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Operational matrices with respect to Hermite polynomials and their applications in solving linear differential equations with variable coefficients

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Abstract. In this paper, a new and efficient approach is applied for numerical approximation of the linear differential equations with variable coefficients based on operational matrices with respect to Hermite polynomials. Explicit formulae which express the Hermite expansion coefficients for the moments of derivatives of any differentiable function in terms of the original expansion coefficients of the function itself are given in the matrix form. The main importance of this scheme is that using this approach reduces solving the linear differential equations to solve a system of linear algebraic equations, thus greatly simplifying the problem. In addition, two experiments are given to demonstrate the validity and applicability of the method.

Keywords: Operational matrices; Hermite polynomials; Linear differential equations with variable coefficients.

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1. Introduction

Orthogonal polynomials play a prominent role in pure, applied and computational mathematics, as well as in the applied sciences and also in many fields of numerical analysis such as quadratures, approximation theory and so on [4, 11, 15, 27]. In particular case, these polynomials have an important role in the spectral methods. These methods (spectral methods) have been successfully applied in the approximation of partial, differential

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and integral equations. Three most widely used spectral versions are the Galerkin, collocation and Tau methods. Their utility is based on the fact that if the solution sought is smooth, usually only a few terms in an expansion of global basis functions are needed to represent it to high accuracy [6–9, 16, 20, 35]. We must note to this point that numerical methods for ordinary, partial and integral differential equations can be classified into the local and global categories. The finite-difference and finite-element methods are based on local arguments, whereas the spectral methods are in the global class [14, 34]. Spectral methods, in the context of numerical schemes for differential equations, belong to the family of weighted residual methods, which are traditionally regarded as the foundation of many numerical methods such as finite element, spectral, finite volume and boundary element methods. Also the linear ODEs with variable coefficients and their solutions play a major role in the branch of modern mathematics and arise frequently in many applied areas. Therefore, a reliable and efficient technique for the solution of them is too important. The analytic results on the existence and uniqueness of solutions to the second order linear ODEs have been investigated by many authors [1, 24], however the existence and uniqueness of the solution for ODEs under their conditions is beyond the scope of this paper. We assume that the ODEs which we consider in this paper with their conditions have solutions. During the last decades, several methods have been used to solve higher order linear ODEs such as Adomian's decomposition method [2, 3, 36], Taylor collocation method [17, 18, 19, 33] Haar functions method [28, 31], Tau method [25, 26], Wavelet method [10], Hybrid function method [21], Legendre wavelet method [30], collocation method based on Jacobi polynomials [22], Taylor polynomial solutions [32], Boubaker polynomial approach [5], and Bernoulli polynomial approach [12]. In this paper, we develop a new and efficient approach to obtain the numerical solution of the general linear differential-difference equations with variable coefficients of the form

$$\begin{aligned}&\sum_{k=1}^{d_j} A_{k,j}(x)y^{(j)}(x) + \sum_{k=1}^{d_{j-1}} A_{k,j}(x)y^{(j-1)}(x) + ... + \sum_{k=1}^{d_0} A_{k,j}(x)y^{(0)}(x) = g(x),\\ &-\infty \leq x \leq +\infty,\\ &j \geq 0, d_t > 0,\ t = 0, ..., j,\end{aligned} \tag{1}$$

with the conditions

$$\sum_{k=0}^{j} \alpha_{ik} y^{(k)}(a_i) = \mu_i,\ \ i = 0, 1, ..., j. \tag{2}$$

The main importance of our work is considering the general linear differential equation (1) with respect to (2) in which the other papers only considered particular cases of our general problem. The remainder of this paper is organized as follows: In section 2, we introduce the properties of Hermite polynomials and the basic formulation of them required for our subsequent development. Section 3, is devoted to the operational matrices of the Hermite polynomials (derivative and moment) with some useful theorems. Section 4, summarizes the application of the Hermite polynomials to the solution of problem (1) and (2). Thus, a set of linear equations is formed and a solution of the considered problem is introduced. Section 5, is devoted to approximations by Hermite polynomials and useful theorems. In section 6, the proposed method is applied for two numerical experiments. Finally, we have monitored a brief conclusion in section 7. Note that we have computed the numerical results by Matlab (version 2013) programming.

