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Research Article

# Integral Formulae of Bernoulli and Genocchi Polynomials

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## Abstract

Recently, some interesting and new identities are introduced in the work of Kim et al. (2012). From these identities, we derive some new and interesting integral formulae for Bernoulli and Genocchi polynomials.

## 1. Introduction

As it is well known, the Bernoulli polynomials are defined by generating functions as follows:

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \tag{1.1}$$

(see [1–5]) with the usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x=0$, $B_n(0)=B_n$ are called the $n$th Bernoulli numbers.

The Genocchi polynomials are also defined by

$$\frac{2t}{e^t+1}e^{xt}=e^{G(x)t}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} \tag{1.2}$$

(see [1, 6–10]) with the usual convention about replacing $G^n(x)$ by $G_n(x)$. In the special case, $x=0$, $G_n(0)=G_n$ are called the $n$th Genocchi numbers.

From (1.1), we note that

$$B_n(x)=\sum_{l=0}^{n}\binom{n}{l}B_l x^{n-l} \tag{1.3}$$

(see [1–5]). Thus, by (1.3), we get

$$\frac{d}{dx}B_n(x)=n\sum_{l=0}^{n-1}\binom{n-1}{l}B_l x^{n-1-l}=nB_{n-1}(x) \tag{1.4}$$

(see [2]). From (1.2), we note that

$$G_n(x)=\sum_{l=0}^{n}\binom{n}{l}G_l x^{n-l}. \tag{1.5}$$

From (1.5), we can derive the following equation:

$$\frac{d}{dx}G_n(x) = n\sum_{l=0}^{n-1}\binom{n-1}{l}G_l x^{n-1-l} = nG_{n-1}(x). \tag{1.6}$$

By the definition of Bernoulli and Genocchi numbers, we get the following recurrence formulae:

$$B_0 = 1, \quad B_n(1) - B_n = \delta_{1,n}, \quad G_0 = 0, \quad G_n(1) + G_n = 2\delta_{1,n}, \tag{1.7}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [2]). From (1.4), (1.6), and (1.7), we note that

$$\int_0^1 B_n(x)\,dx = \frac{\delta_{0,n}}{n+1} \quad (n \geq 0), \qquad \int_0^1 G_n(x)\,dx = -\frac{2G_{n+1}}{n+1} \quad (n \geq 1). \tag{1.8}$$

From the identities of Bernoulli and Genocchi polynomials, we derive some new and interesting integral formulae of an arithmetical nature on the Bernoulli and Genocchi polynomials.

## 2. Integral Formula of Bernoulli and Genocchi Polynomials

From (1.1) and (1.2), we note that

$$\begin{aligned}
\frac{t}{e^t-1}e^{xt} &= \frac{1}{2}\left(\frac{2te^{xt}}{e^t+1}\right) + \frac{1}{t}\left(\frac{t}{e^t-1}\right)\left(\frac{2te^{xt}}{e^t+1}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\right) + \frac{1}{t}\left(\sum_{l=0}^{\infty}B_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} + \frac{1}{t}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}G_l(x)B_{n-l}\frac{t^n}{n!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\left(-\frac{1}{2}G_n(x) + \sum_{\substack{l=0\\ l\neq n}}^{n+1}\frac{\binom{n+1}{l}G_l(x)B_{n+1-l}}{n+1}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq n}}^{n+1}\binom{n+1}{l}\frac{G_l(x)B_{n+1-l}}{n+1}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.1}$$

By comparing the coefficients on the both sides of (2.1), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$B_n(x) = \sum_{l=0\ l\neq n}^{n+1}\binom{n+1}{l}\frac{G_l(x)B_{n+1-l}}{n+1}. \tag{2.2}$$

From (1.1) and (1.2), also notes that

$$\begin{aligned}
\frac{2t}{e^t+1}e^{xt} &= \frac{1}{t}\left(\frac{2t(e^t-1)}{e^t+1}\right)\left(\frac{te^{xt}}{e^t-1}\right) = \frac{1}{t}\left(2t - 2\frac{2t}{e^t+1}\right)\left(\frac{te^{xt}}{e^t-1}\right) \\
&= \frac{1}{t}\left(2t - 2\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\right) \\
&= \frac{1}{t}\left(-2\sum_{l=1}^{\infty}\frac{G_{l+1}}{l+1}\frac{t^{l+1}}{l!}\right)\left(\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\right) \\
&= \sum_{n=1}^{\infty}\left(-2\sum_{l=1}^{n}\binom{n}{l}\frac{G_{l+1}}{l+1}B_{n-l}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

By comparing the coefficients on the both sides of (2.3), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$, one has*

$$G_n(x) = -2\sum_{l=1}^{n}\binom{n}{l}\frac{G_{l+1}}{l+1}B_{n-l}(x). \tag{2.4}$$

Let one take the definite integral from 0 to 1 on both sides of Theorem 2.1. For $n \geq 2$,

$$0 = -2\sum_{\substack{l=1\\ l\neq n}}^{n+1}\binom{n+1}{l}\frac{G_{l+1}}{l+1}\frac{B_{n+1-l}}{n+1} = -B_nG_2 - 2\sum_{\substack{l=1\\ l\neq n-1}}^{n}\binom{n}{l}\frac{B_{n-l}G_{l+2}}{(l+1)(l+2)}. \quad (2.5)$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$B_n = 2\sum_{\substack{l=1\\ l\neq n-1}}^{n}\binom{n}{l}\frac{B_{n-l}G_{l+2}}{(l+1)(l+2)}. \quad (2.6)$$

## 3. $p$-Adic Integral on $\mathbb{Z}_p$ Associated with Bernoulli and Genocchi Numbers

Let $p$ be a fixed odd prime number. Throughout this section, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic $p$-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x) \quad (3.1)$$

