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Advances in Applied Mathematics 40 (2008) 149–167

ADVANCES IN Applied Mathematics

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Hankel determinants for some common lattice paths

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Received 9 July 2006; accepted 7 August 2006

Available online 8 January 2007

Abstract

For a fixed positive integer ℓ, let $f(n,\ell)$ denote the number of lattice paths that use the steps $(1,1)$, $(1,-1)$, and $(\ell,0)$, that run from $(0,0)$ to $(n,0)$, and that never run below the horizontal axis. Equivalently, $f(n,\ell)$ satisfies the quadratic functional equation $F(x)=\sum_{n\geqslant 0} f(n,\ell)x^n = 1 + x^\ell F(x) + x^2 F(x)^2$. Let H_n denote the n by n Hankel matrix, defined so that $(H_n)_{i,j} = f(i+j-2,\ell)$. Here we investigate the values of their determinants where $\ell = 1,2,3$. For $\ell = 1,2$ we are able to employ the Gessel–Viennot–Lindström method. For the case $\ell = 3$, the sequence of determinants forms a sequence of period 14, namely,

$$\big(\det(H_n)\big)_{n\geqslant 1} = (1,1,0,0,-1,-1,-1,-1,-1,0,0,1,1,1,1,1,0,0,-1,-1,-1,\ldots).$$

For this case we are able to use the continued fractions method recently introduced by Gessel and Xin. We also apply this technique to evaluate Hankel determinants for other generating functions satisfying a certain type of quadratic functional equation.

Keywords: Hankel determinants; Lattice paths; Gessel–Viennot–Lindström; Continued fractions

1. Introduction

We will consider lattice paths that use the following three steps: $U=(1,1)$, the up diagonal step; $H=(\ell,0)$, the horizontal step of length ℓ, where ℓ is a fixed positive integer; and $D=(1,-1)$, the down diagonal step. Further, each H-step will be weighted by t, and the others by 1.

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0196-8858/$ – see front matter
doi:10.1016/j.aam.2006.08.007

The weight of a path is the product of the weights of its steps. The weight of a path set is the sum of the weights of its paths.

Let $f(n,t,\ell)$ denote the weight of the set of paths running from (0, 0) to $(n,0)$ that never run below the x-axis. When $t=1$, weight becomes cardinality. (Pergola et al. [9] and Sulanke [11] have considered such paths for various values of ℓ and have given additional references.) For example,

- $f(n,t,1)$ is the weight of a set of Motzkin paths, counted by the Motzkin numbers: $(f(0,1,1), f(1,1,1), f(2,1,1),\dots)=(1,1,2,4,9,21,51,127,323,835\dots)$.
- $f(n,0,1)$ counts the Dyck paths to $(n,0)$ and is an aerated Catalan number: $(f(0,0,0), f(1,0,0), f(2,0,0),\dots)=(1,0,1,0,2,0,5,0,14,0,42,0,132,0,429,\dots)$.
- $f(n,t,2)$ is the weight of a set of large Schröder paths, counted by the aerated large Schröder numbers: $(f(0,1,2), f(1,1,2), f(2,1,2),\dots)=(1,0,2,0,6,0,22,0,90,0,394,0,1806,\dots)$.
- $(f(0,1,3), f(1,1,3), f(2,1,3),\dots)=(1,0,1,1,2,3,6,10,20,36,72,136,273,532,\dots)$.

Let

$$F(x)=\sum_{n\geqslant 0} f(n,t,\ell)x^n$$

denote the generating function for $f(n,t,\ell)$. We find, by a common combinatorial decomposition, that $F(x)$ is the formal power series satisfying

$$F(x)=1+tx^{\ell}F(x)+x^2F(x)^2.$$

Any sequence $A=(a_0,a_1,a_2,\dots)$ defines a sequence of Hankel matrices, $H_1,H_2,H_3,\dots,$ where H_n is an n by n matrix with entries $(H_n)_{i,j}=a_{i+j-2}$. For instance, the sequence $(f(n,1,3))_{n\geqslant 0}$ yields

$$H_1=[1],\qquad H_2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\qquad H_3=\begin{bmatrix}1&0&1\\0&1&1\\1&1&2\end{bmatrix},\qquad H_4=\begin{bmatrix}1&0&1&1\\0&1&1&2\\1&1&2&3\\1&2&3&6\end{bmatrix}.$$

For $F(x)=\sum_{n\geqslant 0} f(n,t,\ell)x^n$, our interest is the sequence of determinants $\det(H_n(F))$ where each n by n matrix $H_n(F)$ has entries

$$\big(H_n(F)\big)_{i,j}=f(i+j-2,t,\ell).$$

The following propositions constitute our main results:

Proposition 1.1. *For $n\geqslant 0$, $\ell=1$, and arbitrary t (including $t=0$, yielding the Dyck path case)*

$$\det\big(H_n(F)\big)=1.$$

Proposition 1.2. *For $n \geqslant 0$, $\ell = 2$, and arbitrary t (including $t = 0$, yielding the Dyck path case),*

$$\det\bigl(H_n(F)\bigr) = \begin{cases} (1+t)^{n^2/4} & \text{if } n \text{ is even}, \\ (1+t)^{(n-1)(n+1)/4} & \text{if } n \text{ is odd}. \end{cases}$$

Proposition 1.3. *For $t = 1$ and $\ell = 3$,*

$$\bigl(\det\bigl(H_n(F)\bigr)\bigr)_{n\geqslant 1}^{14} = (1, 1, 0, 0, -1, -1, -1, -1, -1, 0, 0, 1, 1, 1).$$

Moreover, if $n = m \bmod 14$ and $m, n \geqslant 0$, then $\det(H_m(F)) = \det(H_n(F))$.

In Section 2, using the well-known combinatorial method of Gessel–Viennot–Lindström [3, 5,13], we will prove Propositions 1.1 and 1.2. Our proof of Proposition 1.1 is essentially that of Viennot [13] who also used the method to calculate various other Hankel determinants related to Motzkin paths. Aigner [1] also studied such determinants. We note that earlier Shapiro [10] demonstrated that each Hankel determinant for the usual Catalan numbers has value 1. For the large Schröder numbers $(r_n)_{n\geqslant 0} = 1, 2, 6, 22, 90, 394, \ldots$ whose generating function satisfies

$$R(x) = \sum_{n\geqslant 0} r_n x^n = 1 + xR(x) + xR(x)^2,$$

we show that the n-order Hankel determinant is $2^{n(n-1)/2}$, as stated in Proposition 2.1.

We remark that the problem of evaluating Hankel determinants corresponding to a generating function has received significant attention, in particular by Wall [14]. One of the basic tools for such evaluation is the method of continued fractions, either by J-fractions in Krattenthaler [8] or Wall [14] or by S-fractions in Jones and Thron [7, Theorem 7.2]. However, both of these methods need the condition that the determinant can never be zero, a condition not always present in our study. Recently, Brualdi and Kirkland [4] used the J-fraction expansion to calculate Hankel determinants for various sequences related to the Schröder numbers. A slight modification of their proof of [4, Lemma 4.7] proves our Proposition 2.1 for $t = 1$.

In Section 3 we establish the periodicity of 14 for the case $\ell = 3$ of Proposition 1.3, by the continued fractions method recently developed by Gessel and Xin [6]. In the final section, we review their technique more generally: it yields a transformation for generating functions, satisfying a certain quadratic functional equation, that also transforms the associated Hankel determinants in a simple manner. We apply their technique to evaluate the Hankel determinants for the cases $\ell = 1, 2$ (again) and for other cases related to $\ell = 3$.

