AN ALGEBRAIC INTERPRETATION OF THE CONTINUOUS BIG q-HERMITE POLYNOMIALS

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Abstract

The continuous big q-Hermite polynomials are shown to realize a basis for a representation space of an extended q-oscillator algebra. An expansion formula is algebraically derived using this model.

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Lie algebra theory is well known to provide a unifying framework for discussing special functions. The discovery, some ten years ago, of quantum groups has in turn prompted the undertaking of a systematic investigation of the algebraic properties of the q-analogs of those special functions. One indeed witnesses nowadays intense research activity in this area as q-special functions are seen to have more and more applications.

Within the Askey scheme,[1,2] sets of basic or q-orthogonal polynomials are called continuous because their elements are orthogonal with respect to continuous measures. We have initiated in Ref. [3] a study of these continuous q-polynomials from an algebraic point of view, focusing on the continuous q-Hermite and continuous q-ultraspherical polynomials and, as a result, have shown that various properties of these functions can be derived using symmetry techniques. We indicate here that the class of continuous big q-Hermite polynomials also lends itself to a similar treatment.

We shall be using standard notation.[1,2] The q-hypergeometric series ${}_r\phi_s$ is

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\middle| q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n \ , \tag{1} $$

with

$$ (a_1, a_2, \ldots, a_k; q)_\alpha = (a_1; q)_\alpha (a_2; q)_\alpha \ldots (a_k; q)_\alpha \ , \tag{2a} $$

and

$$ (a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty} \ , \qquad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k) \ , \quad |q| < 1 \ . \tag{2b} $$

Clearly, the series ${}_r\phi_s$ terminates if one of the a_i, $i = 1, \ldots, r$, is equal to q^{-n} with n a positive integer.

The continuous big q-Hermite polynomials $H_n(x; a|q)$ depend on one parameter and are defined as follows:[2]

$$ H_n(x; a|q) = a^{-n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0, \quad 0 \end{matrix}\middle| q; q\right) \tag{3a} $$

$$ = e^{in\theta}\, {}_2\phi_0\left(\begin{matrix} q^{-n}, ae^{i\theta} \\ - \end{matrix}\middle| q; q^n e^{-2i\theta}\right) \ , \qquad x = \cos\theta \ . \tag{3b} $$

When a is real and $|a| < 1$, these polynomials obey the following orthogonality relation:

$$ \frac{1}{2\pi} \int_{-1}^{1} \frac{w(x; a|q)}{\sqrt{1 - x^2}} H_m(x; a|q) H_n(x; a|q)\, dx = \frac{\delta_{mn}}{(q^{n+1}; q)_\infty} \ , \tag{4} $$

where

$$ w(x; a|q) = \left| \frac{(e^{2i\theta}; q)_\infty}{(ae^{i\theta}; q)_\infty} \right|^2 \ . \tag{5} $$

Note that as $q \to 1^-$,

$$ H_n(x; a|q) \to (2x - a)^n \ . \tag{6} $$

The continuous q-Hermite polynomials $H_n(x|q)$ can be defined as the $a \to 0$ limit of the polynomials $H_n(x;a|q)$. This limit can be taken immediately in (3b), leading to

$$H_n(x|q) = e^{in\theta}\ {}_2\phi_0\left(\begin{matrix} q^{-n},\ 0 \\ - \end{matrix}\middle| q; q^n e^{-2i\theta}\right) . \tag{7}$$

The orthogonality relations of these polynomials are obtained by setting $a = 0$ in (4) and (5) and writing $H_n(x;0|q) \equiv H_n(x|q)$.

In the following, we shall show that the continuous big q-Hermite polynomials occur in the realization of a set of basis vectors for a representation space of a q-algebra $\mathcal{G}_q$ that encompasses the q-Heisenberg algebra. This algebraic set up will then be used to derive an expansion formula involving the polynomials $H_n(x;a|q)$.