1.1 *The Hermite polynomials*

In this part, we define the Hermite polynomials and their properties such as their Sturm-Liouville ordinary differential equation, three terms recursion formula and etc. Let $\Lambda = (-\infty, +\infty)$, then the Hermite polynomials are denoted by H_n, and they are the eigenfunctions of the Sturm-Liouville problem

$$e^{x^2}\left(e^{-x^2}\left(H_n(x)\right)'\right)' + \lambda_n H_n(x) = 0, \quad x \in \Lambda, \tag{3}$$

with the eigenvalues $\lambda_n = 2n$ [14, 34].
Laguerre polynomials are orthogonal in $L^2_{w(x)}(\Lambda)$ space with the weight function $w(x) = e^{-x^2}$, satisfy in the following relation

$$\int_{-\infty}^{+\infty} H_n(x)H_m(x)w(x)dx = \gamma_n\delta_{m,n},\ \gamma_n = \sqrt{\pi}2^n n!, \tag{4}$$

where $\delta_{m,n}$ is the Kronecker delta function. The explicit form of these polynomials is in the form

$$H_n(x) = n!\sum_{i=0}^{[n/2]} \frac{(-1)^i}{i!}\frac{(2x)^{n-2i}}{(n-2i)}. \tag{5}$$

These polynomials are satisfied in the following three terms recurrence formula

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),\ n \geq 1. \tag{6}$$

An important property of the Hermite polynomials is the following derivative relation [14, 34]:

$$H_n'(x) = 2nH_{n-1}(x), \quad \forall n \geq 1,\ x \in \Lambda. \tag{7}$$

Further, $(H_i(\,x))'$ are orthogonal with respect to the weight function $w_{\alpha+k}$. i.e.

$$\int_{-\infty}^{+\infty} H_i'(x)H_j'(x)w(x)dx = \gamma_n\delta_{i,j}, \tag{8}$$

where γ_n is defined in (4).

A function $y(x) \in L^2_{w(x)}(-\infty, +\infty)$, can be expressed in terms of Hermite polynomials as

$$y(x) = \sum_{i=0}^{\infty} a_i H_i(x), \tag{9}$$

where the coefficients a_i is given by

$$a_i = \frac{1}{\gamma_i} \int_{-\infty}^{+\infty} H_i(x) y(x) w(x)(x) dx. \tag{10}$$

In practice, only the first $m+1$ terms of the Hermite polynomials are considered. Then we have:

$$y_m(x) = \sum_{i=0}^{m} a_i H_i(x) = (H_i(x))^T A, \tag{11}$$

where the Hermite coefficient vector A and the Hermite vector $H(x)$ are given by

$$\begin{aligned} A &= [a_0, a_1, ..., a_m]^T, \\ H(x) &= [H_0, H_1, ..., H_m]^T . \end{aligned} \tag{12}$$

2. Operational matrices of the Hermite polynomials (derivative and moment)

In this section, we present the operational matrices of the Hermite polynomials (derivative and moment). To do this, first we introduce the concept of the operational matrix.

2.1 *The operational matrix*

Definition 1. Suppose

$$\phi = [\phi_0, \phi_1, ..., \phi_n], \tag{13}$$

where $\phi_0, \phi_1, ..., \phi_n$ are the basis functions on the given interval $[a, b]$. The matrices $E_{n\times n}$ and $F_{n\times n}$ are named as the operational matrices of derivatives and integrals respectively if and only if

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{dt}}\phi(t) \simeq E\phi(t), \\ &\textstyle\int_a^x \phi(t)dt \simeq F\phi(t). \end{aligned} \tag{14}$$

Further assume $g = [g_0, g_1, ..., g_n]$, named as the operational matrix of the product, if and only if

$$\phi(x)\phi^T(x) \simeq G_g \phi(x). \tag{15}$$

In other words, to obtain the operational matrix of a product, it is sufficient to find $g_{i,j,k}$ in the following relation

$$\phi_i(x)\phi_j(x) \simeq \sum_{k=0}^{i+j} g_{i,j,k}\phi_k(x), \tag{16}$$

which is called the linearization formula [13]. Operational matrices are used in several areas of numerical analysis and they hold particular importance in various subjects such as integral equations [29], differential and partial differential equations [23] and etc. Also many textbooks and papers have employed the operational matrices for spectral methods. Now we present the following theorem.