A search for periodicity of Hankel determinants for other integer pairs (ℓ, t) for $3 \leqslant \ell \leqslant 10$ and $1 \leqslant |t| \leqslant 5$ yielded only the cases for $(\ell, t) = (3, -1), (4, -1), (4, -2)$ to be of mild interest. The conclusion of Proposition 1.3 holds for $(\ell, t) = (3, -1)$. Moreover, it is elementary to prove that the sequence of Hankel determinants for any $-t$ agrees with that for t whenever ℓ is odd. One can use the methods of Section 4 to prove sequence of Hankel determinants for $(\ell, t) = (4, -1)$ has period 10 beginning with $1, 1, 0, 0, -1, 0, 0, 1, 1, 1$ and that for $(\ell, t) = (4, -2)$ has period 8 beginning with $1, 1, -1, 1, -1, 1, 1, 1$.

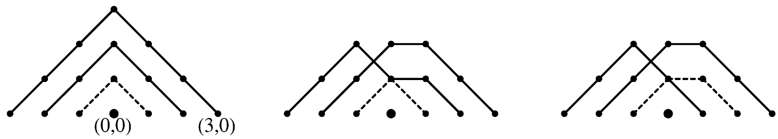

Fig. 1. Some of the 4-tuples of paths for $\ell = 1$ and for I-T-CONFIG with $[(0,0),(-1,0),(-2,0),(-3,0)]$ and $[(0,0),(1,0),(2,0),(3,0)]$. In each of these 4-tuples there is a point path (a path of zero length) at $(0,0)$. The first 4-tuple is the only nonintersecting 4-tuple for $\ell = 1$. The second and third 4-tuples are intersecting only at the point $(0,1)$. The second 4-tuple corresponds to the permutation 1243, having sign of -1, while the third corresponds to the permutation 1342, having sign of 1. These two 4-tuples cancel one another under the Gessel–Viennot–Lindström method.

2. Employing the Gessel–Viennot–Lindström method

Assuming a rudimentary knowledge of the Gessel–Viennot–Lindström method [3,5,13], we reformulate it and use it to prove some of the results indicated above. All lattice paths use the three steps as previously defined. Given an n-tuple of lattice paths on the $\mathbb{Z} \times \mathbb{Z}$ plane, we say that it is *intersecting* if two of the paths meet at a common step end point; otherwise, we say that it is *nonintersecting*. Thus a nonintersecting n-tuple may have paths that cross or touch at a point which are not a step end point. See Fig. 1.

Let $[(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)]$ and $[(x'_1, y'_1), (x'_2, y'_2), \dots, (x'_n, y'_n)]$ denote two lists of distinct lattice points such that

$$x_{k+1} \leqslant x_k \leqslant 0 \quad \text{and} \quad 0 \leqslant y_k \leqslant y_{k+1}$$

and

$$0 \leqslant x'_k \leqslant x'_{k+1} \quad \text{and} \quad 0 \leqslant y'_k \leqslant y'_{k+1}.$$

We will refer to such a pair of lists as an "I-T-CONFIG" of order n as their points will be the *initial* and *terminal* points for each n-tuple of paths being considered.

Let $P_{i,j}$ denote the set of all paths running from (x_i, y_i) to (x'_j, y'_j) that never run below the x-axis, with $|P_{i,j}|$ denoting the sum of the weights of its paths. Let S_n denote the set of permutations on $\{1, 2, 3, \dots, n\}$. For any permutation $\sigma \in S_n$, let P_σ denote the set of all n-tuples of paths $(p_1, p_2, \dots, p_n)$, where $p_i \in P_{i,\sigma(i)}$ for $1 \leqslant i \leqslant n$. The *signed weight* of $(p_1, p_2, \dots, p_n) \in P_\sigma$ is defined to be $\operatorname{sgn}(\sigma)$ times the product of the weights of the n paths. See Figs. 1 and 2.

For our purpose the Gessel–Viennot–Lindström method is formulated in a form similar to that in Viennot's notes [13]:

Lemma 2.1. *Given an* I-T-CONFIG *of order n, the sum of the signed weights of the nonintersecting n-tuples in $\bigcup_{\sigma \in S_n} P_\sigma$ is equal to* $\det((|P_{i,j}|)_{1 \leqslant i,j \leqslant n})$.

Proof of Proposition 1.1. (A similar proof appears in [13].) By Lemma 2.1 $\det(H_n)$ is equal to the sum of the signed weights of the nonintersecting n-tuples in $\bigcup_{\sigma \in S_n} P_\sigma$ for the I-T-CONFIG where $(x_i, y_i) = (-i+1, 0)$ and $(x'_i, y'_i) = (i-1, 0)$, for $1 \leqslant i \leqslant n$. Thus, for this I-T-CONFIG, we seek the nonintersecting n-tuples. First, the 1-tuple $P_{1,1}$ contains just the point path beginning and ending at $(0,0)$. Next, any nonintersecting path from $(-i+1, 0)$, for $1 < i \leqslant n$, must begin with a U-step, while any nonintersecting path to $(j-1, 0)$, for $1 < j \leqslant n$, must end with a D-step. Repeating this analysis at each integer-ordinate level k, shows the nonintersecting path from $(-i+1,0)$, $1 \leqslant i \leqslant k$, is forced to be a sequence of U-steps followed by a sequence of D-steps; moreover, it shows that any nonintersecting path from $(-i+1,0)$ to $(j-1,0)$, $k<i,j$, must start with k U-steps and end with k D-steps. Inductively, each nonintersecting path is a sequence of U-steps followed by a sequence of D-steps. The n-tuple of such paths is the only nonintersecting n-tuple of $\bigcup_{\sigma \in S_n} P_\sigma$, and it has weight 1. □

We will use the following in proving Proposition 1.2:

Lemma 2.2. *For the lattice paths that use the steps U, $H=(2,0)$, and D, that never run below the x-axis, and that have the* I-T-CONFIG*,*

$$(x_i, y_i) = (-2i+2, 0) \quad \textit{and} \quad (x'_i, y'_i) = (2i-2, 0)$$

for $1 \leqslant i \leqslant n$, the sum of the signed weights of the nonintersecting n-tuples in $\bigcup_{\sigma \in S_n} P_\sigma$ equals $(1+t)^{n(n-1)/2}$.

Proof. For $(p_1, p_2, \ldots, p_n) \in (P_{1,\sigma(1)}, P_{2,\sigma(2)}, \ldots, P_{n,\sigma(n)})$, suppose that $(p_1, p_2, \ldots, p_n)$ is a nonintersecting n-tuple of paths for some permutation σ. Since the points in the I-T-CONFIG are spaced two units apart, the horizontal distance at any integer ordinate between any two paths of $(P_{1,\sigma(1)}, P_{2,\sigma(2)}, \ldots, P_{n,\sigma(n)})$ must be even. It follows inductively that, for $1 \leqslant i \leqslant n$, any path of the path set $P_{i,\sigma(i)}$ must begin with a sequence of $i-1$ U-steps and finish with a sequence of $\sigma(i)-1$ D-steps. Thus, computing the weight of the nonintersecting n-tuples is equivalent to computing the weight of the nonintersecting n-tuples for the new ("V" shaped) initial-terminal configuration, denoted by I-T-CONFIG-NEW, defined by

$$(x_i, y_i) = (-i+1, i-1) \quad \text{and} \quad \left(x'_i, y'_i\right) = (i-1, i-1)$$

for $1 \leqslant i \leqslant n$.