We shall now present a realization of $\mathcal{G}_q$ in terms of operators acting on functions of the two variables: $x = (z + z^{-1})/2$, with $z = e^{i\theta}$, and t. To do so, we shall need the q-shift operators T_z and T_t whose powers act as follows:

$$\begin{aligned} T_z^\alpha f\left[(z+z^{-1}), t\right] &= f\left[(q^\alpha z + q^{-\alpha} z^{-1}), t\right] , \\ T_t^\beta f\left[(z+z^{-1}), t\right] &= f\left[(z+z^{-1}), q^\beta t\right] , \qquad \alpha, \beta \in \mathbf{R} . \end{aligned} \tag{8}$$

Let,

$$A_+ = \frac{t}{z - z^{-1}}\left(T_z^{1/2} - T_z^{-1/2}\right) , \tag{9a}$$

$$\begin{aligned} A_- = \frac{q^{-1/2}}{t(z-z^{-1})}\Bigg[& \frac{1}{z^2}(1 - q^{-1/2} z T_t^{1/2}) T_z^{1/2} \\ & - z^2\left(1 - \frac{q^{-1/2}}{z} T_t^{1/2}\right) T_z^{-1/2}\Bigg] , \end{aligned} \tag{9b}$$

$$B_+ = \frac{t}{(z - z^{-1})}(z T_z^{-1/2} - \frac{1}{z} T_z^{1/2}) , \tag{9c}$$

$$\begin{aligned} B_- = \frac{1}{t(z-z^{-1})}\Bigg[& z\left(1 - \frac{q^{-1/2}}{z} T_t^{1/2}\right) T_z^{-1/2} \\ & - \frac{1}{z}\left(1 - q^{-1/2} z T_t^{1/2}\right) T_z^{1/2}\Bigg] , \end{aligned} \tag{9d}$$

$$K = T_t . \tag{9e}$$

Notice that as $q \to 1^-$,

$$\frac{1}{1-q} A_+ \to -\frac{t}{2}\frac{\partial}{\partial x}, \quad A_- \to -\frac{1}{t}(2x - 1), \quad B_+ \to t, \quad B_- \to \frac{1}{t} , \tag{10a}$$

and

$$\frac{1-K}{1-q} \to t\frac{\partial}{\partial t} . \tag{10b}$$

In this limit, B_+, B_- and K enlarge in a simple way the Heisenberg algebra that $A_+/(1-q)$ and A_- realize. Consider the set of functions

$$f_n^m(x,t) = t^m H_n(x;q^{m/2}|q) \ , \qquad n \in \mathbf{Z}^+ \ , \qquad m \in \mathbf{Z} \ . \tag{11}$$

It can be checked[4] that the operators A_+, A_-, B_+, B_- and K transform this ensemble of functions onto itself according to:

$$A_+ f_n^m = -q^{-n/2}(1-q^n) f_{n-1}^{m+1} \ , \tag{12a}$$

$$A_- f_n^m = -q^{-(n+1)/2} f_{n+1}^{m-1} \ , \tag{12b}$$

$$B_+ f_n^m = q^{-n/2} f_n^{m+1} \ , \tag{12c}$$

$$B_- f_n^m = q^{-n/2} f_n^{m-1} \ , \tag{12d}$$

$$K f_n^m = q^m f_n^m \ . \tag{12e}$$

It is also natural to consider two additional operators, namely, multiplication by x and by t^2. Take $P = 2x$ and $Q = t^2$. The three-term recurrence relation of the continuous big q-Hermite polynomials[2]

$$2xH_n(x;a|q) = H_{n+1}(x;a|q) + aq^n H_n(x;a|q) + (1-q^n)H_{n-1}(x;a|q) \ , \tag{13}$$

immediately gives the action of $P = 2x$ on the space of functions spanned by the f_n^m. It reads

$$P f_n^m = f_{n+1}^m + q^{n+m/2} f_n^m + (1-q^n) f_{n-1}^m \ . \tag{14}$$

In order to write down the action of $Q = t^2$ on the basis functions, one first observes that the continuous big q-Hermite polynomials satisfy the following identity

$$H_n(x;a|q) = H_n(x;aq|q) - a(1-q^n)\, H_{n-1}(x;aq|q) \ . \tag{15}$$

This formula is most easily proven by checking that both sides verify the same recurrence relation with the same initial condition. It then follows from (15) that

$$Q f_n^m = f_n^{m+2} - q^{m/2}(1-q^n) f_{n-1}^{m+2} \ . \tag{16}$$

The q-algebra that the operators $A_\pm$, $B_\pm$, K, P and Q realize can be characterized by the following relations:

$$\begin{array}{ll}
A_-A_+ - qA_+A_- = -(1-q) \ , & B_+B_- - B_-B_+ = 0 \ , \\
B_+A_+ - q^{1/2}A_+B_+ = 0 \ , & B_-A_+ - q^{1/2}A_+B_- = 0 \ , \\
A_-B_+ - q^{1/2}B_+A_- = 0 \ , & A_-B_- - q^{1/2}B_-A_- = 0 \ , \\
A_+P - q^{1/2}PA_+ = -q^{-1/2}(1-q)B_+ \ , & A_-P - q^{-1/2}PA_- = q^{-1}(1-q)B_- \ , \\
q^{1/2}B_+P - PB_+ = (1-q)A_+ \ , & B_-P - q^{1/2}PB_- = -(1-q)A_- \ , \\
A_+Q - QA_+ = 0 \ , & (QA_- - A_-Q) = q^{-1}(1-q)B_+K^{1/2} \ , \\
B_+Q - QB_+ = 0 \ , & B_-Q - q\,QB_- = (1-q)B_+ \ , \\
KA_+ - qA_+K = 0 \ , & KA_- - q^{-1}A_-K = 0 \ , \\
KB_+ - qB_+K = 0 \ , & KB_- - q^{-1}B_-K = 0 \ , \\
KP - PK = 0 \ , & KQ - q^2QK = 0 \ .
\end{array} \tag{17}$$

Let us make a few comments on this algebra. First note that A_+ and A_- generate the q-Heisenberg algebra.[5] There are also various interesting q-subalgebras. The generators A_+, B_+ and P, for example, form a closed set. We see that A_+ and B_+ q-commute and are in a certain way rotated one into the other by P. The set $\{A_-, B_-, P\}$ also has a similar structure. When $q \to 1$, the algebra exhibits large abelian sectors. Furthermore, some generators (see (10)) become redundant: B_+^2 and Q for instance, have the same limit, and the same is true of B_+A_- and $-2P+1$.

We now want to illustrate how the model given in (9) and (11) can be used to derive properties of the continuous big q-Hermite polynomials. In the Lie theory approach to ordinary special functions, one considers exponentials of the algebra generators and relates their matrix elements in representation spaces to various functions of interest. One then uses diverse realizations to obtain identities and formulas. To proceed similarly in the case of q-special functions, we need q-analogs of the exponential. It has been appreciated that the q-exponential which is naturally associated to the continuous q-orthogonal polynomials is the one first introduced in Ref.[6] and denoted by $\mathcal{E}_q(x;a,b)$. Indeed, one for example finds[6,7,3] that it generates the continuous q-Hermite polynomials:

$$\mathcal{E}_q\left(x;-i,b/2\right) = \left(-b^2/4;q^2\right)_\infty^{-1} \sum_{k=0}^{\infty} \frac{q^{k^2/4}}{(q;q)_k} \left(\frac{ib}{2}\right)^k H_k(x|q) \ . \tag{18}$$

We shall analogously consider the $\mathcal{E}_q$-exponential of the generator $P = 2x$ and determine some of its matrix elements in the bases $\{f_n^m\}$ to obtain an interesting expansion formula in the polynomials $H_n(x;q^m|q)$. However before doing so, we need to define this q-exponential $\mathcal{E}_q$ and to record some of its properties.

We have[6]

$$\mathcal{E}_q(x;a,b) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} \left(aq^{(1-n)/2}e^{i\theta};q\right)_n \left(aq^{(1-n)/2}e^{-i\theta};q\right)_n b^n, \qquad x = \cos\theta \ . \tag{19}$$

In the limit $q \to 1^-$,

$$\mathcal{E}_q\big(x;a,(1-q)b\big) \to \exp\left[(1+a^2-2a\,x)b\right] \ , \tag{20}$$

and, in particular, for $a = -i$,

$$\lim_{q\to 1^-} \mathcal{E}_q\left(x;-i,(1-q)b/2\right) = e^{ibx} \ . \tag{21}$$

The essential feature of these q-exponentials is that they are eigenfunctions of the divided difference operator

$$\tau = \frac{1}{z-z^{-1}}\left(T_z^{1/2} - T_z^{-1/2}\right) \ . \tag{22}$$

Indeed,

$$\tau\, \mathcal{E}_q(x;a,b) = ab\, q^{-1/4}\, \mathcal{E}_q(x;a,b) \ . \tag{23}$$