Theorem 1. If we consider the Hermite approximation

$$y(x) \cong \sum_{i=0}^{m} a_k H_k(x) = (H(x))^T A, \tag{17}$$

then

$$x^i y^{(j)}(x) \cong B^T H(x) = \left(\left(G^i D^j \right)^T A \right)^T H(x), \tag{18}$$

where

$$D_{i,j} = \begin{cases} 2i, & j = i-1, \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

and

$$G_{i,j} = \begin{cases} 1/2, & j = i+1, \\ -i+1, & j = i-1, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Proof: First, we obtain the operational matrix with respect to the derivative operator. For this goal, we must obtain a matrix D which satisfy in the following formula

$$\begin{bmatrix} (H_0(x))' \\ (H_1(x))' \\ \vdots \\ (H_n(x))' \end{bmatrix} = D \begin{bmatrix} H_0(x) \\ H_1(x) \\ \vdots \\ H_n(x) \end{bmatrix}, \tag{21}$$

but by using (7), we can obtain the matrix D as the following

$$D_{i,j} = \begin{cases} 2i, & j = i-1, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Now by j-times repeating the formula (21), we can obtain the operational matrix with respect to $y^{(j)}(x)$ as the following

$$\begin{bmatrix} (H_0(x))^{(j)} \\ (H_1(x))^{(j)} \\ \vdots \\ (H_n(x))^{(j)} \end{bmatrix} = D^j \begin{bmatrix} H_0(x) \\ H_1(x) \\ \vdots \\ H_n(x) \end{bmatrix}. \tag{23}$$

Also for obtaining the operational matrix with respect to moment opertor we must obtain a matrix G, which satisfy in the following relation

$$\begin{bmatrix} xH_0(x) \\ xH_1(x) \\ \vdots \\ xH_n(x) \end{bmatrix} = G \begin{bmatrix} H_0(x) \\ H_1(x) \\ \vdots \\ H_n(x) \end{bmatrix}, \tag{24}$$

but by using (6), we can obtain the matrix G as the following

$$G_{i,j} = \begin{cases} 1/2, & j = i+1, \\ -i+1, & j = i-1, \\ 0, & \text{otherwise}. \end{cases} \tag{25}$$

Now by i-times repeating the formula (24), we can obtain the operational matrix with respect to $x^i y(x)$, as the following

$$\begin{bmatrix} x^iH_0(x) \\ x^iH_1(x) \\ \vdots \\ x^iH_n(x) \end{bmatrix} = G^i \begin{bmatrix} H_0(x) \\ H_1(x) \\ \vdots \\ H_n(x) \end{bmatrix}. \tag{26}$$

Now using formulae (21) and (24), yields

$$\begin{aligned} x^i y^{(j)}(x) &\simeq \sum_{k=0}^{n} a_k x^i (H_k(x))^{(j)} = A^T x^i \begin{bmatrix} (H_0(x))^{(j)} \\ (H_1(x))^{(j)} \\ \vdots \\ (H_n(x))^{(j)} \end{bmatrix} = \\ A^T G^i D^j &\begin{bmatrix} H_0(x) \\ H_1(x) \\ \vdots \\ H_n(x) \end{bmatrix} = \left(\left(G^i D^j\right)^T A \right)^T H(x), \end{aligned} \tag{27}$$

so the proof is completed. □

3. The method of solution

In this section, we describe our new approach for solving the linear differential equations with variable coefficients (1) with respect to the conditions (2). Our approach is based on approximating the exact solution of (1) by truncated Hermite expansion as

$$y(x) \simeq \sum_{i=0}^{m} a_i H_i(x) = (H(x))^T A, \tag{28}$$

where

$$A = [a_0, a_1, ..., a_m]^T, \tag{29}$$

and

$$H(x) = [H_0(x), H_1(x), ..., H_m(x)]. \tag{30}$$

Also we assume that the coefficients $A_{k,j}(x)$ have the Taylor series expansion in the following form

$$A_{k,j} = \sum_{i=0}^{m_j} e_{k,i}^{(j)} x^i. \tag{31}$$

Now by substituting (28) and (31), into (1), we obtain

$$\sum_{k=1}^{s_j}\sum_{i=0}^{m_j} e_{k,i}^{(j)} x^i y^{(j)}(x) + \sum_{k=1}^{s_{j-1}}\sum_{i=0}^{m_{j-1}} e_{k,i}^{(j-1)} x^i y^{(j-1)}(x) + ... + \sum_{k=1}^{s_0}\sum_{i=0}^{m_0} e_{k,i}^{(0)} x^i y^{(0)}(x) \simeq f(x), \tag{32}$$