Before continuing, we notice, for example when $t=1$ and $n=4$, that the matrix $M(0)$ defined by $(M(0)_{i,j})_{1 \leqslant i,j \leqslant 4} = (|P'_{i,j}|)_{1 \leqslant i,j \leqslant 4}$ for I-T-CONFIG-NEW is an array of Delannoy numbers. (See [2,12].) When $t=0$, $M(0)$ is the initial array from Pascal's triangle. In the following array for $t=1$, the entries count the ways a chess king can move from the north-west corner if it uses only east, south, or south-east steps. Momentarily we will see the role of the argument 0 in $M(0)$:

$$M(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 5 & 13 & 25 \\ 1 & 7 & 25 & 63 \end{bmatrix}.$$

Now for arbitrary t and n, let $M(0)$ be the n by n matrix defined recursively by

$$M(0)_{i,j} = M(0)_{i-1,j} + tM(0)_{i-1,j-1} + M(0)_{i,j-1}$$

for $1<i$ and $1<j$ with $M(0)_{1,j}=1$ and $M(0)_{i,1}=1$ for $1 \leqslant i$ and $1 \leqslant j$. By Lemma 2.1, $\det(M(0))$ is equal to the weight of the nonintersecting n-tuples for I-T-CONFIG. The proof is completed once we show

$$\det\bigl(M(0)\bigr) = (1+t)^{n(n-1)/2}.$$

Given $M(0)$, we recursively define a sequence of n by n matrices

$$M(0),\ M(1),\ M(2),\ \ldots,\ M(n-1)$$

where, for $1 \leqslant k \leqslant n-1$,

$$M(k)_{ij} = \begin{cases} M(k-1)_{i,j} & \text{for } 1 \leqslant i \leqslant k, \\ M(k-1)_{i,j} - M(k-1)_{i-1,j} & \text{for } k+1 \leqslant i \leqslant n. \end{cases}$$

With CLAIM(k) denoting the claim that

$$\begin{aligned} &M(k)_{i,j} = M(k)_{i-1,j} + tM(k)_{i-1,j-1} + M(k)_{i,j-1} \quad \text{for } i, j > k, \\ &M(k)_{i,i} = (1+t)^{i-1} \quad \text{for } i \leqslant k, \\ &M(k)_{i,j} = 0 \quad \text{for } i > j \text{ and } j \leqslant k, \\ &M(k)_{i,k+1} = (1+t)^k \quad \text{for } i \geqslant k+1, \end{aligned}$$

one can establish CLAIM(k) for $1 \leqslant k \leqslant n-1$ by induction. Since $M(n-1)$ is upper triangular, we observe that

$$\det\bigl(M(n-1)\bigr) = (1+t)^{n(n-1)/2}.$$

By the type of row operations used to obtain the sequence $M(0), M(1), M(2), \ldots, M(n-1)$, their determinants are equal. □

Since, by the I-T-CONFIG of Lemma 2.2, $(H)_{i,j} = |P_{i,j}|$ counts the large Schröder paths from $(0,0)$ to $(2i+2j-2,0)$, immediately we have the following corollary for the Hankel determinants of the weighted (non-aerated) large Schröder numbers:

Proposition 2.1. *Let f_n denote the weight of the set of paths from $(0,0)$ to $(2n,0)$ which never run beneath the x-axis and where $H=(2,0)$ is weighted by t. Equivalently, let f_n satisfy*

$$F(x) = \sum_{n \geqslant 0} f_n x^n = 1 + txF(x) + xF(x)^2.$$

Then the determinant of the nth order Hankel matrix H_n, with $(H_n)_{i,j} = f_{i+j-2}$, equals $(1+t)^{n(n-1)/2}$.

As a second corollary to Lemma 2.2, we have

Lemma 2.3. *For the lattice paths that use the steps U, $H=(2,0)$, and D, that never run below the x-axis, and that have the* I-T-CONFIG *with*

$$(x_i, y_i) = (-2i+1, 0) \quad \textit{and} \quad \bigl(x_i', y_i'\bigr) = (2i-1, 0)$$

for $1 \leqslant i \leqslant n$, the sum of the signed weights for the nonintersecting n-tuples in $\bigcup_{\sigma \in S_n} P_\sigma$ is $(1+t)^{n(n+1)/2}$.

Proof. We first translate all paths upwards one unit and then prepend a U-step and append a D-step to every path. Next we add the point path at $(0, 0)$. The sum of the signed weights of the nonintersecting n-tuples in the original configuration equals that of the nonintersecting $(n + 1)$-tuples in this new configuration, which in turn is given by Lemma 2.2. □

Proof of Proposition 1.2. Suppose that n is even; the proof when n is odd is similar. Here the Hankel matrix $(|P_{i,j}|)_{1\leqslant i,j\leqslant n}$ corresponds to the I-T-CONFIG with

$$(x_i, y_i) = (-i + 1, 0) \quad \text{and} \quad \left(x_i', y_i'\right) = (i - 1, 0) \quad \text{for } 1 \leqslant i \leqslant n.$$

Since $\ell = 2$, no endpoint of a step on a path that originates from an oddly indexed initial point (i.e., a point $(-i + 1, 0)$ for odd i) will intersect an endpoint of a step on a path that originates from an evenly indexed initial point. Moreover, for any permutation σ corresponding to a nonintersecting n-tuple, $\sigma(i) - i$ must be even for each i, and hence $\operatorname{sgn}(\sigma) = 1$. Thus the weight of the nonintersecting n-tuples is the product of the weight of those originating from oddly indexed initial points times the weight of those originating from evenly indexed initial points.

Hence, with $m = n/2$, let I-T-CONFIGA have

$$(x_i, y_i) = (-2i + 2, 0) \quad \text{and} \quad \left(x_i', y_i'\right) = (2i - 2, 0) \quad \text{for } 1 \leqslant i \leqslant m,$$

and let I-T-CONFIGB have

$$(x_i, y_i) = (-2i + 1, 0) \quad \text{and} \quad \left(x_i', y_i'\right) = (2i - 1, 0) \quad \text{for } 1 \leqslant i \leqslant m.$$

Applying Lemmas 2.2 and 2.3 to these configurations yields the weight of nonintersecting n-tuples of the original configuration as

$$(1 + t)^{m(m-1)/2}(1 + t)^{m(m+1)/2} = (1 + t)^{m^2} = (1 + t)^{n^2/4}. \qquad \square$$

Next we consider Hankel determinants for sequences of path weights that ignore the initial term. For the sequence $f(1, t, \ell), f(2, t, \ell), \ldots,$ we will let $H_n^1(F)$ denote the matrix where the entries satisfy $(H_n^1(F))_{i,j} = f(i + j - 1, t, \ell)$. See Fig. 2.

Proposition 2.2. *For* $\ell = 1$ (*Motzkin case again*), *the sequence of determinants satisfies the recurrence*

$$\det\bigl(H_n^1(F)\bigr) = t \det\bigl(H_{n-1}^1(F)\bigr) - \det\bigl(H_{n-2}^1(F)\bigr)$$

subject to $\det(H_1^1(F)) = t$ *and* $\det(H_2^1(F)) = (t - 1)(t + 1)$.