The continuous q-polynomials obey second order τ-difference equations. Note also that in our model $A_+ = t\tau$ (see $(9a)$). It is thus not surprising, in view of (23), to see $\mathcal{E}_q$ be the appropriate q-exponential to use in connection with continuous q-polynomials. There is one more property of $\mathcal{E}_q$ that we shall need in the following. Consider the function $g_n(b)$ defined by

$$g_n(b) = \mathcal{E}_q\big(-;0, b\,q^{n/2}\big) = \sum_{k=0}^{\infty} \frac{q^{k(k+2n)/4}}{(q;q)_k}\, b^k \ . \tag{24}$$

It is readily verified that $g_n(b)$ satisfies the 3-term recurrence relation

$$g_{n+1}(b) = g_{n-1}(b) - b\,q^{(2n-1)/4}\, g_n(b) \ . \tag{25}$$

As an example of application of our formalism, we will now derive an expansion formula for $\mathcal{E}_q(x;-i,b/2)$ in terms of continuous big q-Hermite polynomials. This q-exponential of the generator $P/2 = x$ acts, of course, on the representation space of our q-algebra. Recall that $K = T_t$ is diagonal on the basis $\{f_n^m\}$: $K\,f_n^m = q^m f_n^m$. Since P and K commute, we must have

$$\mathcal{E}_q\,(x;-i,b/2)\ f_0^m = \sum_{n=0}^{\infty} W_n^m(b)\, f_n^m \ . \tag{26}$$

Note that we are considering the action of the q-exponential of x on the particular basis vectors $f_0^m(x,t) = t^m$. The expansion coefficients $W_n^m(b)$ will be obtained from the recursion relations that they obey. These relations will be found by exploiting properties of the $\mathcal{E}_q$-exponential and making use of the representation (12), (15), (16). Let us first act on both sides of (26) with $A_+ = t\tau$. With the help of (23), we see on the one hand that

$$\begin{aligned} A_+\mathcal{E}_q\,(x;-i,b/2)\, f_0^m &= -i\frac{b}{2}\, q^{-1/4}\mathcal{E}_q\,(x;-i,b/2)\, f_0^{m+1} \\ &= -i\frac{b}{2}\, q^{-1/4}\sum_{n=0}^{\infty} W_n^{m+1}(b)\, f_n^{m+1} \ , \end{aligned} \tag{27}$$

the last equality following from (26). On the other hand, using $(12a)$, we have

$$\begin{aligned} A_+\mathcal{E}_q\,(x;-i,b/2)\ f_0^m &= \sum_{n=0}^{\infty} W_n^m(b) A_+\, f_n^m \\ &= -\sum_{n=0}^{\infty} q^{-n/2}(1-q^n) W_n^m(b)\, f_{n-1}^{m+1} \ . \end{aligned} \tag{28}$$

Equating the right-hand sides of (27) and (28), we then find

$$i(b/2)q^{-1/4}\, W_n^{m+1}(b) = q^{-(n+1)/2}(1-q^{n+1})\, W_{n+1}^m(b) \ . \tag{29}$$

Second, we act similarly on both sides of (26) with $Q = t^2$. Clearly,

$$Q\,\mathcal{E}_q\,(x;-i,b/2) = \mathcal{E}_q\,(x;-i,b/2)\,f_0^{m+2} \\ = \sum_{n=0}^{\infty} W_n^{m+2}(b)\,f_n^{m+2}\ , \tag{30}$$

while (16) yields

$$Q\,\mathcal{E}_q\,(x;-i,b/2)\ f_0^m = \sum_{n=0}^{\infty} W_n^m(b)\,Q\,f_n^m \\ = \sum_{n=0}^{\infty} W_n^m(b)\left[f_n^{m+2} - q^{m/2}(1-q^n)f_{n-1}^{m+2}\right]\ . \tag{31}$$

Combining (30) and (31), we get

$$W_n^{m+2}(b) = W_n^m(b) - (1-q^{n+1})q^{m/2}\,W_{n+1}^m(b)\ . \tag{32}$$

Finally, we replace in this last equation $W_{n+1}^m(b)$ by the expression that (29) gives for it to find

$$W_n^{m+2}(b) = W_n^m(b) - i\frac{b}{2}\,q^{(n+m+1/2)/2}\,W_n^{m+1}(b)\ . \tag{33}$$