therefore from (32), we must simplify $x^i \left(y^{(j)}(x)\right)$ as the following

$$\begin{aligned} x^i y^{(j)}(x) \simeq \sum_{k=0}^{m} a_k H_k(x) = (H(x))^T B_{(j)}^{(i)} = \\ (\left(G^i D^j\right)^T A)^T (H(x)), \end{aligned} \tag{33}$$

where D and G, are defined in (19) and (20) respectively. Also we approximate the right hand side of (1) as

$$f(x) = \sum_{i=0}^{m} b_i H_i(x) = (H(x))^T B, \tag{34}$$

where

$$B = [b_0, b_1, ..., b_m]^T, \tag{35}$$

and

$$H(x) = [H_0(x), H_1(x), ..., H_m(x)]. \tag{36}$$

Using formulae (33) and (34) into (32), we obtain

$$\begin{aligned} (H(x))^T \left(\sum_{k=1}^{s_j}\sum_{i=0}^{m_j} e_{i,k}^{(j)} B_{(j)}^{(i)} + \sum_{k=1}^{s_j}\sum_{i=0}^{m_j} e_{i,k}^{(j-1)} B_{(j-1)}^{(i)} + ... \sum_{k=1}^{s_j}\sum_{i=0}^{m_j} e_{i,k}^{(0)} B_{(0)}^{(i)} \right) = \\ \left(L^{(\alpha)}(x)\right)^T F \simeq (H(x))^T B. \end{aligned} \tag{37}$$

From linear independency of the Hermite polynomials, we conclude

$$F = B \tag{38}$$

where

$$F = [f_0, f_1, ..., f_m]. \tag{39}$$

Therefore from (38), we have a system of $m+1$ algebraic equations of $m+1$ unknown coefficients a_i. Finally, we must obtain the corresponding matrix form for the boundary conditions. For this purpose from (2), the values $y^{(j)}(a)$ can be written as:

$$y^{(j)}(a) = (H(x))^T \left(D^j\right)^T A, \ a \in (-\infty, +\infty). \tag{40}$$

Substituting (40) in the boundary conditions (2) and then simplifying it, we obtain the following matrix form

$$\sum_{i=0}^{j} b_{i,l} y^{(l)}(a_i) = (H(x))^T \left\{ \sum_{i=0}^{j} b_{i,l} D^i A \right\} = \sigma_l, \ a_i \in (-\infty, +\infty). \tag{41}$$

Now from (38) and (41), we have $m+j+1$ algebraic equations of $m+1$ unknown coefficients. Thus for obtaining the unknown coefficients, we must eliminate j arbitrary equations from these $m+j+1$ equations. But because of the necessity of holding the boundary conditions, we eliminate the last j equations from (38). Finally, replacing the last j equations of (38) by the j equations of (41), we obtain a system of $m+1$ equations of $m+1$ unknowns a_i.

4. Approximations by Hermite polynomials

Now in this section, we present some useful theorems which show the approximations of functions by Hermite polynomials. For this purpose, let us define $\Lambda = \{x \mid -\infty < x < \infty\}$ and

$$J_N = span\{H_0(x), H_1(x), ..., H_N(x)\}.$$

The $L^2_{w(x)}(\Lambda)-$ orthogonal projection $\pi_N : L^2(\Lambda) \to J_N$ is a mapping in a way that for any $y(x) \in L^2(\Lambda)$, we have:

$$\langle \pi_N(y) - y, \Phi \rangle = 0, \qquad \forall \Phi \in J_N.$$

Due to the orthogonality, we can write

$$\pi_N(y) = \sum_{k=0}^{N-1} c_k H_k(x), \tag{42}$$

where $c_i \ \ (i = 0, 1, ..., N-1)$ are constants in the following form

$$c_i = \frac{1}{\gamma_k} < y(x), H_k >_{L^2_{w(x)}} .$$

In the literature of spectral methods, $\pi_N(y)$ is named as Hermite expansion of $y(x)$ and approximates $y(x)$ on $(-\infty, +\infty)$. In the spectral methods, by substituting the Hermite expansion $\pi_N(y)$ in the ordinary differential equations and their boundary conditions, we obtain a residual term which symbolically is showed by $Res(x)$ as a function of x and N. Different strategies for minimizing a residual term $Res(x)$, leads to different versions of the spectral methods such as Galerkin, Tau and collocation methods. For instance, in the collocation methods the residual term $Res(x)$ is vanished in particular points named as collocated points. Also estimating the distance between $y(x)$ and it's Hermite expansion as measured in the weighted norm $\|.\|_{w^{(\alpha)}}$ is an important problem in numerical analysis. The following theorem provide the basic approximation results for Hermite expansion.