Proof. For arbitrary t (Aigner [1] considered the case for $t = 1$), our proof considers how the particular paths must look like in the nonintersecting case. Observe that $\det(H_n^1(F))$ is the sum of the weights of the nonintersecting n-tuples for the I-T-CONFIG(n) taken as

$$\bigl[(0, 0), (-1, 0), \ldots, (n - 1, 0)\bigr] \quad \text{and} \quad \bigl[(1, 0), (2, 0), \ldots, (n, 0)\bigr].$$

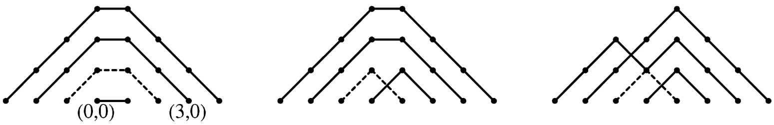

Fig. 2. Three of the 4-tuples of paths for $\ell = 1$ and for I-T-CONFIG with $[(0,0), (-1,0), (-2,0), (-3,0)]$ and $[(1,0), (2,0), (3,0), (4,0)]$. The first and second 4-tuples are both nonintersecting. The first has a signed weight of t^4 while the second has a signed weight of $-t^2$. The third is intersecting only at the point $(0,1)$.

Each of these nonintersecting n-tuples belongs to one of two types: (1) those containing the path from $(0,0)$ to $(1,0)$ with all other paths forced to begin with U, end with D, and have ordinate at least one elsewhere; (2) those containing the path UD from $(0,0)$ to $(2,0)$ and the path UD from $(-1,0)$ to $(1,0)$ with all other paths forced to begin with UU, end with DD, and have ordinate at least two elsewhere. The set of the first type has a total weight t times the sum of the weights of the nonintersecting $(n-1)$-tuples on the I-T-CONFIG$(n-1)$, which is $t\det(H^1_{n-1}(F))$. Since each n-tuple of the second type has the defined crossing of the path from $(0,0)$ with that from $(-1,0)$, the set has a total weight the sign of the corresponding permutation times the sum of the weights of the nonintersecting $(n-2)$-tuples on the I-T-CONFIG$(n-2)$, which is $-\det(H^1_{n-2}(F))$. □

For $\ell = 2$, we will indicate how Lemma 2.3 proves

Proposition 2.3. *For $n \geqslant 0$, $\ell = 2$, and arbitrary t, the sequence of determinants satisfies*

$$\det\bigl(H^1_n(F)\bigr) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ (-1)^{n/2}(1+t)^{n(n+2)/4} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Here the Hankel matrix can correspond to I-T-CONFIG with

$$(x_i, y_i) = (-i+1, 0) \quad \text{and} \quad \bigl(x'_i, y'_i\bigr) = (i, 0) \quad \text{for } 1 \leqslant i \leqslant n.$$

Since $\ell = 2$, if there is a path from (x_i, y_i) to (x'_j, y'_j), then $i - j$ is odd. It follows that, if n is odd, there can be no n-tuples of paths for the configuration. If n is even and $m = n/2$, the sign of any permutation for a nonintersecting n-tuples can be shown to be $(-1)^m$. Thus the weight of the nonintersecting n-tuples is $(-1)^m$ times the weight of those originating from oddly indexed initial points times the weight of those originating from evenly indexed initial points. The proof is completed by applying 2.3 to I-T-CONFIGA with

$$(x_i, y_i) = (-2i+2, 0) \quad \text{and} \quad \bigl(x'_i, y'_i\bigr) = (2i, 0) \quad \text{for } 1 \leqslant i \leqslant m,$$

and to I-T-CONFIGB with

$$(x_i, y_i) = (-2i+1, 0) \quad \text{and} \quad \bigl(x'_i, y'_i\bigr) = (2i-1, 0) \quad \text{for } 1 \leqslant i \leqslant m. \qquad \square$$

3. Periodicity fourteen and continued fractions

Here we will repeatedly apply the "continued fractions method" recently developed by Gessel and Xin [6] to establish the periodicity of the sequence of Hankel determinants for $\ell = 3$ and

$t=1$. This method, presented more formally in the next section, uses the following two lemmas in transforming a generating function and its Hankel determinants into a new generating function and its Hankel determinants. We start with the generating function $F_0(x)$ satisfying

$$F_0(x) = 1 + x^3 F_0(x) + x^2 F_0(x)^2.$$

We remark that from this functional equation, or from the related recurrence for its coefficients, there appears to be no clue why the associated sequence of Hankel determinants should have a period of 14.

For an arbitrary generating bivariate function $D(x,y) = \sum_{i,j=0}^{\infty} d_{i,j} x^i y^j$, let $[D(x,y)]_n$ denote the n by n determinant $\det((d_{i,j})_{0\leqslant i,j\leqslant n-1})$. For any $A(x) = \sum_{n\geqslant 0} a_n x^n$, define the *Hankel matrix for A of order n*, $n \geqslant 1$, by $H_n(A) = (a_{i+j-2})_{1\leqslant i,j\leqslant n}$. The proofs of the following are elementary.

Lemma 3.1. *The Hankel determinant satisfies*

$$\det\bigl(H_n(A)\bigr) = \left[\frac{xA(x) - yA(y)}{x-y}\right]_n.$$

Lemma 3.2 *("A product rule"). If $u(x)$ is a formal power series with $u(0)=1$, then*

$$\bigl[u(x)D(x,y)\bigr]_n = \bigl[D(x,y)\bigr]_n = \bigl[u(y)D(x,y)\bigr]_n.$$

We will make five transformations which introduce the generating functions, labeled $F_i(x)$, $1 \leqslant i \leqslant 5$, in showing

$$\det\bigl(H_n(F_0)\bigr) = \det\left(\operatorname{diag}\left([1],[1],\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix},[1],[1],H_{n-7}(F_0)\right)\right),$$

for $n \geqslant 8$, where the right side is the determinant of a block-diagonal matrix consisting of six blocks along the diagonal (four of which are 1 by 1 identity matrices) and having entry 0 elsewhere. It then follows that $\det(H_n(F_0)) = -\det(H_{n-7}(F_0))$. This implies that the period for $\det(H_n(F_0))$ is 14, and Proposition 1.3 will be proved.