(see [2, 5, 11]). From (3.1), we can derive the following integral equation:

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \quad (n \in \mathbb{N}), \quad (3.2)$$

where $f_n(x) = f(x+n)$ and $f'(i) = ((df(x))/dx)|_{x=i}$ (see [2]). Let us take $f(y) = e^{t(x+y)}$. Then we have

$$\int_{\mathbb{Z}_p} e^{t(x+y)}d\mu(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \quad (3.3)$$

(see [2, 5]). From (3.3), we have

$$\int_{\mathbb{Z}_p}(x+n)^n d\mu(y) = B_n(x), \qquad \int_{\mathbb{Z}_p} y^n d\mu(y) = B_n \quad (3.4)$$

(see [2, 5]). Thus, by (3.2) and (3.4), we get

$$\int_{\mathbb{Z}_p}(x+n)^m d\mu(x) = \int_{\mathbb{Z}_p} x^m d\mu(x) + m\sum_{i=0}^{n-1} i^{m-1}, \quad (3.5)$$

(see [2]). From (3.5), we have

$$B_m(n) - B_m = m\sum_{i=0}^{n-1} i^{m-1} \quad (n \in \mathbb{Z}_+) \quad (3.6)$$

(see [2]). The fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows [2, 8, 9]:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x)(-1)^x. \quad (3.7)$$

From (3.7), we obtain the following integral equation:

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1}(-1)^{n-l-1} f(l) \quad (3.8)$$

(see [2]), where $f_n(x) = f(x+n)$. Thus, by (3.8), we have

$$\int_{\mathbb{Z}_p}(x+n)^m d\mu_{-1}(x) = (-1)^n\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) + 2\sum_{l=0}^{n-1}(-1)^{n-l-1} l^m \quad (3.9)$$

(see [2]). Let us take $f(y) = e^{t(x+y)}$. Then we have

$$t\int_{\mathbb{Z}_p} e^{t(x+y)}d\mu_{-1}(y) = \frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}. \quad (3.10)$$

From (3.10), we have

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}, \int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) = \frac{G_{n+1}}{n+1}. \tag{3.11}$$

Thus, by (3.9) and (3.11), we get

$$\frac{G_{m+1}(n)}{m+1} = (-1)^n \left( \frac{G_{n+1}}{n+1} + 2\sum_{l=0}^{n-1} (-1)^{l-1} l^m \right). \tag{3.12}$$

Let us consider the following $p$-adic integral on $\mathbb{Z}_p$:

$$K_1 = \int_{\mathbb{Z}_p} B_n(x)\, d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} x^l d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l. \tag{3.13}$$

From Theorem 2.1 and (3.13), one has

$$\begin{aligned} K_1 &= \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \binom{n+1}{k} \frac{B_{n+1-k}}{n+1} \sum_{l=0}^{k} \binom{k}{l} G_{k-l} \int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} \frac{B_{n+1-k} B_l G_{k-l}}{n+1}. \end{aligned} \tag{3.14}$$

Therefore, by (3.13) and (3.14), we obtain the following theorem.

Theorem 3.1. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l = \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} \frac{B_{n+1-k} B_l G_{k-l}}{n+1}. \tag{3.15}$$

Now, one sets

$$K_2 = \int_{\mathbb{Z}_p} B_n(x)\, d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1}. \tag{3.16}$$

By Theorem 2.1, one gets

$$\begin{aligned} K_2 &= \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \binom{n+1}{k} \frac{B_{n+1-k}}{n+1} \sum_{l=0}^{k} \binom{k}{l} G_{k-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} \frac{B_{n+1-k} G_{k-l} G_{l+1}}{(n+1)(l+1)}. \end{aligned} \tag{3.17}$$

Therefore, by (3.16) and (3.17), we obtain the following theorem.

Theorem 3.2. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1} = \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} \frac{B_{n+1-k} G_{k-l} G_{l+1}}{(n+1)(l+1)}. \tag{3.18}$$

Let us consider the following $p$-adic integral on $\mathbb{Z}_p$:

$$K_3 = \int_{\mathbb{Z}_p} G_n(x)\, d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \frac{G_{l+1}}{l+1}. \tag{3.19}$$

From Theorem 2.2, one has

$$K_3 = -2\sum_{l=1}^{n}\binom{n}{l}\frac{G_{l+1}}{l+1}\sum_{k=0}^{n-l}\binom{n-l}{k}B_{n-l-k}\int_{\mathbb{Z}_p}x^k d\mu_{-1}(x)$$
$$= -2\sum_{l=1}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}B_{n-l-k}\frac{G_{l+1}G_{k+1}}{(l+1)(k+1)}. \quad (3.20)$$

Therefore, by (3.19) and (3.20), we obtain the following theorem.

Theorem 3.3. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}\frac{G_{n-l}G_{l+1}}{l+1} = -2\sum_{l=1}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{B_{n-l-k}G_{l+1}G_{k+1}}{(l+1)(k+1)}. \quad (3.21)$$

Now, one sets

$$K_4 = \int_{\mathbb{Z}_p}G_n(x)\,d\mu(x) = \sum_{l=0}^{n}\binom{n}{l}G_{n-l}B_l. \quad (3.22)$$

By Theorem 2.2, one gets

$$K_4 = -2\sum_{l=1}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{G_{l+1}}{l+1}B_{n-l-k}B_k. \quad (3.23)$$

Therefore, by (3.22) and (3.23), we obtain the following corollary.

Corollary 3.4. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}G_{n-l}B_l = -2\sum_{l=1}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{G_{l+1}B_{n-l-k}B_k}{l+1}. \quad (3.24)$$

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