Theorem 2. we have

$$\| \frac{d^l}{dx^l}(\pi_N(y) - y) \|_{w(x)} \leqslant N^{(l-m)/2} \| \frac{d^m}{dx^m}y(x) \|_{w(x)},$$

$$0 \leqslant l \leqslant m, \quad \forall y \in B^m(\Lambda),$$

where

$$B^m(\Lambda) = \{\forall y \in L^2_w : \frac{d^l y}{dx^l} \in L^2_w(\Lambda), \quad 0 \leqslant l \leqslant m\}.$$

Proof: See [14]. □

5. The test experiments

In this section, two numerical experiments are given to illustrate the properties of the method and all of them were performed on the computer using a program written in Matlab 2013.

Experiment 1. Consider the second-order differential equation

$$(x^2 + 1)y''(x) + y'(x) = 1, \tag{43}$$

with the boundary conditions

$$y(0) = 0, y(1) = 1. \tag{44}$$

The exact solution is $y(x) = x$.

Now we approximate the exact solution of (43) by

$$y(x) \simeq \sum_{i=0}^{5} a_i H_i(x) = (H(x))^T A, \tag{45}$$

where

$$A = [a_0, a_1, .., a_5]. \tag{46}$$

Also we expand the right hand side of (43) as

$$1 \simeq \sum_{i=0}^{5} b_i H_i(x) = (H(x))^T B, \tag{47}$$

where

$$B = [1, 0, .., 0]. \tag{48}$$

First we reduce the equation (43) into the following matrix form

$$\left(G^2 D^2 + D^2 + D\right)^T A = 0. \tag{49}$$

Also its boundary conditions as

$$\sum_{i=0}^{5} a_i H_i(0) = (H(0))^T A = 0. \tag{50}$$

and

$$\sum_{i=0}^{5} a_i H_i(1) = (H(1))^T A = 1. \tag{51}$$

By implementation of our method which is presented in section 4, and also after the augmented matrices of the system and boundary conditions are computed, we obtain the solution

$$y(x) = x, \tag{52}$$

which is the exact solution.

Experiment 2. Consider the third-order linear difference equation

$$x^2 y'''(x) + y''(x) = 2, \tag{53}$$

with boundary conditions

$$y(0) = 0,\ y(1) = 1,\ y(-1) = 1. \tag{54}$$

Now we approximate the exact solution of (53) by

$$y(x) \simeq \sum_{i=0}^{5} a_i H_i(x) = (H(x))^T A. \tag{55}$$

Also we expand the right hand side of (53) as

$$2 \simeq \sum_{i=0}^{5} b_i H_i(x) = (H(x))^T B, \tag{56}$$

where

$$B = [2, 0, .., 0]. \tag{57}$$

No we must reduce the equation (53) into the following matrix form

$$\left(G^2 D^3 + D^2\right)^T A = B. \tag{58}$$

and also its boundary conditions as

$$\sum_{i=0}^{5} a_i H_i(0) = (H(0))^T A = 0, \tag{59}$$

$$\sum_{i=0}^{5} a_i H_i(1) = (H(1))^T A = 1. \tag{60}$$

and

$$\sum_{i=0}^{5} a_i H_i(-1) = (H(-1))^T A = 1. \tag{61}$$

After the augmented matrices of the system and boundary conditions are computed, we obtain the solution

$$y(x) = x^2, \tag{62}$$

which is the exact solution.

6. Conclusion

In this paper, we have introduced a new and efficient approach for numerical approximation of linear differential equations with variable coefficients. The method is based on the approximation of the exact solution with Hermite polynomials approximation and also variable coefficients with Taylor series expansion. Implementation of the method reduces the problem to a system of algebraic equations. Two test experiments are presented for showing the accuracy and efficiency of the method.

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