Here $F_i(x)$ will always be a power series satisfying a quadratic functional equation

$$a(x)F_i(x)^2 + b(x)F_i(x) + c(x) = 0,$$

which is equivalent to the continued fraction

$$F_i(x) = \frac{-c(x)}{b(x) + a(x)F_i(x)}.$$

In particular, for $\ell = 3$ and $t = 1$,

$$F_0(x) = \frac{1}{1 - x^3 - x^2 F_0(x)}.$$

Transformation 1. Using this continued fraction of F_0, substitution, and simplification we obtain

$$\det\bigl(H_n(F_0)\bigr) = \left[\frac{xF_0(x) - yF_0(y)}{x - y}\right]_n$$
$$= \left[\frac{-xy^2F_0(y) + yx^2F_0(x) + (x - y)(yx^2 + xy^2 + 1)}{(1 - x^3 - x^2F_0(x))(1 - y^3 - y^2F_0(y))(x - y)}\right]_n.$$

Multiplying by $(1 - x^3 - x^2F_0(x))(1 - y^3 - y^2F_0(y))$, which will not affect the value of the determinant by the product rule, we can write the determinant as

$$\left[1 + xy\frac{xF_1(x) - yF_1(y)}{x - y}\right]_n$$

where

$$F_1(x) = F_0(x) + x. \tag{3.1}$$

The associated matrix is block-diagonal with two blocks: the matrix [1] and the Hankel matrix for $F_1(x)$. Certainly,

$$\det\bigl(H_n(F_0)\bigr) = \det\bigl(H_{n-1}(F_1)\bigr).$$

From (3.1) and the functional equation for $F_0(x)$, we obtain the functional equation

$$F_1(x) = \frac{1 + x}{1 + x^3 - x^2F_1(x)}.$$

Transformation 2. Using this continued fraction for F_1, substituting in $\frac{xF_1(x)-yF_1(y)}{x-y}$, and multiplying by $(1 + x^3 - x^2F_1(x))(1 + y^3 - y^2F_1(y))$ yields

$$\left[\frac{xF_1(x) - yF_1(y)}{x - y}\right]_n$$
$$= \left[\frac{-xy^2(x + 1)F_1(y) + yx^2(y + 1)F_1(x) - (y + 1)(x + 1)(xy - 1)(x - y)}{x - y}\right]_n.$$

Upon multiplying by $(1 + x)^{-1}(1 + y)^{-1}$, the determinant is equal to

$$\left[1 + xy\frac{xF_2(x) - yF_2(y)}{x - y}\right]_n,$$

where

$$F_2(x) = F_1(x)/(1 + x) - 1. \tag{3.2}$$

The associated matrix being block-diagonal shows

$$\det\big(H_{n-1}(F_1)\big) = \det\big(H_{n-2}(F_2)\big).$$

From (3.2) and the functional equation for $F_1(x)$, we obtain

$$F_2(x) = \frac{x^2}{1 - 2x^2 - x^3 - (x^3 + x^2)F_2(x)}.$$

Transformation 3. Substituting for F_2 with the above fraction, simplifying, and multiplying by $(1+x)(1-x-x^2-x^2F_2(x))(1+y)(1-y-y^2-y^2F_2(y))$ shows that the determinant $[\frac{xF_2(x)-yF_2(y)}{x-y}]_n$ equals

$$\left[\frac{y^2x^3(y+1)F_2(y) - x^2y^3(x+1)F_2(x) - (x-y)(2y^2x^2 - x^2 - xy - y^2)}{x-y}\right]_n$$

which can be rewritten as

$$\left[x^2 + xy + y^2 - 2x^2y^2 + x^3y^3\frac{xF_3(x) - yF_3(y)}{x-y}\right]_n,$$

where $F_3(x)$ is indeed a power series satisfying

$$F_3(x) = (x+1)F_2(x)/x^2. \tag{3.3}$$

This time the corresponding matrix is a block-diagonal matrix with the block $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ followed by the Hankel matrix for $F_3(x)$. Hence

$$\det\big(H_{n-2}(F_2)\big) = -\det\big(H_{n-5}(F_3)\big).$$

From (3.3) and the functional equation for $F_2(x)$, we obtain

$$F_3(x) = \frac{1+x}{1 - 2x^2 - x^3 - x^4F_3(x)}.$$

Transformation 4. Substituting for F_3 with the fraction, simplifying, and multiplying by $(1 - 2x^2 - x^3 - x^4F_3(x))(1 - 2y^2 - y^3 - y^4F_3(y))$ the determinant $[\frac{xF_3(x)-yF_3(y)}{x-y}]_n$ equals

$$\left[\frac{-xy^4(x+1)F_3(y) + yx^4(y+1)F_3(x) + (y+1)(x+1)(xy+1)(x-y)}{x-y}\right]_n.$$

By multiplying the generating function by $(1+x)^{-1}(1+y)^{-1}$, this determinant becomes

$$\left[1 + xy\frac{xF_4(x) - yF_4(y)}{x-y}\right]_n,$$

where

$$F_4(x) = 1 + x^2 F_3(x)/(1+x). \tag{3.4}$$

Therefore,

$$\det\bigl(H_{n-5}(F_3)\bigr) = \det\bigl(H_{n-6}(F_4)\bigr).$$

From (3.4) and the functional equation for $F_3(x)$, we obtain

$$F_4(x) = \frac{1}{1 + x^3 - (x^3 + x^2)F_4(x)}.$$

Transformation 5. Substituting for F_4 with the above fraction, simplifying, and multiplying by $(1 - x^2 F_4(x))(1 - y^2 F_4(y))$ the determinant $[\frac{xF_4(x) - yF_4(y)}{x-y}]_n$ equals

$$\left[\frac{-xy^2(y+1)F_4(y) + x^2y(x+1)F_4(x) - (x-y)(yx^2 + xy^2 - 1)}{x-y}\right]_n$$
$$= \left[1 + xy\frac{xF_5(x) - yF_5(y)}{x-y}\right]_n,$$

where $F_5(x) = (1+x)F_4(x) - x$. Hence, $\det(H_{n-6}(F_4)) = \det(H_{n-7}(F_5))$.

Finally, it is routinely checked that $F_5(x) = F_0(x)$.

4. The quadratic transformation for Hankel determinants

One can use the method introduced in the previous section to evaluate the Hankel determinants for generating functions satisfying a certain type of quadratic functional equation. The generating functions $F(x)$ in this section are the unique solution of a quadratic functional equation satisfying

$$F(x) = \frac{x^d}{u(x) + x^k v(x) F(x)}, \tag{4.1}$$

where $u(x)$ and $v(x)$ are rational power series with nonzero constants, d is a nonnegative integer, and k is a positive integer. Note that if $k = 0$, $F(x)$ is not unique. Our task now is to derive a transformation $\mathcal{T}$ so that $\det(H_n(F)) = a\det(H_{n-d-1}(\mathcal{T}(F)))$ for some value a and nonnegative integer d. In addition to Hankel matrices for the power series $A = \sum_{n\geqslant 0} a_i x^i$, we will consider *shifted Hankel matrices*: $H_n^k(A)$ denotes the matrix $(a_{i+j+k-2})_{1\leqslant i,j\leqslant n}$. Shifted matrices have appeared in Propositions 2.2 and 2.3.

The first proposition is elementary:

Proposition 4.1. *If F satisfies* (4.1)*, then $G = u(0)F$ satisfies*

$$\det\bigl(H_n(G)\bigr) = u(0)^n \det\bigl(H_n(F)\bigr), \quad \textit{and} \quad G(x) = \frac{x^d}{u(0)^{-1}u(x) + x^k u(0)^{-2} v(x) G(x)}.$$

Proposition 4.2. *Suppose F satisfies (4.1) with $u(0) = 1$. We separate $u(x)$ uniquely as $u(x) = u_L(x) + x^{d+2}u_H(x)$, where $u_L(x)$ is a polynomial of degree at most $d + 1$ and $u_H(x)$ is a power series.*

(i) *If $k = 1$, then there is a unique G such that*

$$G(x) = \frac{-v(x) - xu_L(x)u_H(x)}{u_L(x) - x^{d+2}u_H(x) - x^{d+1}G(x)}.$$

Moreover,

$$G(x) = -xu_H(x) - x^{-d}v(x)F(x)$$

and a shifted matrix appears with

$$\det\bigl(H^1_{n-d-1}\bigl(G(x)\bigr)\bigr) = (-1)^{d(d+1)/2}\det\bigl(H_n\bigl(F(x)\bigr)\bigr).$$