The matrix elements $W_n^m(b)$ can now be explicitly determined from the two recurrence relations (29) and (33), that we have found for them. Separation of the discrete variables is readily achieved in these equations by taking $W_n^m(b)$ of the form

$$W_n^m(b) = u_n(b)\,y_{m+n}(b)\ . \tag{34}$$

Indeed, substitution of (34) in (29) and (33), respectively, gives

$$u_{n+1}(b) = \frac{q^{(2n+1)/4}}{1-q^{n+1}}\left(\frac{ib}{2}\right)u_n(b)\ , \tag{35}$$

and

$$y_{m+n+2}(b) = y_{m+n}(b) - \frac{ib}{2}\,q^{(m+n+1/2)/2}\,y_{m+n+1}(b)\ . \tag{36}$$

The recurrence relation (35) is easily solved and fixes $u_n(b)$ up to a function $u_0(b)$:

$$u_n(b) = u_0(b)\,\frac{q^{n^2/4}}{(q;q)_n}\left(\frac{ib}{2}\right)^n\ . \tag{37}$$

The 3-term recurrence relation (36) is recognized as the one already given in (25) and $y_{m+n}(b)$ is thus immediately identified:

$$y_{m+n}(b) = \mathcal{E}_q\left(-;0,\frac{ib}{2}q^{(m+n)/2}\right)\ . \tag{38}$$

(It is understood that the overall arbitrary function of b in the solution of (36), is to be absorbed in $u_0(b)$.) If we use the realization $f_n^m(x,t) = t^m H_n\big(x; q^{m/2}|q\big)$ and factor out the t-dependence, (26) becomes

$$\mathcal{E}_q\,(x; -i, b/2) = \sum_{n=0}^{\infty} W_n^m(b)\, H_n\big(x; q^{m/2}|q\big)\ . \tag{39}$$

At this point,

$$W_n^m(b) = u_0(b)\,\frac{q^{n^2/4}}{(q;q)_n}\,\mathcal{E}_q\left(-;0,\frac{ib}{2}q^{(m+n)/2}\right)\left(\frac{ib}{2}\right)^n\ . \tag{40}$$

We will therefore have obtained the identity we are looking for, once we will have determined $u_0(b)$. To this end, notice that as $m \to \infty$, or $q^m \to 0$,

$$W_n^m(b) \to u_0(b)\,\frac{q^{n^2/4}}{(q;q)_n}\left(\frac{ib}{2}\right)^n\ ; \tag{41}$$

recall also that $H_n(x;0|q) \equiv H_n(x|q)$. Hence, in the limit $q^m \to 0$, (39) must coincide with the expansion formula for $\mathcal{E}_q(x;-i,b/2)$ in terms of continuous q-Hermite polynomials already given in (18). Comparison immediately shows that

$$u_0(b) = (-b^2/4; q^2)_\infty^{-1}\ . \tag{42}$$

Putting everything together finally gives the following expansion formula of the $\mathcal{E}_q$-exponential of x in terms of continuous big q-Hermite polynomials:

$$\begin{aligned}\mathcal{E}_q(x;-i,b/2) = (-b^{2/4};q^2)_\infty^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n}\,\mathcal{E}_q\left(-;0,\frac{ib}{2}q^{(m+n)/2}\right)\left(\frac{ib}{2}\right)^n \\ \times H_n(x, q^{m/2}|q)\ .\end{aligned} \tag{43}$$

The constructive derivation of this identity illustrates well the usefulness of algebraic techniques for obtaining and interpreting properties of continuous q-polynomials. We plan to pursue investigations in this direction.

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References

1. Gasper, G. and Rahman, M., *Basic Hypergeometric Series*, (Cambridge University Press, Cambridge, 1990)
2. Koekoek, R. and Swarttouw, R. F., The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 94–05, Delft University of Technology (1994).
3. Floreanini, R. and Vinet, L., A model for the continuous q-ultraspherical polynomials, CRM–2233, Université de Montréal (1995)
4. Kalnins, E. G. and Miller, W., Symmetry techniques for q-series: Askey-Wilson polynomials, Rocky Mountain J. Math. **19** (1989), 223–230
5. Floreanini, R. and Vinet, L., q-Orthogonal polynomials and the oscillator quantum group, Lett. Math. Phys. **22** (1991), 45–54
6. Ismail, M. E. H. and Zhang, R., Diagonalization of certain integral operators, Adv. Math. **109** (1994), 1–33.
7. Al-Salam, W., A characterization of the Rogers q-Hermite polynomials, University of Alberta, preprint, 1994