, and $E_n \propto -1/\eta^2(n)$ the energy levels, with $\eta(n) \equiv n + (D-3)/2$. (The energies are degenerate, meaning that they are independent of ℓ here.) For the central potential, the wavefunction is separable, so that $\psi(x) = YR_{n\ell}(r)$, where the function Y is

independent of r , and explicitly we have [21]

$$R_{n\ell}(r) = \kappa_{n\ell} r^\ell e^{-r/\eta(n)} L_{n-\ell-1}^{2\ell+D-2} \left[\frac{2r}{\eta(n)} \right], \quad (45)$$

where the normalizing constant $\kappa_{n\ell}$ is independent of r . We have

$$\int_{R^D} \psi(x) r^q dx = \int_{S^{D-1}} Y d\Omega \int_0^\infty R_{n\ell}(r) r^q r^{D-1} dr, \quad (46)$$

where S^{n-1} is the sphere in R^n and $d\Omega$ the measure on it. We put Eq. (45) into the radial integration of this equation and change variable to $u = 2r/\eta(n)$. We apply Proposition 3 with $q = s - 1 - D/2$ and Proposition 7 follows.

Remarks. (i) We have just considered the standard Coulomb problem in D dimensions. However, it should be noted that only in three dimensions is the potential $V(r) \propto 1/r$ the same as the Greens function of the Poisson equation $-\nabla^2 \phi = 4\pi\rho$, where ϕ is the electrostatic potential and ρ the charge density [21]. (ii) The relativistic pi-mesic atom is described by the Klein-Gordon equation. This problem may be transformed to the hydrogenic atom in any dimension, and the wave equation solutions are functionally identical [21]. In particular, the same radial wavefunction (45) appears. Therefore, Proposition 7 could instead be stated in terms of the wavefunction of a pi-mesic atom. (iii) The 3-dimensional hydrogenic atom is also separable in parabolic coordinates (ξ, η, ϕ) [24, 4]. In this case the wavefunction $u_{n_1 n_2 m}(\xi, \eta, \phi) \propto e^{-\xi/2} \xi^{|m|/2} L_{n_1+m}^{|m|}(\xi) e^{-\eta/2} \eta^{|m|/2} L_{n_2+m}^{|m|}(\eta) e^{im\phi}$, $m = 0, \pm 1, \pm 2, \dots$, contains Laguerre functions in both of the confocal paraboloid coordinates.

Summary and brief discussion

We have explicitly evaluated the Mellin transforms of Hermite and associated Laguerre functions in terms of several other special functions. In particular, our approach permits the arsenal of known results for hypergeometric functions to be applied. For instance, the transformation formulae and symmetries of the function ${}_2F_1$ directly lead to the functional equation and reciprocity law of the polynomial factor of the Mellin transform. The Hermite and associated Laguerre functions and their Mellin transforms are of much interest in analytic number theory due to their connections with generalizations of Riemann's second proof of the functional equation of his zeta function [5, 6, 17]. This interest is reinforced since the polynomial factors of the Mellin transforms have simple zeros that occur only on the critical line $\text{Re } s = 1/2$.

We have described links of the Mellin transforms to fundamental problems of quantum mechanics, including the isotropic harmonic oscillator and Coulomb problems. Our Proposition 7 evidences such connections for the solution of the Schrodinger equation with central potential. Classically, only for the harmonic oscillator and Kepler problem are the orbits always closed. Transformation between these two problems in quantum mechanics was noted by Schrodinger himself and there is continuing interest in this subject. In addition to our work, known algebraic relations (e.g., [18]) may yield additional insight into the group representation aspects behind the Mellin transforms of interest. We also mention that character sum analogs over finite fields are known for the Hermite polynomials [11].

The orthogonality of factors of Mellin transforms of solutions of the Schrodinger equation may be viewed as follows. Since the Schrodinger equation with suitable boundary conditions is self-adjoint, its eigenfunctions corresponding to distinct eigenvalues (eigenenergies) are orthogonal. Then the isometry of the Fourier (Mellin) transform takes the orthogonality in real space to that in momentum space.

Our Proposition 1 suggests several questions. The most begging question may be the following. If we put $f(s) = {}_2F_1(\beta, s + \alpha/2; \alpha + 1; 2)$, Eq. (7) shows its functional equation to be $f(s) = (-1)^\beta f(1 - s)$. Therefore, we ask what conditions must be placed upon the parameter β in order for f to have zeros only on the critical line? The development of an orthogonality relation for such functions with respect to an appropriate measure would probably address this. At least two other hypergeometric extensions may be possible. We first consider all cases of the extended hypergeometric function ${}_pF_q$ that reduce to polynomials (e.g., [20], p. 64) and correspond to a self-adjoint boundary value problem. We then ask which of these have a functional equation relating argument z to $1 - z$ and possess zeros only along $\text{Re } z = 1/2$. Another direction is to consider multivariate hypergeometric functions. This includes the two-variable Kampé de Fériet function and its generalizations.

Another avenue of generalization would be through the connection of the extended hypergeometric function to the Meijer G -function ([12], p. 1071). In particular, interesting candidate functions are offered by the G -function at argument 1 or -1 , as then the functional relations of G show there are functional relations at parameters a_r , b_s and $1 - b_s$ and $1 - a_r$.

Within the Askey scheme [15], the Wilson polynomials $W_n(x^2; a, b, c, d)$ with $\text{Re}(a, b, c, d) > 0$ are orthogonal on $[0, \infty)$ with respect to the measure $(2\pi)^{-1} |\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)\Gamma(d + ix)/\Gamma(2ix)|^2 dx$. Therefore it seems worthwhile to consider in the future the Mellin transform of the function

$$\mathcal{W}_n(x^2; a, b, c, d) = \frac{1}{\sqrt{2\pi}} \left| \frac{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)\Gamma(d + ix)}{\Gamma(2ix)} \right| W_n(x^2; a, b, c, d). \quad (47)$$

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