(ii) *If $k \geqslant 2$, then there is a unique G such that*

$$G(x) = \frac{-x^{k-2}v(x) - u_L(x)u_H(x)}{u_L(x) - x^{d+2}u_H(x) - x^{d+2}G(x)}.$$

Moreover,

$$G(x) = -u_H(x) - x^{k-d-2}v(x)F(x)$$

and

$$\det\bigl(H_{n-d-1}\bigl(G(x)\bigr)\bigr) = (-1)^{d(d+1)/2}\det\bigl(H_n\bigl(F(x)\bigr)\bigr).$$

Proof. We prove only part (ii) as part (i) is similar. The generating function for $H_n(F)$ is given by

$$\begin{aligned}\frac{xF(x) - yF(y)}{x - y} &= \frac{1}{x - y}\left(\frac{x^{d+1}}{u(x) + x^k v(x)F(x)} - \frac{y^{d+1}}{u(y) + y^k v(y)F(y)}\right)\\ &= \frac{-y^{d+1}u(x) - y^{d+1}x^k v(x)F(x) + x^{d+1}u(y) + x^{d+1}y^k v(y)F(y)}{(u(x) + x^k v(x)F(x))(u(y) + y^k v(y)F(y))(x - y)}.\end{aligned}$$

We can multiply by $(u(x) + x^k v(x)F(x))$ and by $(u(y) + y^k v(y)F(y))$ without changing the above determinant by the product rule. Next we observe that x^d divides $F(x)$, and write $u(x) = u_L(x) + x^{d+2}u_H(x)$ as in the proposition. The resulting generating function can be written as

$$\begin{aligned}&\frac{-y^{d+1}u_L(x) + x^{d+1}u_L(y)}{x - y}\\ &\quad + (xy)^{d+1}\frac{-x(u_H(x) - x^{k-d-2}v(x)F(x)) + y(u_H(y) + y^{k-d-2}v(y)F(y))}{x - y}.\end{aligned}$$

We now set $G(x) = -u_H(x) - x^{k-d-2}v(x)F(x)$, which can be straightforwardly shown to agree with the defining functional equations. Suppose that $u_L(x) = 1 + a_1x + \cdots + a_{d+1}x^{d+1}$, then $[\frac{xF(x)-yF(y)}{x-y}]_n$ is equal to the determinant of the block-diagonal matrix

$$\operatorname{diag}\left(\begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \vdots & a_{d-1} \\ 1 & a_1 & \cdots & a_d \end{pmatrix}, H_{n-d-1}\big(G(x)\big)\right).$$

The determinant of the first block is easily seen to be $(-1)^{d(d+1)/2}$. □

Given these propositions and that $H^1(A) = H(x^{-1}(A(x) - A(0)))$ for any series A, we can now define our transformation $\mathcal{T}(F)$: *For F satisfying* (4.1),

- *if $u(0) \neq 1$, then $\mathcal{T}(F) = G$, as given in Proposition* 4.1;
- *if $u(0) = 1$ and $k = 1$, then $\mathcal{T}(F) = x^{-1}(G(x) - G(0))$, with G given in Proposition* 4.2(i);
- *if $u(0) = 1$ and $k \geqslant 2$, then $\mathcal{T}(F) = G$, as given in Proposition* 4.2(ii).

Moreover, the relation between $\det(H_n(F))$ *and* $\det(H_n(\mathcal{T}(F)))$ *is given in Propositions* 4.1 *and* 4.2.

Example 1 *(Other proofs of Propositions 1.1 and 2.2).* For Motzkin paths with arbitrary t, the generating function $F(x)$ satisfies

$$F(x) = \frac{1}{1 - tx - x^2F(x)}.$$

Applying Proposition 4.2, $F_1 = \mathcal{T}(F)$ gives

$$\det\big(H_{n-1}(F_1)\big) = \det\big(H_n(F)\big), \quad \text{where } F_1(x) = \frac{1}{1 - tx - x^2F_1(x)}.$$

Hence, $F(x) = F_1(x)$, and consequently $\det(H_n(F(x))) = 1$ for all n.

Whereas the Gessel–Viennot–Lindström method leads to a proof in the shifted case for arbitrary t, as in Proposition 2.2, we have been able to use the continued fractions technique only for $t = 1$ and $t = 2$.

For $t = 1$ we will show that $(\det(H_n^1(F)))_{n\geqslant 1} = (1, 0, -1, -1, 0, 1, 1, \ldots)$, continuing with period 6. Let $G_1(x) = (F(x) - 1)/x$, so that $\det(H_n^1(F)) = \det(H_n(G_1))$. Let $G_2 = \mathcal{T}(G_1)$ and $G_3 = \mathcal{T}(G_2)$, both under Proposition 4.2(ii). Since

$$G_1(x) = \frac{1 + x}{1 - x - 2x^2 - x^3G_1(x)}$$

with $d = 0$, $k = 3$, $u(x) = u_L(x) = 1 - 2x$, $u_H = 0$, and $v(x) = -(1 + x)^{-1}$, we find that

$$G_2(x) = \frac{x}{1 - x - 2x^2 - x^2(1 + x)G_2(x)}$$

with $d = 1$, $k = 2$, $u(x) = u_L(x) = 1 - x - 2x^2$, $u_H = 0$, and $v(x) = -(1 + x)$. Applying Proposition 4.2(ii) shows

$$\begin{aligned} G_3(x) &= -x^{-1}\bigl(-(1+x)\bigr)G_2(x) \\ &= -x^{-1}\bigl(-(1+x)\bigr)\bigl(-x\bigl(-(1+x)^{-1}\bigr)\bigr)G_1(x) \\ &= G_1(x) \end{aligned}$$

and $\det(H_{n-3}(G_3)) = -\det(H_{n-1}(G_2)) = -\det(H_n(G_1))$, which yields the periodicity of the sequence of determinants.

For $t = 2$ we will show that $\det(H_n^1(F)) = n + 1$ for $n \geqslant 1$. Define G_1 to satisfy

$$G_1(x) = \frac{2+x}{1-2x-2x^2-x^3G_1(x)}.$$

One can easily see that $G_1(x) = (F(x) - 1)/x$ with $G_1(0) = u_1(0)^{-1} = \det(H_1(G_1)) = \det(H_1^1(F)) = 2$. For $n \geqslant 2$, define G_n to satisfy

$$G_n(x) = \frac{(n-1)^2(n^2+n+x)}{(n^2-n)(n^2-2n^2x-2x^2) - n^2(n^2-n+x)x^2G_n(x)}.$$

By induction one can show that $G_n = \mathcal{T} \circ \mathcal{T}(G_{n-1})$ (under Proposition 4.1 then under Proposition 4.2), and that $G_n(0) = u_n(0)^{-1} = (n-1)(n+1)/n^2$. Also by induction and Proposition 4.1, for $n \geqslant 2$,

$$\begin{aligned} \det\bigl(H_n(G_1)\bigr) &= \left[2^n \prod_{i=2}^{n-1}\left(\frac{(i-1)(i+1)}{i^2}\right)^{n+1-i}\right]\det\bigl(H_1(G_n)\bigr) \\ &= 2^n \prod_{i=2}^{n}\left(\frac{(i-1)(i+1)}{i^2}\right)^{n+1-i} \end{aligned}$$

which simplifies to $\det(H_n(G_1)) = n + 1$.

Example 2 *(Another proof of Proposition 1.2).* For large Schröder paths arbitrary t, we have

$$F(x) = \frac{1}{1-tx^2-x^2F(x)}.$$

Applying $\mathcal{T}$ gives

$$\det\bigl(H_{n-1}(F_1)\bigr) = \det\bigl(H_n(F)\bigr), \quad \text{where } F_1(x) = \frac{1+t}{1+tx^2-x^2F_1(x)}.$$

Applying $\mathcal{T}$ again, we obtain

$$(1+t)^n \det\bigl(H_{n-1}(F_2)\bigr) = \det\bigl(H_n(F_1)\bigr), \quad \text{where } F_2(x) = \frac{1}{1-tx^2-x^2F_2(x)}.$$

This implies $F_2 = F$, and hence the recurrence $\det(H_n(F)) = (1+t)^{n-1}\det(H_{n-2}(F))$, with initial condition $\det(H_1(F)) = 1$, and $\det(H_2(F)) = 1+t$.

Example 3 *(Another proof of Proposition 2.1).* Consider the continued fraction

$$F(x) = \frac{1}{1 - tx - xF(x)},$$

where $F(x)$ is the generating function for the Catalan numbers for $t = 0$ and the large Schröder numbers for $t = 1$.

Under Proposition 4.2(i) we have a unique G_1 such that $G_1(x) = F(x)$ and $\det(H^1_{n-1}(G_1)) = \det(H_n(F))$. Taking $G_2 = (G_1(x) - 1)/x = (F(x) - 1)/x$, we have

$$G_2(x) = \frac{(1+t)}{1 - (2+t)x - x^2 G_2(x)}$$

where $\det(H_{n-1}(G_2)) = \det(H^1_{n-1}(F))$ and $u(x) = (1 - (2+t)x)/(1+t)$.

Under Proposition 4.1 we have a unique G_3,

$$G_3(x) = \frac{1}{1 - (2+t)x - (1+t)x^2 G_3(x)},$$

with $G_3(x) = G_2/(1+t)$ and $\det(H_{n-1}(G_3)) = (1+t)^{-(n-1)}\det(H_{n-1}(G_2))$.

Under Proposition 4.2(ii) we have a unique G_4 such that $G_4(x) = (1+t)G_3(x)$ and $\det(H_{n-2}(G_4)) = \det(H_{n-1}(G_3))$.

We see that $G_4(x) = G_2(x)$; thus $\det(H_{n-1}(G_2)) = (1+t)^{n-1}\det(H_{n-2}(G_2))$ with $\det(H_1(G_2)) = 1+t$. Hence $\det(H_n(F)) = \det(H_{n-1}(G_2)) = (1+t)^{n(n-1)/2}$.

Example 4 *(Another proof of Proposition 2.3).* To compute $\det(H^1_n(F))$, first we consider

$$H^1_n(F) = H_n(F_1), \quad \text{where } F_1 = \frac{(t+1)x}{1 - (2+t)x^2 - x^3 F_1}.$$

Applying $\mathcal{T}$ shows that $\det(H_n(F_1)) = -(1+t)^n \det(H_{n-2}(F_1))$.

Example 5. For $\ell = 3$, recall the functional equation

$$F_0(x) = \frac{1}{1 - tx^3 - x^2 F_0(x)}.$$

For arbitrary t, our transformation gives more and more complicated expressions. This is not surprising since the Hankel determinants do not factor nicely. However, for $t = 1$ and for $k = 1, 2, 3$, the transformation gives nice results similar to that of Proposition 1.3: indeed, sequences of $\det(H^k_n(F_0))$ also have period 14. For $k = 4$ there is an interesting result.

Subexample 5i. The sequence for $\det(H^1_n(F_0))$ starts with $0, -1, 0, 1, 1, 0, -1, 0, 1, 0, -1, -1, 0, 1$. If we define F_1 so that $F_0(x) = 1 + xF_1(x)$, then

$$\det\bigl(H_n(F_1)\bigr) = \det\bigl(H^1_n(F_0)\bigr), \quad \text{with } F_1 = \frac{x(x+1)}{1 - 2x^2 - x^3 - x^3 F_1} \text{ and } d = 1.$$

Then applying $\mathcal{T}$ repeatedly so $\mathcal{T}(F_i) = F_{i+1}$, we obtain

$$\det\bigl(H_{n-2}(F_2)\bigr) = -\det\bigl(H_n(F_1)\bigr), \quad \text{where } F_2 = \frac{x}{(x+1)(1-x-x^2-x^3F_2)} \text{ and } d=1;$$

$$\det\bigl(H_{n-2}(F_3)\bigr) = -\det\bigl(H_n(F_2)\bigr), \quad \text{where } F_3 = \frac{1+x-x^2}{1-2x^2+x^3-x^3F_3} \text{ and } d=0;$$

$$\det\bigl(H_{n-1}(F_4)\bigr) = \det\bigl(H_n(F_3)\bigr), \quad \text{where } F_3 = \frac{x}{(1+x-x^2)(1-x-x^2F_3)} \text{ and } d=1;$$

$$\det\bigl(H_{n-2}(F_5)\bigr) = -\det\bigl(H_n(F_4)\bigr), \quad \text{where } F_5 = \frac{x(x+1)}{1-2x^2-x^3-x^3F_5}.$$

The periodicity is established by noticing that $F_5 = F_1$ and $\det(H_{n-7}(F_5)) = -\det(H_n(F_1))$.

Subexample 5ii. The sequence for $\det(H_n^2(F_0))$ starts with $1, 1, 1, 1, 0, 0, -1, -1, -1, -1, -1, 0, 0, 1$. If we define G_0 so that $F_0(x) = 1 + x^2G_0(x)$, then $\det(H_n(G_0)) = \det(H_n^2(F_0))$,

$$G_0 = \frac{1+x}{1-2x^2-x^3-x^4G_0}.$$

One can establish the periodicity using Proposition 4.2. However, this generating function has appeared in Transformation 3 of Section 3, where one can see that

$$\det\bigl(H_n(G_0)\bigr) = -\det\bigl(H_{n+5}(F_0)\bigr). \tag{4.2}$$

Subexample 5iii. The sequence for $\det(H_n^3(F_0))$ starts with $1, -1, -1, 0, 0, 0, -1, -1, 1, 1, 0, 0, 0, 1$ and continues with period 14. The verification for this case uses Proposition 4.2(ii) occasionally interspersed with Proposition 4.1. Here we will only sketch the verification. By defining F_1 so that $F_0(x) = 1 + x^2 + x^3F_1(x)$, one finds that

$$F_1 = \frac{1+2x+x^2+x^3}{1-2x^2-x^3-2x^4-x^5F_1}.$$

For the first transformation, with $F_2 = \mathcal{T}F_1$, we find

$$F_2 = \frac{1-2x+x^3}{-1+4x^2+x^3+2x^4-x^2(1+2x+x^2+x^3)F_2},$$

in which $u(x) = (-1+4x^2+x^3+2x^4)/(1-2x+x^3)$. Now, since $u(0) = -1$, one needs to apply Proposition 4.1 for the next transformation. One proceeds until a generating function equal to F_1 appears to establish the periodicity. We remark that $d = 0$ for each transformation until the final one which uses Proposition 4.2(ii) with $d = 3$ (this corresponds to a fourth order block).

Subexample 5iv. The sequence for $\det(H_n^4(F_0))$ begins with

$$2, 3, 4, 0, 0, -4, -5, -6, -7, -8, 0, 0, 8, 9, 10, 11, 12, 0, 0, -12, -13, -14, -15, -16,$$
$$0, 0, 16, \ldots.$$

For $n \geqslant 8$, an essence of periodicity can be gleaned from the recurrence

$$\det\bigl(H_n^4(F_0)\bigr) = 4\det\bigl(H_{n-1}(F_0)\bigr) - \det\bigl(H_{n-7}^4(F_0)\bigr),$$

for which we sketch a proof, often omitting the functional equations.

We will be applying the transformation $\mathcal{T}$ eight times, alternating its definition to be first under Proposition 4.1 and then under Proposition 4.2(ii). Let F_1 satisfy $F_0 = 1 + x^2 + x^3 + x^4 F_1$. Hence, $\det(H_n(F_1)) = \det(H_n^4(F_0))$, and

$$F_1 = \frac{2 + 3x + 2x^2 + 2x^3 + x^4}{1 - 2x^2 - x^3 - 2x^4 - 2x^5 - x^6 F_1}.$$

Here $u(0) = \frac{1}{2}$, where $u(x)$ is for F_1. Thus, with $F_2 = \mathcal{T} F_1$, $\det(H_n(F_2)) = (\frac{1}{2})^n \det(H_n(F_1))$. Now $d = 0$, where d is for F_2. With $F_3 = \mathcal{T} F_2$, $\det(H_{n-1}(F_3)) = \det(H_n(F_2))$.

Here $u(0) = \frac{4}{3}$, where $u(x)$ is for F_3. Thus, with $F_4 = \mathcal{T} F_3$, $\det(H_{n-1}(F_4)) = (\frac{4}{3})^{n-1} \det(H_{n-1}(F_3))$. Now $d = 0$, where d is for F_4. With $F_5 = \mathcal{T} F_4$, $\det(H_{n-2}(F_5)) = \det(H_{n-1}(F_4))$.

Here $u(0) = \frac{9}{8}$, where $u(x)$ is for F_5. Thus, with $F_6 = \mathcal{T} F_5$, $\det(H_{n-2}(F_6)) = (\frac{9}{8})^{n-2} \det(H_{n-2}(F_5))$. Now $d = 0$, where d is for F_6. With $F_7 = \mathcal{T} F_6$, $\det(H_{n-3}(F_7)) = \det(H_{n-2}(F_6))$.

Here $u(0) = \frac{4}{3}$, where $u(x)$ is for F_7. Thus, with $F_8 = \mathcal{T} F_7$, $\det(H_{n-3}(F_8)) = (\frac{4}{3})^{n-3} \det(H_{n-3}(F_7))$.

Now $d = 2$, where d is for F_8. With $F_9 = \mathcal{T} F_8$, $\det(H_{n-6}(F_9)) = -\det((H_{n-3}(F_8)) = (\frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \frac{8}{4}, 0, 0, -\frac{8}{4}, -\frac{9}{4}, -\frac{10}{4}, \ldots)$.

Thus (surprisingly),

$$\begin{aligned}\det\bigl(H_{n-6}(F_9)\bigr) &= -\left(\frac{1}{2}\right)^n \left(\frac{4}{3}\right)^{n-1} \left(\frac{9}{8}\right)^{n-2} \left(\frac{4}{3}\right)^{n-3} \det\bigl(H_n^4(F_0)\bigr) \\ &= -\frac{1}{4}\det\bigl(H_n^4(F_0)\bigr). \end{aligned} \tag{4.3}$$

Moreover,

$$F_9 = \frac{20 + 16x - 8x^2 - 4x^3 + x^4}{8(2 - 4x^2 - 2x^3 + x^4) - 16x^4 F_9} = \frac{5}{4} + x + 2x^2 + 3x^3 + 6x^4 + 10x^5 + \cdots.$$

It is easily verified that $F_9(x)$ and $\frac{1}{4} + G_0(x)$, where G_0 appears in Subexample 5ii, satisfy the same functional equation, and hence are equal. Therefore,

$$\begin{aligned}\det\bigl(H_{n-6}(F_9)\bigr) &= \left[\frac{xF_9(x) - yF_9(y)}{x - y}\right]_{n-6} \\ &= \left[\frac{1}{4} + \frac{xG_0(x) - yG_0(y)}{x - y}\right]_{n-6} \\ &= \frac{1}{4}\det\bigl(H_{n-7}^4(F_0)\bigr) + \det\bigl(H_{n-6}(G_0)\bigr)\end{aligned}$$

where $\frac{1}{4}\det(H^4_{n-7}(F_0))$ is $\frac{1}{4}$ times the determinant of the $(1,1)$-minor of $H_{n-6}(G_0)$, equivalently of $H^2_{n-6}(F_0)$. Combining this with identity (4.3) and noting $\det(H_n(G_0)) = -\det(H_{n+5}(F_0))$ from (4.2) proves the initial recurrence of this subexample.

Acknowledgments

We, particularly Xin, are most grateful to Ira Gessel. We also appreciate the suggestions made by Christian Krattenthaler, Lou Shapiro, and the referee of FPSAC-06.

References

[1] M. Aigner, Motzkin numbers, European J. Combin. 19 (1998) 663–675.
[2] C. Banderier, S. Schwer, Why Delannoy numbers?, J. Statist. Plann. Inference 135 (2005) 40–54.
[3] D. Bressoud, Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture, Cambridge Univ. Press, 1999.
[4] R.A. Brualdi, S. Kirkland, Aztec diamonds and digraphs, and Hankel determinants of Schröder numbers, J. Combin. Theory Ser. B 94 (2005) 334–351.
[5] I.M. Gessel, G.X. Viennot, Binomial determinants, paths, and hook length formulae, Adv. Math. 58 (3) (1985) 300–321.
[6] I.M. Gessel, G. Xin, The generating function of ternary trees and continued fractions, Electron. J. Combin. 13 (2006) R53 (electronic).
[7] W.B. Jones, W.J. Thron, Continued Fractions: Analytic Theory and Applications, Encyclopedia Math. Appl., vol. 11, Addison–Wesley, Reading, MA, 1980.
[8] C. Krattenthaler, Advanced determinant calculus: A complement, Linear Algebra Appl. 411 (2005) 68–166; The Andrews Festschrift, Maratea, 1998, Sem. Lothar. Combin. 42 (1999), Article B42q, 67 pp. (electronic).
[9] E. Pergola, R. Pinzani, S. Rinaldi, R.A. Sulanke, Lattice path moments by cut and paste, Adv. in Appl. Math. 30 (1–2) (2003) 208–218.
[10] L.W. Shapiro, A Catalan triangle, Discrete Math. 14 (1) (1976) 83–90.
[11] R.A. Sulanke, Moments of generalized Motzkin paths, J. Integer Seq. 3 (2000), Article 00.1.1.
[12] R.A. Sulanke, Objects counted by the central Delannoy numbers, J. Integer Seq. 6 (1) (2003), Article 03.1.5, 19 pp.
[13] G. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux, Lecture notes, Univ. Quebec, Montréal, Quebec, 1983.
[14] H.S. Wall, Analytic Theory of Continued Fractions, Van Nostrand, New York, 1948.